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SOME TYPES OF SLANT SUBMANIFOLDS OF BRONZE RIEMANNIAN MANIFOLDS

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Abstract. The aim of this article is to examine some types of slant submanifolds of bronze Riemannian manifolds. We introduce hemi-slant submanifolds of a bronze Riemannian manifold. We obtain integrability conditions for the distribution involved in quasi hemi-slant submanifold of a bronze Riemannian manifold. Also, we give some examples about this type submanifolds.

1. Introduction

In 1990, B.Y. Chen introduced the geometry of slant submanifolds in complex manifolds [3]. Then this topic was extended to semi-slant, pseudo-slant and bi-slant in different structure. Semi-slant submanifolds in almost Hermitian manifolds were introduced by N. Papagiuc [8]. Semi-slant submanifolds in Sasakian manifolds were studies by J.L. Cabrerizo [7].

Metallic structure was introduced V. W. de Spinadel [12]. Let p and q be positive integers. So, members of the metallic means family are positive solution

$$x^2 - px - q = 0,$$

and this number, which are called $(p, q)-$metallic numbers denoted by [4]

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

By use of metallic means family, in [4], the authors introduced the metallic structure which is given by J of type $(1, 1)-$tensor field satisfying

$$J^2 = pJ + qI.$$

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Some remarks on this structure were studied by many geometers (see [1, 5, 11, 2, 6]).

If $p = 3$ and $q = 1$, then $\tilde{J}$ is called bronze structure which satisfies

$$\tilde{J}^2 = 3\tilde{J} + I,$$

where I is the identity tensor [9]. In [9], the authors studied the notion of bronze structure on manifolds, using the bronze mean is defined by

$$\phi_{br} = \frac{3+\sqrt{13}}{2},$$

which is the positive solution

$$x^2 - 3x - 1 = 0.$$

Recently, in [10], twin bronze Riemannian metric was studies and some geometric charecterization was given by author.

In this article, we introduced the notion of hemi-slant and quasi-hemi-slant submanifolds of bronze Riemannian manifolds. Especially, we defined a new example of this structure and we provide some non-trivial examples of this types submanifolds.

2. Preliminaries

Firstly, we give definitions and theorems then we give a new example of bronze Riemannian manifolds.

Let $\tilde{N}$ be a differentiable manifold with (1,1)-tensor field $\tilde{J}$. Then we say that $\tilde{J}$ is a bronze structure if

$$\tilde{J}^2 = 3\tilde{J} + I. \tag{2.1}$$

So, $(\tilde{N}, \tilde{J})$ is called a bronze manifold.

If $(\tilde{N}, \tilde{g})$ is a Riemannian manifold with $\tilde{J}$ bronze structure, such that $\tilde{g}$ is $\tilde{J}$-compatible

$$\tilde{g}(\tilde{J}U, V) = \tilde{g}(U, \tilde{J}V), \tag{2.2}$$

then $(\tilde{N}, \tilde{g}, \tilde{J})$ is a bronze Riemannian manifold. From (2.2), one can write that

$$\tilde{g}(\tilde{J}U, \tilde{J}V) = 3\tilde{g}(\tilde{J}U, V) + \tilde{g}(U, V), \tag{2.3}$$

for any $U, V \in \Gamma(T\tilde{N})$.

Proposition 2.1 ([10]). *If $\tilde{J}$ is a bronze structure on manifold $\tilde{N}$, then*

$$\tilde{F} = \frac{1}{\sqrt{13}}(2\tilde{J} - 3I),$$

is an almost product structure on $\tilde{N}$. Conversely every almost product structure $\tilde{F}$ on $\tilde{N}$ induces two bronze structure satisfies that

$$\tilde{J}_1 = \frac{1}{2}(3I + \sqrt{13}\tilde{F}) \quad \textit{and} \quad \tilde{J}_2 = \frac{1}{2}(3I - \sqrt{13}\tilde{F}).$$

Now, we give a new example of bronze structure

Example 1. Let $\mathbb{R}^4$ be a real space and give a map by

$$\begin{aligned} \tilde{J} \quad &: \quad \mathbb{R}^4 \to \mathbb{R}^4 \\ (\omega_1, \omega_2, \omega_3, \omega_4) \quad &\to \quad (\phi_{br}\omega_1, \bar{\phi}_{br}\omega_2, \phi_{br}\omega_3, \bar{\phi}_{br}\omega_4), \end{aligned}$$

where $\phi_{br} = \frac{3+\sqrt{13}}{2}$ and $\bar{\phi}_{br} = \frac{3-\sqrt{13}}{2}$. In this case $\tilde{J}$ satisfies the equation (2.1). So, we can say that $(\mathbb{R}^4, \tilde{J})$ is an example of bronze structure.

Let N^n be a submanifold of m-dimensional bronze Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$.The tangent space of $\tilde{N}$ in a point $x \in N$ can be decomposed by

$$T_x\tilde{N} = T_xN \perp T_x^{\perp}N.$$

If we show fU and tU, the tangential and normal parts of $\tilde{J}U$, respectively, we can write

$$\tilde{J}U = fU + tU, \tag{2.4}$$

for any $U \in \Gamma(TN)$.

Similarly for $Z \in \Gamma(T^{\perp}N)$, the tangential and normal parts of $\tilde{J}Z$ satisfiy

$$\tilde{J}Z = BZ + CZ. \tag{2.5}$$

If we consider the properties of f and C, we have

$$\tilde{g}(fU, V) = \tilde{g}(U, fV), \tag{2.6}$$

$$\tilde{g}(CZ, W) = \tilde{g}(Z, CW), \tag{2.7}$$

for any $U, V \in \Gamma(TN)$, $Z, W \in \Gamma(T^{\perp}N)$. Also, we have

$$\tilde{g}(tU, Z) = \tilde{g}(U, BZ). \tag{2.8}$$

Proposition 2.2. *Let N be a submanifold of bronze Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$.Then for any $U \in \Gamma(TN)$, $Z \in \Gamma(T^{\perp}N)$, we have*

$$f^2U = 3fU + U - BCU, \tag{2.9}$$

$$3tU = ftU + CtU, \tag{2.10}$$

$$C^2Z = 3CZ + Z - BtZ, \tag{2.11}$$

$$3BZ = BfZ + BCZ. \tag{2.12}$$

Also, Gauss and Weingarten equations are definmed by

$$\tilde{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.13}$$

$$\tilde{\nabla}_U Z = -A_Z U + \nabla^t_U Z, \tag{2.14}$$

where $\tilde{\nabla}$ and ∇ are Levi-Civita connection on $(\tilde{N},\tilde{g})$ and (N,g), respectively. Moreover

$$\tilde{g}(h(U,V),Z) = g(A_Z U, V). \tag{2.15}$$

In this paper, we suppose that

$$\tilde{\nabla}\tilde{J} = 0, \tag{2.16}$$

i.e., $\tilde{N}$ is a locally bronze Riemannian manifold.

Lemma 2.3. *If $\tilde{N}$ is a locally bronze Riemannian manifold, then*

$$(\nabla_U f)V = A_{ZV}U + Bh(U,V), \tag{2.17}$$

$$(\nabla_U t)V = Ch(U,V) - h(U,fV), \tag{2.18}$$

$$(\nabla_U B)Z = A_{CZ}U - fA_Z U, \tag{2.19}$$

$$(\nabla_U C)Z = -h(U,BZ) - tA_Z U, \tag{2.20}$$

for any $U,V \in \Gamma(TN)$, $Z \in \Gamma(T^\perp N)$.

3. Hemi-slant Submanifolds of $(\tilde{N},\tilde{g},\tilde{J})$

In this section, we give main results and new type examples for hemi-slant submanifolds of bronze Riemannian manifold $(\tilde{N},\tilde{g},\tilde{J})$.

Definition 3.1. Let (N,g) be a submanifold of $(\tilde{N},\tilde{g},\tilde{J})$. Then N is called a *hemi-slant submanifold* if the following conditions are satisfied:

i) There exist two orthogonal distributions D_α and D_β such that

$$TN = D_\alpha \perp D_\beta,$$

ii) The distributions D_α is slant with angle $\theta \in [0, \frac{\pi}{2}]$

iii) The distributions D_β is anti-invariant, $\tilde{J}D_\beta \subseteq \Gamma(T^\perp N)$.

Also, if $\dim(D_\alpha)\cdot\dim(D_\beta) \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then N is called a *proper hemi-slant submanifolds.*

Example 2. Let $\mathbb{R}^4$ be the Euclidean space with the usual Euclidean metric. We define the bronze structure

$$\begin{aligned} \tilde{J} \quad &: \quad \mathbb{R}^4 \to \mathbb{R}^4 \\ (\omega_1, \omega_2, \omega_3, \omega_4) \quad &\to \quad (\phi_{br}\omega_1, \bar{\phi}_{br}\omega_2, \omega_3 + \frac{3}{2}\omega_4, -\omega_3), \end{aligned}$$

where $\phi_{br} = \frac{3+\sqrt{13}}{2}$ and $\bar{\phi}_{br} = \frac{3-\sqrt{13}}{2}$. One can easily verify that the equation (2.1). So, $(\mathbb{R}^4, \tilde{J})$ is a new example of bronze Riemannian manifold.

Assume that N is a submanifold of $(\mathbb{R}^4, \tilde{J})$ defined by

$$x_1 = u\cos s, \qquad x_2 = u\sin s,$$

$$x_3 = v, \qquad x_4 = -2v,$$

Then, a local orthonormal frame on TN given by

$$\tilde{\Psi}_1 = \cos s\partial x_1 + \sin s\partial x_2,$$

$$\tilde{\Psi}_2 = \partial x_3 - 2\partial x_4,$$

Thus, we arrive at $\tilde{J}(\tilde{\Psi}_2)\perp Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2\}$ and $cos\theta = \frac{\phi_{br}\cos^2 s+\bar{\phi}_{br}\sin^2 s}{\sqrt{\phi_{br}\cos^2 s+\bar{\phi}_{br}\sin^2 s}}$. If we consider $D_\beta = Sp\{\tilde{\Psi}_2\}$ $(\tilde{J}(\tilde{\Psi}_2) \subseteq \Gamma(T^\perp N))$ and $D_\alpha = Sp\{\tilde{\Psi}_1\}$ then N is a hemi-slant submanifold in $(\tilde{N}, \tilde{g}, \tilde{J})$.

Example 3. Let $\mathbb{R}^6$ be the Euclidean space with the usual Euclidean metric. We define the bronze structure

$$\begin{aligned} \tilde{J} \quad &: \quad \mathbb{R}^6 \to \mathbb{R}^6 \\ (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) \quad &\to \quad \begin{pmatrix} \phi_{br}\omega_1, \bar{\phi}_{br}\omega_2, \bar{\phi}_{br}\omega_3, \phi_{br}\omega_4 \\ \omega_5 + \frac{3}{2}\omega_6, -\omega_5 \end{pmatrix}, \end{aligned}$$

where $\phi_{br} = \frac{3+\sqrt{13}}{2}$ and $\bar{\phi}_{br} = \frac{3-\sqrt{13}}{2}$. Thus $(\mathbb{R}^6, \tilde{J})$ is a bronze Riemannian manifold.

Assume that N is a submanifold of $(\mathbb{R}^6, \tilde{J})$ defined by

$$x_1 = u\cos s, \qquad x_2 = u\sin s,$$

$$x_3 = v, \qquad x_4 = \bar{\phi}_{br}v,$$

$$x_5 = \frac{1}{\sqrt{3}}w, \qquad x_6 = -\frac{2}{\sqrt{3}}w.$$

Then, we can obtain a local orthonormal frame on TN given by

$$\tilde{\Psi}_1 = \cos s\partial x_1 + \sin s\partial x_2,$$

$$\tilde{\Psi}_2 = \partial x_3 + \bar{\phi}_{br}\partial x_4,$$

$$\tilde{\Psi}_3 = \frac{1}{\sqrt{3}}\partial x_5 - \frac{2}{\sqrt{3}}\partial x_6$$

So, we get $\tilde{J}(\tilde{\Psi}_2)\perp Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3\}$ and $\tilde{J}(\tilde{\Psi}_3)\perp Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3\}$ which gives $cos\theta = \frac{\phi_{br}\cos^2 s + \bar{\phi}_{br}\sin^2 s}{\sqrt{\phi_{br}\cos^2 s + \bar{\phi}_{br}\sin^2 s}}$. Therefore $D_\alpha = Sp\{\tilde{\Psi}_1\}$ and $D_\beta = Sp\{\tilde{\Psi}_2, \tilde{\Psi}_3\}$, then N is a hemi-slant submanifold in $(\mathbb{R}^6, \tilde{g}, \tilde{J})$.

Let P_α and P_β orthogonal projections on D_α and D_β, respectively. For $U \in \Gamma(TN)$, we can state

$$U = P_\alpha U + P_\beta U,$$

where $P_\alpha U \in \Gamma(D_\alpha)$ and $P_\beta U \in \Gamma(D_\beta)$.

From the definition of hemi-slant submanifold, we have;

Lemma 3.2. *Let (N, g) hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then*

$$\begin{aligned} \tilde{J}U &= fP_\alpha U + tP_\alpha U + tP_\beta U \\ &= fP_\alpha U + tU, \end{aligned} \tag{3.1}$$

$$\tilde{J}P_\beta U = tP_\beta U, \quad fP_\beta U = 0, \quad fP_\alpha U \in \Gamma(D_\alpha), \tag{3.2}$$

for any $U \in \Gamma(TN)$.

Remark 1. If N is a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$, then $\tilde{g}(\tilde{J}P_\alpha U, fP_\alpha U) = \cos\theta(X)\|\tilde{J}P_\alpha U\|\|fP_\alpha U\|$ and slant angle $\theta(X)$ of the distribution D_α is constant.

So, for $U \in \Gamma(TN)$, we have

$$\begin{aligned} \cos\theta &= \frac{\tilde{g}(\tilde{J}P_\alpha U, fP_\alpha U)}{\|\tilde{J}P_\alpha U\|\|fP_\alpha U\|} \\ &= \frac{\|fP_\alpha U\|}{\|\tilde{J}P_\alpha U\|}. \end{aligned} \tag{3.3}$$

Proposition 3.3. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. For every $U, Y \in \Gamma(TN)$, we get*

$$\tilde{g}(fP_\alpha U, fP_\alpha Y) = \cos^2\theta(3\tilde{g}(fP_\alpha U, P_\alpha Y) + \tilde{g}(P_\alpha U, P_\alpha Y), \tag{3.4}$$

$$\tilde{g}(tU, tY) = \sin^2\theta(3\tilde{g}(fP_\alpha U, P_\alpha Y) + \tilde{g}(P_\alpha U, P_\alpha Y). \tag{3.5}$$

Proposition 3.4. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$ with slant angle θ of D_β. In this case*

$$(fP_\alpha)^2 = \cos^2\theta(3fP\alpha + I), \tag{3.6}$$

$$\nabla(fP_\alpha)^2 = 3\cos^2\theta\nabla(fP_\alpha). \tag{3.7}$$

Now, we give the conditions for the integrability of the distribution of $(\tilde{N}, \tilde{g}, \tilde{J})$.

Theorem 3.5. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. In this case, for any $U, V \in \Gamma(D_\alpha)$*

$$\nabla_U FV - \nabla_V fU + A_{NU}V - A_{NV}U \in \Gamma(D_\alpha). \tag{3.8}$$

Proof. From (2.6), for any $U, V \in \Gamma(D_\alpha)$ and $W \in \Gamma(D_\beta)$, we find

$$\tilde{g}(f[U, V], Z) = \tilde{g}([U, V], fZ) = 0,$$

which gives $fZ = 0$. So, we arrive at $f[U, V] \in \Gamma(D_\alpha)$ and (3.8). □

Theorem 3.6. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_α is integrable.*

Proof. By using (2.3), for every $U, V \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$, we

$$\tilde{g}(\tilde{\nabla}_U V, Z) = \tilde{g}(\tilde{J}\tilde{\nabla}_U V, \tilde{J}Z) - 3\tilde{g}(\tilde{\nabla}_U V, \tilde{J}Z).$$

From the definition of hemi-slant submanifold of a bronze Riemannian manifold we get $\tilde{J}Z = tZ$, $Z \in \Gamma(D_\beta)$. So, we have

$$\tilde{g}(\tilde{\nabla}_U V, Z) = \tilde{g}(\tilde{\nabla}_U \tilde{J}V, tZ) - 3\tilde{g}(\tilde{\nabla}_U V, tZ).$$

From (2.13), we get

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_U V, Z) &= \tilde{g}(h(U, fV), tZ) + \tilde{g}(\nabla^\perp_U tV, tZ) \\ &\quad -3\tilde{g}(h(U, V), tZ). \end{aligned}$$

In view of (2.18), we can write $\nabla^\perp_U tV = Ch(U, V) - h(u, fV) + t\nabla_U V$ for any $U, V \in \Gamma(D_\alpha)$, which gives

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_U V, Z) &= \tilde{g}(Ch(U, V), tZ) + \tilde{g}(t\nabla_U V, tZ) \\ &\quad -3\tilde{g}(h(U, V), tZ). \end{aligned} \tag{3.9}$$

From (2.18) and symmetric properties of h, we arrive at

$$\begin{aligned} \tilde{g}([U, V], Z) &= \tilde{g}(t\nabla_U V, tZ) - \tilde{g}(t\nabla_V U, tZ) \\ &= \tilde{g}(t[U, V], tZ). \end{aligned}$$

Thus from (3.5), weobtain

$$\tilde{g}([U,V],Z) = \sin^2\theta \begin{pmatrix} 3\tilde{g}(P_\alpha[U,V], fP_\alpha Z) \\ +\tilde{g}(P_\alpha[U,V], P_\alpha Z \end{pmatrix}.$$

Since $P_\alpha Z$ is the projection of Z on $\Gamma(D_\alpha)$ then $P_\alpha Z = 0$, for $Z \in \Gamma(D_\beta)$. o, we arrive at

$$\tilde{g}([U,V],Z) = 0,$$

which gives proof of our assertion. □

Theorem 3.7. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_β is integrable if and only if*

$$A_{tZ}W = 0, \tag{3.10}$$

for $Z, W \in \Gamma(D_\beta)$.

Proof. For $Z, W \in \Gamma(D_\beta)$, we can write $fZ = fW = 0$. So

$$\nabla_Z fW = \nabla_W fZ = 0.$$

In view of (3.2) for $Z, W \in \Gamma(D_\beta)$, $f([Z,W]) = 0$ iff $A_{tZ}W = A_{tW}Z = 0$. From (2.17), we have

$$\begin{aligned} 0 &= \tilde{g}((\nabla_U f)Z, W) = \tilde{g}(A_{tZ}U, W) + \tilde{g}(Bh(U,Z), W) \\ &= \tilde{g}(\nabla_U Z, fW), \end{aligned}$$

from which we find $\tilde{g}(A_{tZ}U, W) = -\tilde{g}(Bh(U,Z), W)$.

If we consider for $U \in \Gamma(TN)$, $Z, W \in \Gamma(D_\beta)$, we get

$$\begin{aligned} \tilde{g}(A_{tZ}U, W) &= \tilde{g}(A_{tW}U, Z) = \tilde{g}(A_{tW}Z, U) \\ &= \tilde{g}(h(U,Z), tW) = \tilde{g}(th(U,Z), W), \end{aligned}$$

then we arrive at (3.10).

Contrarily, we suppose that $A_{tW}Z = 0$, for $Z, W \in \Gamma(D_\beta)$. In this case, we get $\tilde{g}(A_{tW}Z, U) = \tilde{g}(Bh(U,Z), W)$. From (2.17) with last equation for $U \in \Gamma(D_\alpha)$, $Z, W \in \Gamma(D_\beta)$, we find

$$0 = \tilde{g}((\nabla_Z f)W, U) = \tilde{g}(f\nabla_Z W, U) = \tilde{g}(\nabla_Z W, fU).$$

Also, we know that $f(D_\alpha) = D_\alpha$ we arrive at $\nabla_Z W \in \Gamma(D_\beta)$. So the assertion was proved. □

Now, we consider mixed totally geodesic hemi-slant submanifolds of Bronze Riemannian manifold. Firstly we give following.

Definition 3.8. Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then N is called a *mixed totally geodesic submanifold* if for $U \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$

$$h(U, Z) = 0. \tag{3.11}$$

Theorem 3.9. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then N is a mixed totally geodesic submanifold if*

i) $A_N U \in \Gamma(D_\alpha)$,

ii) $A_N Z \in \Gamma(D_\beta)$,

for $U \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$.

Proof. For any $U \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$, from (2.15), we have

$$\tilde{g}(h(U, Z), N) = \tilde{g}(A_N U, Z) = \tilde{g}(A_N Z, U),$$

which yields N is a mixed totally geodesic submanifold if and only if $A_N U \in \Gamma(D_\alpha)$ and $A_N Z \in \Gamma(D_\beta)$. □

Theorem 3.10. *Let N be a hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. If $(\tilde{\nabla}_U t)Z = 0$, for $U \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$, then N is a mixed totally geodesic submanifold in $\tilde{N}$.*

Proof. From $(\tilde{\nabla}_U t)Z = 0$ and (2.18) with $fZ = 0$ we find

$$h(Z, fU) = nh(U, Z) = h(U, fZ) = 0,$$

for $U \in \Gamma(D_\alpha)$ and $Z \in \Gamma(D_\beta)$.

In view of (3.6), we get

$$3\cos^2\theta C h(Z, fU) + \cos^2\theta h(Z, U) = 0.$$

By use of $Ch(Z, fU) = 0$ and $\theta \neq \frac{\pi}{2}$ we arrive at $h(U, Z) = 0$. □

4. Quasi Hemi-slant Submanifolds of $(\tilde{N}, \tilde{g}, \tilde{J})$

Now, we introduce and characterize quasi hemi-slant submanifolds of bronze Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$ and give an examples of this type submanifold.

Definition 4.1. Let (N, g) be a submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then N is called a *quasi hemi-slant submanifold* if the following conditions are satisfied:

i) There exist orthogonal distributions D_γ, D_α and D_β such that

$$TN = D_\gamma \perp D_\alpha \perp D_\beta,$$

ii) The distributions D_γ is invariant i.e., $\tilde{J}(D_\gamma) = D_\gamma$,
iii) The distributions D_α is slant with angle θ,
iv) The distributions D_β is anti-invariant, $\tilde{J}D_\beta \subseteq \Gamma(T^\perp N)$.

Also, we say that quasi hemi-slant submanifolds are *proper* if $D_\gamma \neq \{0\}$, $D_\beta \neq \{0\}$ and $\theta \neq (0, \frac{\pi}{2})$.

Example 4. Let $\mathbb{R}^8$ be the Euclidean space with the usual Euclidean metric. We define the bronze structure

$$\begin{array}{rcl} \tilde{J} & : & \mathbb{R}^8 \to \mathbb{R}^8 \\ (\omega_1, \omega_2, ..., \omega_8) & \to & \left(\begin{array}{c} \phi_{br}\omega_1, \phi_{br}\omega_2, \bar{\phi}_{br}\omega_3, \phi_{br}\omega_4, \\ \omega_5 + \frac{3}{2}\omega_6, -\omega_5, \bar{\phi}_{br}\omega_7, \bar{\phi}_{br}\omega_8 \end{array} \right), \end{array}$$

where $\phi_{br} = \frac{3+\sqrt{13}}{2}$ and $\bar{\phi}_{br} = \frac{3-\sqrt{13}}{2}$. One can easily verify that the equation (2.1). So, $(\mathbb{R}^8, \tilde{J})$ is a new example of bronze Riemannian manifold.

Assume that N is a submanifold of $(\mathbb{R}^8, \tilde{J})$ defined by

$$x_1 = \frac{u_1 + 3u_2}{2}, \quad x_2 = \frac{u_1 + 3u_2}{2},$$

$$x_3 = u_3 \cos s, \quad x_4 = u_3 \sin s,$$

$$x_5 = u_4, \quad x_6 = -u_4,$$

$$x_7 = \frac{u_5 + \sqrt{13}u_6}{2}, \quad x_8 = \frac{u_5 + \sqrt{13}u_6}{2}$$

Then, a local orthonormal frame on TN given by

$$\tilde{\Psi}_1 = \frac{1}{2}\left(\partial x_1 + \partial x_2\right),$$

$$\tilde{\Psi}_2 = \frac{3}{2}\left(\partial x_1 + \partial x_2\right),$$

$$\tilde{\Psi}_3 = \cos s \partial x_3 + \sin s \partial x_4,$$

$$\tilde{\Psi}_4 = \partial x_5 - 2\partial x_6,$$

$$\tilde{\Psi}_5 = \frac{1}{2}\left(\partial x_7 + \partial x_8\right),$$

$$\tilde{\Psi}_6 = \frac{\sqrt{13}}{2}\left(\partial x_7 + \partial x_8\right),$$

Thus, we get $\tilde{J}(\tilde{\Psi}_4) \perp Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3, \tilde{\Psi}_4, \tilde{\Psi}_5, \tilde{\Psi}_6\}$ and $cos\theta = \frac{\phi_{br}\cos^2 s + \bar{\phi}_{br}\sin^2 s}{\sqrt{\phi_{br}\cos^2 s + \bar{\phi}_{br}\sin^2 s}}$. If we consider $D_\gamma = Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_5, \tilde{\Psi}_6\}$, $D_\alpha = Sp\{\tilde{\Psi}_3\}$ and $D_\beta = Sp\{\tilde{\Psi}_4\}$ ($\tilde{J}(\tilde{\Psi}_4) \subseteq \Gamma(T^\perp N)$) and then N is a quasi hemi-slant submanifold in $(\tilde{N}, \tilde{g}, \tilde{J})$.

Let D_γ, P_α and P_β orthogonal projections on D_γ, D_α and D_β, respectively. For $U \in \Gamma(TN)$, we have

$$U = P_\gamma U + P_\alpha U + P_\beta U, \tag{4.1}$$

where $P_\gamma U \in \Gamma(D_\gamma)$, $P_\alpha U \in \Gamma(D_\alpha)$ and $P_\beta U \in \Gamma(D_\beta)$.

In view of (4.1), we get

$$\tilde{J}U = fP_\gamma U + tP_\gamma U + fP_\alpha U + tP_\alpha U + fP_\beta U + tP_\beta U.$$

Since $\tilde{J}(D_\gamma) = D_\gamma$, $\tilde{J}D_\beta \subseteq \Gamma(T^\perp N)$, then $tP_\alpha U = 0 = tP_\beta U$. So, we obtain

$$\tilde{J}U = fP_\gamma U + tP_\gamma U + tP_\alpha U + tP_\beta U. \tag{4.2}$$

For $U, V \in \Gamma(TN)$, we have

$$\nabla_U fV - A_{tV}U - f\nabla_U V - Bh(U, V) = 0,$$

$$h(U, fV) + \nabla^\perp_U tV - t(\nabla_U V) - Ch(U, V) = 0.$$

Also, $Z, W \in \Gamma(T^\perp N)$, we get

$$f([Z, W]) = A_{\tilde{J}Z}W - A_{\tilde{J}W}Z,$$

$$t([Z, W]) = \nabla^\perp_Z \tilde{J}W - \nabla^\perp_W \tilde{J}Z.$$

Now, we examine integrability conditions for the distribution involved in submanifold.

Theorem 4.2. *Let N be a quasi hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_γ is integrable if and only if*

$$g(\nabla_U fV - \nabla_V fU, fP_\alpha X) = g(h(V, fU) - h(U, fV), tP_\alpha X + tP_\beta X),$$

and

$$g(\nabla_U V - \nabla_V U, fP_\alpha X) = g(h(V, U) - h(U, V), tP_\alpha X + tP_\beta X),$$

for $U, V \in \Gamma(D_\gamma)$, $X \in \Gamma(D_\alpha \perp D_\beta)$.

Proof. For $U, V \in \Gamma(D_\gamma)$, $X = P_\alpha X + P_\beta X \in \Gamma(D_\alpha \perp D_\beta)$, using (2.3),(2.13), (2.16) with (2.4),we get

$$\begin{aligned}
g([U,V],X) &= g(\tilde{J}\tilde{\nabla}_U V, \tilde{J}X) - g(\tilde{J}\tilde{\nabla}_V U, \tilde{J}X) \\
&\quad -3g(\tilde{J}\tilde{\nabla}_U V, X) + 3g(\tilde{J}\tilde{\nabla}_V U, X) \\
&= g(\tilde{\nabla}_U \tilde{J}V, \tilde{J}X) - g(\tilde{\nabla}_V \tilde{J}U, \tilde{J}X) \\
&\quad -3g(\tilde{\nabla}_U V, \tilde{J}X) + 3g(\tilde{\nabla}_V U, \tilde{J}X) \\
&= g(\nabla_U fV - \nabla_V fU, fP_\alpha X) \\
&\quad -g(\nabla_U V - \nabla_V U, fP_\alpha X) \\
&\quad +g(h(V,fU) - h(U,fV), tP_\alpha X + tP_\beta X) \\
&\quad -g(h(V,U) - h(U,V), tP_\alpha X + tP_\beta X).
\end{aligned}$$

So, the results follows from above equation. □

Theorem 4.3. *Let N be a quasi hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_α is integrable if and only if*

$$g(A_{tW}Z - A_{tZ}W, fP_\alpha Y) = g(\nabla^\perp_Z tW - \nabla^\perp_W tZ, tP_\beta Y),$$

and

$$g(\nabla_Z fW - \nabla_W fZ, P_\gamma Y + P_\beta Y) = g(A_{tW}Z - A_{tZ}W, P_\gamma Y + P_\beta Y),$$

for $Z, W \in \Gamma(D_\alpha)$, $Y \in \Gamma(D_\gamma \perp D_\beta)$.

Proof. For $Z, W \in \Gamma(D_\alpha)$, $Y \in \Gamma(D_\alpha \perp D_\beta)$, using (2.3), (2.16) with (2.4),we have

$$\begin{aligned}
g([Z,W],Y) &= g(\tilde{J}\tilde{\nabla}_Z W, \tilde{J}Y) - g(\tilde{J}\tilde{\nabla}_W Z, \tilde{J}Y) \\
&\quad -3g(\tilde{J}\tilde{\nabla}_Z W, Y) + 3g(\tilde{J}\tilde{\nabla}_W Z, Y) \\
&= g(\tilde{\nabla}_Z \tilde{J}W, \tilde{J}Y) - g(\tilde{\nabla}_W \tilde{J}Z, \tilde{J}Y) \\
&\quad -3g(\tilde{\nabla}_Z \tilde{J}W, Y) + 3g(\tilde{\nabla}_W \tilde{J}Z, Y) \\
&= g(\tilde{\nabla}_Z fW, \tilde{J}Y) + g(\tilde{\nabla}_Z tW, \tilde{J}Y) \\
&\quad -g(\tilde{\nabla}_W fZ, \tilde{J}Y) - g(\tilde{\nabla}_W tZ, \tilde{J}Y) \\
&\quad -3g(\tilde{\nabla}_Z \tilde{J}W, Y) + 3g(\tilde{\nabla}_W \tilde{J}Z, Y) \\
&= -g(A_{tW}Z - A_{tZ}W, \tilde{J}Y) + g(\nabla^\perp_Z tW - \nabla^\perp_W tZ, \tilde{J}Y) \\
&\quad +g(\tilde{\nabla}_Z \tilde{J}fW, Y) - g(\tilde{\nabla}_W \tilde{J}fZ, Y) \\
&\quad -3g(\tilde{\nabla}_Z \tilde{J}W, Y) + 3g(\tilde{\nabla}_W \tilde{J}Z, Y)
\end{aligned}$$

$$\begin{aligned}
&= -g(A_{tW}Z - A_{tZ}W, fP_\gamma Y) \\
&\quad +g(\nabla^\perp_Z tW - \nabla^\perp_W tZ, tP_\beta Y) \\
&\quad -3g(\tilde{\nabla}_Z \tilde{J}W, Y) + 3g(\tilde{\nabla}_W \tilde{J}Z, Y) \\
&= -g(A_{tW}Z - A_{tZ}W, fP_\gamma Y) \\
&\quad +g(\nabla^\perp_Z tW - \nabla^\perp_W tZ, tP_\beta Y) \\
&\quad -3g(\nabla_Z fW - \nabla_W fZ, P_\gamma Y + P_\beta Y) \\
&\quad +3g(A_{tW}Z - A_{tZ}W, P_\gamma Y + P_\beta Y).
\end{aligned}$$

So, the proof is completed. □

Theorem 4.4. *Let N be a quasi hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_β is integrable if and only if*

$$g(A_{\tilde{J}Z}W - A_{\tilde{J}W}Z - \nabla_Z W + \nabla_W Z, fP_\gamma Y + fP_\alpha Y) = g(\nabla^\perp_W \tilde{J}Z - \nabla^\perp_Z \tilde{J}W, tP_\alpha Y),$$

for $Z, W \in \Gamma(D_\beta)$, $Y \in \Gamma(D_\gamma \perp D_\alpha)$.

Proof. For $Z, W \in \Gamma(D_\beta)$, $Y \in \Gamma(D_\gamma \perp D_\alpha)$, using (2.3), (2.16) with (2.4),we find

$$\begin{aligned}
g([Z,W],Y) &= g(\tilde{J}\tilde{\nabla}_Z W, \tilde{J}Y) - g(\tilde{J}\tilde{\nabla}_W Z, \tilde{J}Y) \\
&\quad -3g(\tilde{J}\tilde{\nabla}_Z W, Y) + 3g(\tilde{J}\tilde{\nabla}_W Z, Y) \\
&= g(\tilde{\nabla}_Z \tilde{J}W, \tilde{J}Y) - g(\tilde{\nabla}_W \tilde{J}Z, \tilde{J}Y) \\
&\quad -3g(\tilde{\nabla}_Z \tilde{J}W, Y) + 3g(\tilde{\nabla}_W \tilde{J}Z, Y) \\
&= g(A_{\tilde{J}Z}W, fP_\gamma Y + fP_\alpha Y) - g(A_{\tilde{J}W}Z, fP_\gamma Y + fP_\alpha Y) \\
&\quad +g(\nabla^\perp_Z \tilde{J}W, tP_\alpha Y) - g(\nabla^\perp_W \tilde{J}Z, tP_\alpha Y) \\
&\quad -3g(\nabla_Z W, fP_\gamma Y + fP_\alpha Y) \\
&\quad +3g(\nabla_W Z, fP_\gamma Y + fP_\alpha Y),
\end{aligned}$$

which gives proof of our assertion. □

Theorem 4.5. *Let N be a quasi hemi-slant submanifold of $(\tilde{N}, \tilde{g}, \tilde{J})$. Then D_γ defines a totally geodesic foliation on N if and only if*

$$g(\nabla_U fV - 3\nabla_U V, fP_\alpha X) = g(h(U, fV) - 3h(U, V), tP_\alpha X + tP_\beta X),$$

and

$$g(\nabla_U fV - 3\nabla_U V, BW) = g(h(U, fV) - 3h(U, V), CW)$$

for $U, V \in \Gamma(D_\gamma)$, $X \in \Gamma(D_\alpha \perp D_\beta)$ and $W \in \Gamma(T^\perp N)$.

Proof. For $U, V \in \Gamma(D_\gamma)$, $X \in \Gamma(D_\alpha \perp D_\beta)$ using (2.3), (2.16) with (2.4),we obtain

$$\begin{aligned} g(\tilde{\nabla}_U V, X) &= g(\tilde{\nabla}_U fV, \tilde{J}X) - 3(\tilde{\nabla}_U V, \tilde{J}X) \\ &= g(\nabla_U fV, fP_\alpha X) + g(h(U, fV), tP_\alpha X + tP_\beta X) \\ & \quad -3(\nabla_U V, fP_\alpha X) - 3g(h(U, V), tP_\alpha X + tP_\beta X). \end{aligned} \tag{4.3}$$

Now, for $W \in \Gamma(T^\perp N)$ and $U, V \in \Gamma(D_\gamma)$, we get

$$\begin{aligned} g(\tilde{\nabla}_U V, W) &= g(\tilde{\nabla}_U fV, \tilde{J}W) - 3(\tilde{\nabla}_U V, \tilde{J}W) \\ &= g(\nabla_U fV, BW) + g(h(U, fV), CW) \\ & \quad -g(\nabla_U V, BW) + g(h(U, V), CW). \end{aligned} \tag{4.4}$$

So from (4.3) and (4.4), the result follows. □

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