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NOTE ON THE MULTIFRACTAL MEASURES OF CARTESIAN PRODUCT SETS

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Abstract. In this paper, we shall be concerned with evaluation of multifractal Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and multifractal packing measure $\mathcal{P}_\mu^{q,t}$ of Cartesian product sets by means of the measure of their components. This is done by investigating the density result introduced in [34]. As a consequence, we get the inequalities related to the multifractal dimension functions, proved in [35], by using a unified method for all the inequalities. Finally, we discuss the extension of our approach to studying the multifractal Hewitt-Stromberg measures of Cartesian product sets.

1. Introduction

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of probability measures on $\mathbb{R}^n$ $(n \geq 1)$. Given $\mu \in \mathcal{P}(\mathbb{R}^n)$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we define the upper and lower t-densities of μ at x by

$$\overline{d}_\mu^t(x) = \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^t} \quad \text{and} \quad \underline{d}_\mu^t(x) = \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^t},$$

where $B(x,r)$ is a closed ball of center x and radius $r > 0$. Several results have been obtained by studying these quantities and connecting them to Hausdorff and packing measures, where one can cite for example [8, 10, 11, 39]. In particular, when the measure μ is equals to the restriction of $\mathcal{H}^t$ or $\mathcal{P}^t$ on a set E one can determine under suitable conditions the inverse inequality connecting $\mathcal{H}^t$ and $\mathcal{P}^t$ and thus we get $\mathcal{H}^t(E) = \mathcal{P}^t(E)$, where E is said to be strong regular. More generally, one can define regular sets by density with respect to the Hausdorff measure [8, 11, 12, 32, 33], to packing measure [39–41] or also to Hewitt-Stromberg measure [4, 27, 28]. In particular Tricot et al. [39, 41] managed to show that a subset of $\mathbb{R}^n$ has an integer Hausdorff and packing dimension if it is strongly regular. Then, the results of [39] were improved to a generalized ϕ-Hausdorff measure in a Polish space by Mattila and Mauldin in [33]. In this paper, we investigate more general density theorem (Theorem

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5) introduced in [34, 36]. This theorem was used in [5] to prove the decomposition theorem for the regularities of a generalized centered Hausdorff measure $\mathcal{H}_\mu^{q,t}$ (see definition in Section 2) and a generalized packing measure $\mathcal{P}_\mu^{q,t}$ (see definition in Section 2) in an Euclidean space. More precisely, we study the multifractal extensions of the following product inequalities for the Hausdorff measure $\mathcal{H}^t$ and the packing measure $\mathcal{P}^t$ in an Euclidean space. We fix $s, t \geq 0$ and let E, F be two Borel sets in $\mathbb{R}^n$. Then there exists a number $c > 0$ such that

$$\mathcal{H}^s(E)\mathcal{H}^t(F) \leq c\mathcal{H}^{s+t}(E \times F), \tag{1.1}$$

$$\mathcal{H}^{s+t}(E \times F) \leq c\mathcal{H}^s(E)\mathcal{P}^t(F), \tag{1.2}$$

$$\mathcal{H}^s(E)\mathcal{P}^t(F) \leq c\mathcal{P}^{s+t}(E \times F), \tag{1.3}$$

$$\mathcal{P}^{s+t}(E \times F) \leq c\mathcal{P}^s(E)\mathcal{P}^t(F). \tag{1.4}$$

Inequality (1.1) was shown in [6] under certain conditions and later in [30] without any restrictions. The inequality (1.2) is stated explicitly in [23] and it was shown for subsets of arbitrary metric spaces (and not just for subsets of Euclidean spaces). Inequality (1.3) is proved in [19, 20], and inequality (1.4) in [24].

We can think of certain density inequalities as "local versions" of the product inequalities. The inequality, for example, $\mathcal{P}^{s+t}(E \times F) \leq c\mathcal{P}^t(E)\mathcal{P}^s(F)$ is a consequence of a density inequality

$$c\, \underline{d}_{\mu_1 \times \mu_2}^{s+t}\big((x, y)\big) \geq \underline{d}_{\mu_1}^t(x)\underline{d}_{\mu_2}^s(y).$$

In this paper, using the density approach, we will study the multifractal Hausdorff and packing measures of product sets. The disadvantage of this approach includes the inability to handle sets of measure ∞. To overcome this problem and study the sets of infinite measure we will require semifiniteness of the multifractal measure. The measure $\mu \in \mathcal{P}(\mathbb{R}^n)$ is said to be semifinite if every Borel set of infinite measure has a Borel subset of finite positive measure. We say that μ is semifinite on A if the restriction of μ to A is semifinite. In this case we have, for every Borel set $E \subseteq \mathbb{R}^n$,

$$\mu(E) = \sup\Big\{\mu(F) : F \subseteq E,\ F \text{ is compact}\Big\}.$$

Denote by $supp\,\mu$ the topological support of μ. We now define the family of doubling measures. For $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $a > 1$, we write

$$P_a(\mu) = \limsup_{r \searrow 0}\left(\sup_{x \in supp\,\mu} \frac{\mu\big(B(x, ar)\big)}{\mu\big(B(x, r)\big)}\right).$$

We say that the measure μ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter a is unimportant:

$$P_a(\mu) < \infty \ \text{ for some } \ a > 1 \quad \text{if and only if} \quad P_a(\mu) < \infty \quad \text{for all} \ \ a > 1.$$

Also, we denote by $\mathcal{P}_D(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$ which satisfy the doubling condition. We can cite as classical examples of doubling measures, the self-similar measures and the self-conformal ones [34]. In particular, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then $\mathcal{H}_\mu^{q,t} \leq \mathcal{P}_\mu^{q,t}$. We will manage to prove the following results.

Theorem 1. *Let $q, s, t \in \mathbb{R}$, $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$ $(k \geq 1)$. If $E \subseteq \operatorname{supp}\mu_1$ and $F \subseteq \operatorname{supp}\mu_2$ are two Borel sets, then*

$$\mathcal{H}_{\mu_1}^{q,t}(E)\mathcal{H}_{\mu_2}^{q,s}(F) \leq \mathcal{H}_{\mu_1\times\mu_2}^{q,t+s}(E\times F). \tag{1.5}$$

Assume that $\mathcal{P}_{\mu_2}^{q,s}(F) < \infty$ or $\mathcal{P}_{\mu_2}^{q,s}$ is semifinite on F. Then

$$\mathcal{H}_{\mu_1}^{q,t}(E)\mathcal{P}_{\mu_2}^{q,s}(F) \leq \mathcal{P}_{\mu_1\times\mu_2}^{q,t+s}(E\times F). \tag{1.6}$$

Theorem 2. *Let $q, s, t \in \mathbb{R}$, $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$. Let $E \subseteq \operatorname{supp}\mu_1$ and $F \subseteq \operatorname{supp}\mu_2$ be two Borel sets. Then:*

(1) *We have*

$$\mathcal{H}_{\mu_1\times\mu_2}^{q,t+s}(E\times F) \leq \mathcal{H}_{\mu_1}^{q,t}(E)\mathcal{P}_{\mu_2}^{q,s}(F), \tag{1.7}$$

provided it is true in the case when the term on the right side is not $0\times\infty$ or $\infty\times 0$.

(2) *We have*

$$\mathcal{P}_{\mu_1\times\mu_2}^{q,t+s}(E\times F) \leq \mathcal{P}_{\mu_1}^{q,t}(E)\mathcal{P}_{\mu_2}^{q,s}(F), \tag{1.8}$$

provided it is true in the case when the term on the right side is not $0\times\infty$ or $\infty\times 0$.

The advantages of the density approach over the traditional one (see [35]) include the local nature of the inequality and the use of a unified method for the four inequalities. The disadvantage includes the inability to handle sets of measure ∞. The previous theorems give natural consequences on the multifractal dimension functions b_μ and B_μ (see definition in Section 2). This will be discussed in Sections 3 and 4. Our density approach can be applied to get a partial result of the multifractal Hewitt-Stromberg measures $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ (see definition in Section 2) of Cartesian product sets by means of the measure of their components (see Section 5).

2. Preliminaries

Let $n, k \geq 1$. We suppose that $\mathbb{R}^n$ is equipped with a metric d and $\mathbb{R}^{n+k}$ is endowed with a metric which is Cartesian product of the metrics in $\mathbb{R}^n$ and $\mathbb{R}^k$ so that for all $r > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$ we have

$$B\big((x,y),r\big) = B(x,r)\times B(y,r).$$

We define the closed ball of center x and radius $r > 0$ by

$$B(x,r) = \Big\{y \in \mathbb{R}^n : d(x,y) \leq r\Big\}.$$

2.1. Multifractal Hausdorff and packing measures and separator functions

We start by introducing the definition of the Hausdorff measure, the centered Hausdorff measure and the packing measure. Let $E \subseteq \mathbb{R}^n$ and $\delta > 0$. A countable family $\mathcal{B} = \{B(x_i, r_i)\}$ of closed balls in $\mathbb{R}^n$ is called a centered δ-covering of E if

$$E \subseteq \bigcup_i B(x_i, r_i), \quad x_i \in E \quad \text{and} \quad 0 < r_i < \delta$$

for all i. The family $\mathcal{B}$ is called a centered δ-packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$ for all $i \neq j$. Let $E \subseteq \mathbb{R}^n$, $t \geq 0$ and $\delta > 0$. We define

$$\mathcal{H}^t_\delta(E) = \inf \left\{ \sum_i \Big(\mathrm{diam}(E_i)\Big)^t : E \subseteq \bigcup_i E_i, \ \ \mathrm{diam}(E_i) < \delta \right\}.$$

This allows to define the t-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of E by

$$\mathcal{H}^t(E) = \sup_{\delta > 0} \mathcal{H}^t_\delta(E).$$

Next we define the centered Hausdorff measure introduced in [39]. We define

$$\overline{\mathcal{C}}^t_\delta(E) = \sup \left\{ \sum_i \Big(2r_i\Big)^t : \Big\{B(x_i, r_i)\Big\} \text{ is a centered } \delta\text{-covering of } E \right\}.$$

The t-dimensional centered Hausdorff pre-measure $\overline{\mathcal{C}}^t(E)$ of E is defined by

$$\overline{\mathcal{C}}^t(E) = \sup_{\delta > 0} \overline{\mathcal{C}}^t_\delta(E).$$

This function is not necessarily an outer measure because it is not necessarily monotone. This is why we define the t-dimensional centered Hausdorff measure $\mathcal{C}^t(E)$ of E as

$$\mathcal{C}^t(E) = \sup_{F \subseteq E} \overline{\mathcal{C}}^t(F).$$

We will now define the packing measure. We set,

$$\overline{\mathcal{P}}^t_\delta(E) = \sup \left\{ \sum_i \Big(2r_i\Big)^t : \Big\{B(x_i, r_i)\Big\} \text{ is a centered } \delta\text{-packing of } E \right\}.$$

Now, we define the t-dimensional packing pre-measure $\overline{\mathcal{P}}^t(E)$ of E by

$$\overline{\mathcal{P}}^t(E) = \sup_{\delta > 0} \overline{\mathcal{P}}^t_\delta(E).$$

This function is not necessarily an outer measure because it is not necessarily countable subadditive. Therefore, we define the t-dimensional packing measure

$\mathcal{P}^t(E)$ of E as

$$\mathcal{P}^t(E) = \inf\left\{\sum_i \overline{\mathcal{P}}^t(E_i) : E \subseteq \bigcup_i E_i\right\}.$$

This measure was first introduced in [40] using centered δ-packings of open balls and in [39] using centered δ-packings of closed balls.

The measures $\mathcal{H}^t$ and $\mathcal{P}^t$ assign a multifractal dimension to each subset E of $\mathbb{R}^n$. They are, respectively, denoted by $\dim(E)$ and $\mathrm{Dim}(E)$ and satisfy

$$\dim(E) = \inf\Big\{t \in \mathbb{R} : \mathcal{H}^t(E) = 0\Big\} = \sup\Big\{t \in \mathbb{R} : \mathcal{H}^t(E) = +\infty\Big\},$$
$$\mathrm{Dim}(E) = \inf\Big\{t \in \mathbb{R} : \mathcal{P}^t(E) = 0\Big\} = \sup\Big\{t \in \mathbb{R} : \mathcal{P}^t(E) = +\infty\Big\}.$$

We refer the reader to [12,13,33,39] for more information about these notions of measures and dimensions.

Now, we introduce the multifractal centered Hausdorff measure $\mathcal{H}^{q,t}_\mu$ and the multifractal packing measure $\mathcal{P}^{q,t}_\mu$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$. We define the multifractal packing pre-measure,

$$\overline{\mathcal{P}}^{q,t}_\mu(E) = \inf_{\delta>0} \sup\Big\{\sum_i \mu\big(B(x_i, r_i)\big)^q (2r_i)^t : \Big\{B(x_i, r_i)\Big\}_i \text{ is a centered } \delta\text{-packing of } E\Big\}$$

if $E \neq \emptyset$ and $\overline{\mathcal{P}}^{q,t}_\mu(\emptyset) = 0$. In a similar way, we define the multifractal Hausdorff pre-measure,

$$\overline{\mathcal{H}}^{q,t}_\mu(E) = \sup_{\delta>0} \inf\Big\{\sum_i \mu\big(B(x_i, r_i)\big)^q (2r_i)^t : \Big\{B(x_i, r_i)\Big\}_i \text{ is a centered } \delta\text{-covering of } E\Big\}$$

if $E \neq \emptyset$ and $\overline{\mathcal{H}}^{q,t}_\mu(\emptyset) = 0$, with the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

The function $\overline{\mathcal{H}}^{q,t}_\mu$ is σ-subadditive but not increasing and the function $\overline{\mathcal{P}}^{q,t}_\mu$ is increasing but not σ-subadditive. That is the reason for which Olsen in [34] introduced the following modifications of the multifractal Hausdorff and packing measures $\mathcal{H}^{q,t}_\mu$ and $\mathcal{P}^{q,t}_\mu$:

$$\mathcal{H}^{q,t}_\mu(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}^{q,t}_\mu(F) \quad \text{and} \quad \mathcal{P}^{q,t}_\mu(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_\mu(E_i).$$

The functions $\mathcal{H}^{q,t}_\mu$ and $\mathcal{P}^{q,t}_\mu$ are metric outer measures (see Propositions 2.2. and 2.3 in [34]) and thus measures on the Borel family of subsets of $\mathbb{R}^n$. In addition, there exists an integer $\xi \in \mathbb{N}$ such that

$$\mathcal{H}^{q,t}_\mu \leq \xi \mathcal{P}^{q,t}_\mu.$$

The measure $\mathcal{H}_\mu^{q,t}$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $\mathcal{P}_\mu^{q,t}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that, for $t \geq 0$, one has

$$2^{-t}\mathcal{H}_\mu^{0,t} \leq \mathcal{H}^t \leq \mathcal{H}_\mu^{0,t} \quad \text{and} \quad \mathcal{P}_\mu^{0,t} = \mathcal{P}^t.$$

Proposition 1. *Let $q, t \in \mathbb{R}$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$. There exists a constant $C(q,t)$ depending on q and t such that for all Borel set $E \subseteq \mathbb{R}^n$ we have,*

$$\overline{\mathcal{H}}_\mu^{q,t}(E) \leq \mathcal{H}_\mu^{q,t}(E) \leq C(q,t)\, \overline{\mathcal{H}}_\mu^{q,t}(E).$$

In particular, $\overline{\mathcal{H}}_\mu^{q,t}(E) = 0$ if and only if $\mathcal{H}_\mu^{q,t}(E) = 0$.

Proof. It is clear from the definition that $\overline{\mathcal{H}}_\mu^{q,t}(E) \leq \mathcal{H}_\mu^{q,t}(E)$. Let $\delta > 0$, $A \subseteq E$ and $\mathcal{B} = \big\{B_i = B(x_i, r_i)\big\}$ be a δ-cover of E. Then take $\mathcal{B}_1 = \big\{B_i \in \mathcal{B} : B \cap A \neq \emptyset\big\}$. Choose, for each $B_i \in \mathcal{B}_1$, $y(x_i) \in B_i \cap A$. Then,

$$B_i = B(x_i, r_i) \subset B(y(x_i), 2r_i) \subset B(x_i, 3r_i) \tag{2.1}$$

and $\mathcal{B}_2 = \big\{B = B(y(x_i), 2r_i)\big\}$ be a 2δ-cover of A. It follows that:

(1) If $q \leq 0$, then

$$\overline{\mathcal{H}}_{\mu,2\delta}^{q,t}(A) \leq \sum_{\mathcal{B}_2} (4r_i)^t \mu\big(B(y(x_i), 2r_i)\big)^q \leq 2^t \sum_{\mathcal{B}} (2r_i)^t \mu\big(B(x_i, r_i)\big)^q.$$

Take the infimum on all δ-covering of E, we get $\overline{\mathcal{H}}_{\mu,2\delta}^{q,t}(A) \leq 2^t \overline{\mathcal{H}}_{\mu,\delta}^{q,t}(E)$, take the limit on δ to get

$$\overline{\mathcal{H}}_\mu^{q,t}(A) \leq 2^t \overline{\mathcal{H}}_\mu^{q,t}(E).$$

Now, take the supremum on A to get $\mathcal{H}_\mu^{q,t}(E) \leq 2^t \overline{\mathcal{H}}_\mu^{q,t}(E)$.

(2) For $q > 0$ and μ satisfy the doubling condition, let k be a constant such that $\mu(B(x, 3r)) \leq k\mu(B(x, r))$. Applying (2.1) we get

$$\overline{\mathcal{H}}_{\mu,2\delta}^{q,t}(A) \leq \sum_{\mathcal{B}_2} (4r_i)^t \mu\big(B(y(x_i), 2r_i)\big)^q \leq 2^t k^q \sum_{\mathcal{B}} (2r_i)^t \mu\big(B(x_i, r_i)\big)^q$$

and then $\mathcal{H}_\mu^{q,t}(E) \leq 2^t k^q \overline{\mathcal{H}}_\mu^{q,t}(E)$. $\square$

Remark 1. The measure $\mathcal{P}_\mu^{q,t}$ is regular that is for $E \subseteq \mathbb{R}^n$, we can find a Borel set B such that $\mathcal{P}_\mu^{q,t}(B) = \mathcal{P}_\mu^{q,t}(E)$ and $E \subseteq B$ (Lemma 5.4.2 [35]). As a consequence (Lemma 5.4.3 in [35]), we have, for all $E_k \nearrow E$,

$$\mathcal{P}_\mu^{q,t}(E) = \sup_k \mathcal{P}_\mu^{q,t}(E_k).$$

This implies also that (see [8, Theorem 3.11] for the idea of the proof)

$$\mathcal{P}_\mu^{q,t}(E) = \inf\Big\{\sup_k \overline{\mathcal{P}}_\mu^{q,t}(E_k) : E_k \nearrow E\Big\}.$$

The measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ assign in a usual way a multifractal dimension to each subset E of $\mathbb{R}^n$. They are, respectively, denoted by $b_\mu^q(E)$ and $B_\mu^q(E)$ (see [34]) and satisfy

$$b_\mu^q(E) = \inf\Big\{t \in \mathbb{R} : \mathcal{H}_\mu^{q,t}(E) = 0\Big\} = \sup\Big\{t \in \mathbb{R} : \mathcal{H}_\mu^{q,t}(E) = +\infty\Big\},$$
$$B_\mu^q(E) = \inf\Big\{t \in \mathbb{R} : \mathcal{P}_\mu^{q,t}(E) = 0\Big\} = \sup\Big\{t \in \mathbb{R} : \mathcal{P}_\mu^{q,t}(E) = +\infty\Big\}.$$

The number $b_\mu^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E whereas $B_\mu^q(E)$ is an obvious multifractal analogue of the packing dimension $\mathrm{Dim}(E)$ of E. In fact, it follows immediately from the definitions that

$$\dim(E) = b_\mu^0(E) \quad \text{and} \quad \mathrm{Dim}(E) = B_\mu^0(E).$$

Next we define the multifractal separator functions

$$b_\mu(q) = b_\mu^q(supp\,\mu) \quad \text{and} \quad B_\mu(q) = B_\mu^q(supp\,\mu).$$

It is well known [34] that b_μ is decreasing and B_μ is decreasing and convex. Moreover,

$$b_\mu \leq B_\mu.$$

2.2. Multifractal Hewitt-Stromberg measures and separator functions

Hewitt-Stromberg measures were introduced in [21, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [1, 14–18, 26]. In particular, Edgar's textbook [10, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures appear also explicitly, for example, in Pesin's monograph [37, 5.3] and implicitly in Mattila's text [31]. One of the purposes of this paper is to define and study a class of natural multifractal generalizations of the Hewitt-Stromberg measures. While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number ε, say, the Hewitt-Stromberg measures are defined using packings of balls with a fixed diameter ε.

In the following, we will set up, for $q, t \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R}^n)$, the lower and upper multifractal Hewitt-Stromberg measures $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$.

For $E \subseteq supp\,\mu$, we define

$$\mathsf{C}_\mu^{q,t}(E) = \limsup_{r\to 0} M_{\mu,r}^q(E)(2r)^t,$$

where

$$M_{\mu,r}^q(E) = \sup\left\{\sum_i \mu(B(x_i,r))^q : \Big\{B(x_i,r)\Big\}_i \text{ is a centered packing of } E\right\}.$$

It's clear that $\mathsf{C}_\mu^{q,t}$ is increasing and $\mathsf{C}_\mu^{q,t}(\emptyset)=0$. However it's not σ-additive. For this, we introduce the $\mathsf{P}_\mu^{q,t}$-measure defined by

$$\mathsf{P}_\mu^{q,t}(E)=\inf\left\{\sum_i \mathsf{C}_\mu^{q,t}(E_i) : E\subseteq \bigcup_i E_i \text{ and the } E_i\text{'s are bounded}\right\}.$$

In a similar way we define

$$\mathsf{L}_\mu^{q,t}(E)=\liminf_{r\to 0} M_{\mu,r}^q(E)(2r)^t.$$

Since $\mathsf{L}_\mu^{q,t}$ is not countably subadditive, one needs a standard modification to get an outer measure. Hence, we modify the definition as follows:

$$\mathsf{H}_\mu^{q,t}(E)=\inf\left\{\sum_i \mathsf{L}_\mu^{q,t}(E_i) : E\subseteq \bigcup_i E_i \text{ and the } E_i\text{'s are bounded}\right\}.$$

The measure $\mathsf{H}_\mu^{q,t}$ is of course a multifractal generalization of the lower t-dimensional Hewitt-Stromberg measure H^t, whereas $\mathsf{P}_\mu^{q,t}$ is a multifractal generalization of the upper t-dimensional Hewitt-Stromberg measures P^t. In fact, it is easily seen that, for $t>0$, one has

$$\mathsf{H}_\mu^{0,t}=\mathsf{H}^t \quad\text{and}\quad \mathsf{P}_\mu^{0,t}=\mathsf{P}^t.$$

The following result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ are Borel metric outer measures and summarises the basic inequalities satisfied by the multifractal Hewitt-Stromberg measures, the generalized Hausdorff measure and the generalized packing measure.

Theorem 3. *Let $q,t\in\mathbb{R}$ and $\mu\in\mathcal{P}(\mathbb{R}^n)$. Then, for every set $E\subseteq\mathbb{R}^n$, we have*:

(1) *The set functions $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ are outer measures and thus they are measures on the Borel algebra.*

(2) *There exists an integer $\xi\in\mathbb{N}$ such that*

$$\mathcal{H}_\mu^{q,t}(E)\le \xi\mathsf{H}_\mu^{q,t}(E)\le \xi\mathsf{P}_\mu^{q,t}(E)\le \xi\mathcal{P}_\mu^{q,t}(E). \tag{2.2}$$

(3) *When $q\le 0$ or $q>0$ and $\mu\in\mathcal{P}_D(\mathbb{R}^n)$, we have*

$$\mathcal{H}_\mu^{q,t}(E)\le \mathsf{H}_\mu^{q,t}(E)\le \mathsf{P}_\mu^{q,t}(E)\le \mathcal{P}_\mu^{q,t}(E).$$

The proof of the first part is straightforward and mimics that in Theorem 2.1 in [2]. The proof of second part is a straightforward application of Besicovitch's covering theorem and we omit it here (we can see also Theorem 2.1 in [2]. The measures $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ assign in the usual way a multifractal dimension to each subset E of $\mathbb{R}^n$. They are, respectively, denoted by $\mathsf{b}_\mu^q(E)$ and $\mathsf{B}_\mu^q(E)$.

Proposition 2. *Let $q\in\mathbb{R}$, $\mu\in\mathcal{P}(\mathbb{R}^n)$ and $E\subseteq\mathbb{R}^n$.*

(1) *There exists a unique number* $\mathsf{b}_\mu^q(E) \in [-\infty, +\infty]$ *such that*

$$\mathsf{H}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \mathsf{b}_\mu^q(E), \\ 0 & \text{if} \quad \mathsf{b}_\mu^q(E) < t. \end{cases}$$

(2) *There exists a unique number* $\mathsf{B}_\mu^q(E) \in [-\infty, +\infty]$ *such that*

$$\mathsf{P}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \mathsf{B}_\mu^q(E), \\ 0 & \text{if} \quad \mathsf{B}_\mu^q(E) < t. \end{cases}$$

In addition, we have

$$\mathsf{b}_\mu^q(E) \le \mathsf{B}_\mu^q(E).$$

The number $\mathsf{b}_\mu^q(E)$ is an obvious multifractal analogue of the lower Hewitt-Stromberg dimension $\underline{\dim}_{MB}(E)$ of E whereas $\mathsf{B}_\mu^q(E)$ is an obvious multifractal analogue of the upper Hewitt-Stromberg dimension $\overline{\dim}_{MB}(E)$ of E. In fact, it follows immediately from the definitions that

$$\mathsf{b}_\mu^0(E) = \underline{\dim}_{MB}(E) \quad \text{and} \quad \mathsf{B}_\mu^0(E) = \overline{\dim}_{MB}(E).$$

Remark 2. It follows from Theorem 3 that

$$b^q(E) \le \mathsf{b}_\mu^q(E) \le \mathsf{B}_\mu^q(E) \le B_\mu^q(E).$$

The definition of these dimension functions makes it clear that they are counterparts of the τ_μ-function which appears in the *multifractal formalism.* This being the case, it is important that they have the properties described by the physicists. The next theorem shows that these functions do indeed have some of these properties.

Theorem 4. *Let* $q \in \mathbb{R}$ *and* $E \subseteq \mathbb{R}^n$.

(1) *The functions* $q \mapsto \mathsf{H}_\mu^{q,t}(E)$, $\mathsf{P}_\mu^{q,t}(E)$ *are decreasing.*
(2) *The functions* $t \mapsto \mathsf{H}_\mu^{q,t}(E)$, $\mathsf{P}_\mu^{q,t}(E)$ *are decreasing.*
(3) *The functions* $q \mapsto \mathsf{b}_\mu^q(E)$, $\mathsf{B}_\mu^q(E)$ *are decreasing.*
(4) *The function* $q \mapsto \mathsf{B}_\mu^q(E)$ *is convex.*

The proof of this is straightforward and mimics that in Theorem 3 in [3]. We note that for all $q \in \mathbb{R}$,

$$\mathsf{b}_\mu^q(\emptyset) = \mathsf{B}_\mu^q(\emptyset) = -\infty,$$

and if $\mu(E) = 0$, then

$$\mathsf{b}_\mu^q(E) = \mathsf{B}_\mu^q(E) = -\infty \quad \text{for} \quad q > 0.$$

Next we define the multifractal separator functions b_μ, $\mathsf{B}_\mu : \mathbb{R} \to [-\infty, +\infty]$ by

$$\mathsf{b}_\mu : \ q \to \mathsf{b}_\mu^q(supp\, \mu) \quad \text{and} \quad \mathsf{B}_\mu : \ q \to \mathsf{B}_\mu^q(supp\, \mu).$$

We also obtain, from the above theorem and definitions, that

$$\begin{cases} \mathsf{b}_\mu(q) \ge 0 & q < 1, \\ \mathsf{b}_\mu(q) = \mathsf{B}_\mu(q) = 0 & q = 1, \\ \mathsf{B}_\mu(q) \le 0 & q > 1. \end{cases}$$

2.3. Preliminaries results related to density theorem

Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, $q, t \in \mathbb{R}$ and $x \in supp\,\mu$, we define the upper and lower (q,t)-densities of ν at x with respect to μ by

$$\overline{d}_\mu^{q,t}(x,\nu) = \limsup_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t} \quad \text{and} \quad \underline{d}_\mu^{q,t}(x,\nu) = \liminf_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t}.$$

If $\underline{d}_\mu^{q,t}(x,\nu) = \overline{d}_\mu^{q,t}(x,\nu)$, we denote $d_\mu^{q,t}(x,\nu)$ the commune value. Let E be a Borel set of $\mathbb{R}^n$ and denote by $\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}$ (resp. $\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}$) the measure $\mathcal{H}_\mu^{q,t}$ (resp. measure $\mathcal{P}_\mu^{q,t}$) restricted to E. We define

$$\begin{cases} \overline{d}_\mu^{q,t}(x,E) = \overline{d}_\mu^{q,t}\left(x, \mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right), \\ \underline{d}_\mu^{q,t}(x,E) = \underline{d}_\mu^{q,t}\left(x, \mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right), \end{cases} \quad \text{and} \quad \begin{cases} \overline{D}_\mu^{q,t}(x,E) = \overline{d}_\mu^{q,t}\left(x, \mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right), \\ \underline{D}_\mu^{q,t}(x,E) = \underline{d}_\mu^{q,t}\left(x, \mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right). \end{cases}$$

If $\overline{d}_\mu^{q,t}(x,E) = \underline{d}_\mu^{q,t}(x,E)$ $\big($resp. $\overline{D}_\mu^{q,t}(x,E) = \underline{D}_\mu^{q,t}(x,E)\big)$, we write $d_\mu^{q,t}(x,E)$ $\big($resp. $D_\mu^{q,t}(x,E)\big)$ for the common value.

Remark 3. The upper and lower (q,t)-densities of ν are related to the multifractal Hausdorff and packing measures. One can, for example, show that $\mathcal{P}_\mu^{q,t}(E) < \infty$ whenever $\inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) > 0$. Indeed, let

$$0 < \gamma < \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu).$$

We consider, for each $k \geq 1$, the set

$$E_k = \Big\{ x \in E : \nu(B(x,r)) \geq \gamma \mu(B(x,r))^q (2r)^t \quad \text{for all} \quad 0 < r \leq 1/k \Big\}.$$

For any δ-packing $\Big\{B(x_i,r_i)\Big\}_i$ of E_k with $0 < \delta < 2/k$, we have

$$\sum_i \mu(B(x_i,r_i))^q\ (2r_i)^t \leq \gamma^{-1} \sum_i \nu(B(x_i,r_i) \leq \gamma^{-1} \nu(E_k(\delta),$$

where $E_k(\delta) = \big\{x : d(x,E) \leq \delta)\big\}$. Letting $\delta \to 0$, we obtain

$$\overline{\mathcal{P}}_\mu^{q,t}(E_k) \leq \gamma^{-1}.$$

Therefore, we get the desired result using Remark 1 and the fact that $E_k \nearrow E$.

The density result was proven with respect to multifractal Hausdorff measure and packing measure in [34, 36]. More precisely, we have the following result.

Theorem 5. *Let $q,t \in \mathbb{R}$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of* $supp\,\mu$*.*

(1) *Assume that $\mathcal{H}_\mu^{q,t}(E) < \infty$. Then we have*

$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \leq \nu(E) \leq \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu). \tag{2.3}$$

(2) *If* $\mathcal{H}^{q,t}_\mu(E) < \infty$ *and* $\mu \in \mathcal{P}_D(\mathbb{R}^n)$*, then*

$$\mathcal{H}^{q,t}_\mu(E) \inf_{x\in E} \overline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \mathcal{H}^{q,t}_\mu(E) \sup_{x\in E} \overline{d}^{q,t}_\mu(x,\nu). \tag{2.4}$$

(3) *If* $\mathcal{P}^{q,h}_\mu(E) < \infty$*, then*

$$\mathcal{P}^{q,t}_\mu(E) \inf_{x\in E} \underline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \mathcal{P}^{q,t}_\mu(E) \sup_{x\in E} \underline{d}^{q,t}_\mu(x,\nu), \tag{2.5}$$

where ξ *is the constant which appear in* (2.2)*.*

As a consequence we have the following result.

Corollary 1. *Let* $\mu,\nu \in \mathcal{P}(\mathbb{R}^n)$ *and* E *be a Borel subset of* $\operatorname{supp}\mu$*. If* $\mathcal{P}^{q,t}_\mu(E) < \infty$*, then*

$$\mathsf{H}^{q,t}_\mu(E) \inf_{x\in E} \underline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \xi \mathsf{H}^{q,t}_\mu(E) \sup_{x\in E} \overline{d}^{q,t}_\mu(x,\nu), \tag{2.6}$$

$$\mathsf{P}^{q,t}_\mu(E) \inf_{x\in E} \underline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \xi \mathsf{P}^{q,t}_\mu(E) \sup_{x\in E} \overline{d}^{q,t}_\mu(x,\nu). \tag{2.7}$$

In addition, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$*, we have*

$$\mathsf{H}^{q,t}_\mu(E) \inf_{x\in E} \underline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \mathsf{H}^{q,t}_\mu(E) \sup_{x\in E} \overline{d}^{q,t}_\mu(x,\nu), \tag{2.8}$$

$$\mathsf{P}^{q,t}_\mu(E) \inf_{x\in E} \underline{d}^{q,t}_\mu(x,\nu) \le \nu(E) \le \mathsf{P}^{q,t}_\mu(E) \sup_{x\in E} \overline{d}^{q,t}_\mu(x,\nu). \tag{2.9}$$

Now, we will end this section by useful lemmas which are direct consequences of Theorem 5 and Corollary 1.

Lemma 1. *Let* $\mu \in \mathcal{P}(\mathbb{R}^n)$ *and* E *be a Borel subset of* $\operatorname{supp}\mu$*.*

(1) *If* $\mathcal{P}^{q,t}_\mu(E) < +\infty$*, then* $\underline{D}^{q,t}_\mu(x,E) = 1$*,* $\mathcal{P}^{q,h}_\mu$*-a.a. on* E*.*
(2) *If* $\mathcal{H}^{q,t}_\mu(E) < +\infty$*, then* $1 \le \overline{d}^{q,t}_\mu(x,E) \le \xi$*,* $\mathcal{H}^{q,h}_\mu$*-a.a. on* E*.*
(3) *If* $\mathcal{H}^{q,t}_\mu(E) < +\infty$ *and* $\mu \in \mathcal{P}_D(\mathbb{R}^n)$*, then* $\overline{d}^{q,t}_\mu(x,E) = 1$*,* $\mathcal{H}^{q,t}_\mu$*-a.a. on* E*.*

Proof. We will prove the assertion (2), for the other assertions see Corollaries 4.5 and 4.6 in [34]. Assume that $\mathcal{H}^{q,t}_\mu(E) < \infty$ and let, for $n \in \mathbb{N}$, the set

$$F_n = \Big\{ x \in E : \overline{d}^{q,t}_\mu(x,E) < 1 - 1/n \Big\}.$$

Apply (2.3) to $\nu = \mathcal{H}^{q,t}_{\llcorner E}$ on F_n to get

$$\mathcal{H}^{q,t}_\mu(F_n) \le \left(1 - \frac{1}{n}\right) \mathcal{H}^{q,t}_\mu(F_n).$$

This implies that $\mathcal{H}^{q,t}_\mu(F_n) = 0$. Therefore,

$$\overline{d}^{q,t}_\mu(x,E) \ge 1 \quad \text{for} \quad \mathcal{H}^{q,t}_\mu\text{-a.a. } x \in E.$$

Now consider, $n \in \mathbb{N}$, the set

$$\widetilde{F}_n = \left\{ x \in E : \overline{d}_\mu^{q,h}(x, E) \geq \xi\Big(1 + \frac{1}{n}\Big) \right\}.$$

We apply (2.3) again to get $\mathcal{H}_\mu^{q,t}(\widetilde{F}_n) = 0$. This implies that

$$\overline{d}_\mu^{q,t}(x, E) \leq \xi \quad \text{for } \mathcal{H}_\mu^{q,h}\text{-a.a. } x \in E. \qquad \square$$

Lemma 2. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of $\operatorname{supp}\mu$ such that $\mathcal{P}_\mu^{q,t}(E) < +\infty$. Then,*

(1) $\overline{d}_\mu^{q,t}\left(x, \mathsf{P}_{\mu\ \llcorner E}^{q,t}\right) \geq 1/\xi$ *and* $\underline{d}_\mu^{q,t}\left(x, \mathsf{P}_{\mu\ \llcorner E}^{q,t}\right) \leq 1$, $\mathsf{P}_\mu^{q,h}$*-a.a. on* E.

(2) $\overline{d}_\mu^{q,t}\left(x, \mathsf{H}_{\mu\ \llcorner E}^{q,t}\right) \geq 1/\xi$ *and* $\underline{d}_\mu^{q,t}\left(x, \mathsf{H}_{\mu\ \llcorner E}^{q,t}\right) \leq 1$, $\mathsf{H}_\mu^{q,h}$*-a.a. on* E.

(3) $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, *then*
- $\overline{d}_\mu^{q,t}\left(x, \mathsf{P}_{\mu\ \llcorner E}^{q,t}\right) \geq 1$, $\mathsf{P}_\mu^{q,h}$*-a.a. on* E.
- $\overline{d}_\mu^{q,t}\left(x, \mathsf{H}_{\mu\ \llcorner E}^{q,t}\right) \geq 1$, $\mathsf{H}_\mu^{q,h}$*-a.a. on* E.

Proof. The proof of this lemma is straightforward and mimics that in Lemma 1. $\square$

Lemma 3. *Let $q, t \in \mathbb{R}$, $\mu \in \mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of $\operatorname{supp}\mu$.*

(1) *If there exists $\nu \in \mathcal{P}(\mathbb{R}^n)$ such that $\sup_{x\in E} \overline{d}_\mu^{q,t}(x, \nu) \leq \gamma < \infty$, then*

$$\mathsf{H}_\mu^{q,t}(E) \geq \mathcal{H}_\mu^{q,t}(E)/\xi \geq \nu(E)/(\gamma\xi).$$

In addition, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then $\mathsf{H}_\mu^{q,t}(E) \geq \mathcal{H}_\mu^{q,t}(E) \geq \nu(E)/\gamma$.

(2) *If there exists $\nu \in \mathcal{P}(\mathbb{R}^n)$ such that $\inf_{x\in E} \overline{d}_\mu^{q,t}(x, \nu) \geq \gamma > 0$, then*

$$\mathcal{H}_\mu^{q,t}(E) \leq \xi\nu(E)/\gamma.$$

In addition, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then $\mathcal{H}_\mu^{q,t}(E) \leq \nu(E)/\gamma$.

(3) *If there exists $\nu \in \mathcal{P}(\mathbb{R}^n)$ such that $\inf_{x\in E} \underline{d}_\mu^{q,t}(x, \nu) \geq \gamma > 0$, then*

$$\mathsf{P}_\mu^{q,t}(E) \leq \mathcal{P}_\mu^{q,t}(E) \leq \nu(E)/\gamma.$$

(4) *If there exists $\nu \in \mathcal{P}(\mathbb{R}^n)$ such that $\sup_{x\in E} \underline{d}_\mu^{q,t}(x, \nu) \leq \gamma < +\infty$, then*

$$\mathcal{P}_\mu^{q,t}(E) \geq \nu(E)/\gamma.$$

Theorem 6 (Besicovitch covering Theorem, [31])**.** *There exists an integer $\xi \in \mathbb{N}$ such that, for any subset A of $\mathbb{R}^n$ and any sequence $(r_x)_{x\in A}$ satisfying*

(1) $r_x > 0, \ \forall\, x \in A$,

(2) $\sup\limits_{x\in A} r_x < \infty$.

Then, there exist ξ countable finite families $B_1, \ldots, B_\xi$ of $\big\{B_x(r_x) : x \in A\big\}$, such that

(1) $A \subset \bigcup_i \bigcup_{B\in B_i} B$.

(2) *B_i is a family of disjoint sets.*

3. Proof of Theorem 1

Let $\mu_1 \in \mathcal{P}(\mathbb{R}^n)$, $\mu_2 \in \mathcal{P}(\mathbb{R}^k)$, $q, t, s \in \mathbb{R}$, $E \subseteq supp\, \mu_1$ and $F \subseteq supp\, \mu_2$ be Borel sets. In this section, we will prove Theorem 1. First, in the next proposition, we give the result for finite measure sets.

Proposition 3. (1) *If* $\mathcal{H}^{q,t}_{\mu_1}(E) < +\infty$ *and* $\mathcal{H}^{q,s}_{\mu_2}(F) < +\infty$, *then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{H}^{q,s}_{\mu_2}(F) \le \xi^2 \mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E \times F).$$

In addition, if $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ *and* $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$, *then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{H}^{q,s}_{\mu_2}(F) \le \mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E \times F).$$

(2) *If* $\mathcal{H}^{q,t}_{\mu_1}(E) < +\infty$ *and* $\mathcal{P}^{q,s}_{\mu_2}(F) < +\infty$, *then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F) \le \xi \mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E \times F).$$

In addition, if $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ *and* $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$, *then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F) \le \mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E \times F).$$

Proof. (1) Let ν_1 be the restriction of $\mathcal{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathcal{H}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \left\{x \in E : \overline{d}^{q,t}_{\mu_1}(x, E) \le \xi\right\}$$

and

$$\widetilde{F} = \left\{x \in F : \overline{d}^{q,s}_{\mu_2}(x, F) \le \xi\right\}.$$

Then, using Lemma 1, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. Now, the product measure $\nu_1 \times \nu_2 \in \mathcal{P}(\mathbb{R}^{n+k})$ then, for $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned}\overline{d}^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y), \nu_1\times\nu_2\Big) &= \limsup_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right]\\ &\le \overline{d}^{q,t}_{\mu_1}\big(x,\nu_1\big)\ \overline{d}^{q,s}_{\mu_2}\big(y,\nu_2\big) \le \xi^2.\end{aligned}$$

Therefore, by Lemma 3,

$$\xi^2\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \ge \nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big) = \nu_1(E)\nu_2(F).$$

Hence,

$$\xi^2\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \ge \xi^2\mathcal{H}^{t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \ge \nu_1(E)\nu_2(F) = \mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{H}^{q,s}_{\mu_2}(F).$$

(2) Let ν_1 be the restriction of $\mathcal{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathcal{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \left\{x \in E : \overline{d}^{q,t}_{\mu_1}(x, E) \le \xi\right\}$$

and

$$\widetilde{F} = \left\{x \in F : \underline{D}^{q,s}_{\mu_2}(x, E) = 1\right\}.$$

Then, using Lemma 1, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. For $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned}\underline{d}^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y),\nu_1\times\nu_2\Big) &= \liminf_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right]\\ &\le \overline{d}^{q,t}_{\mu_1}\big(x,\nu_1\big)\ \underline{d}^{q,s}_{\mu_2}\big(y,\nu_2\big) \le \xi.\end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned}\xi\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\ge \nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big)\\ &= \nu_1(E)\nu_2(F) = \mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F).\end{aligned}$$

Hence,

$$\xi\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \ge \xi\mathcal{P}^{t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \ge \nu_1(E)\nu_2(F) = \mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F). \qquad \square$$

As a consequence, since $supp\,(\mu_1 \times \mu_2) = supp\,\mu_1 \times supp\,\mu_2$, we get the following result.

Corollary 2. (1) *Assume that* $0 < \mathcal{H}^{q,b^q_{\mu_1}(E)}_{\mu_1}(E) < \infty$ *and* $0 < \mathcal{H}^{q,b^q_{\mu_2}(F)}_{\mu_2}(F) < \infty$. *Then*

$$b^q_{\mu_1}(E) + b^q_{\mu_2}(F) \le b^q_{\mu_1\times\mu_2}(E\times F).$$

(2) *Assume that* $0 < \mathcal{H}^{q,b^q_{\mu_1}(E)}_{\mu_1}(E) < \infty$ *and* $0 < \mathcal{P}^{q,B^q_{\mu_2}(F)}_{\mu_2}(F) < \infty$. *Then*

$$b^q_{\mu_1}(E) + B^q_{\mu_2}(F) \le B^q_{\mu_1\times\mu_2}(E\times F).$$

Taking $E = supp\,\mu_1$ and $F = supp\,\mu_2$ then under the conditions of the previous corollary, we get

$$b_{\mu_1} + b_{\mu_2} \le b_{\mu_1\times\mu_2} \quad \text{and} \quad b_{\mu_1} + B_{\mu_2} \le B_{\mu_1\times\mu_2}, \tag{3.1}$$

In (3.1) we do not assume that the measures μ_1 and μ_2 satisfy the doubling condition. In addition, the assumptions in the previous corollary come from the fact that Proposition 3 does not deal with the sets of infinite measure. To overcome this problem we will require semifiniteness of the multifractal measures. Thus, with an additional hypothesis, we may prove these inequalities with no restriction on the finite measure sets.

Corollary 3. (1) *Assume that* $\mathcal{H}^{q,t}_{\mu_1}$ *is semifinite on* E *and* $\mathcal{H}^{q,s}_{\mu_2}$ *is semifinite on* F. *Then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{H}^{q,s}_{\mu_2}(F) \le \xi^2\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

In addition, if $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ *and* $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$, *then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{H}^{q,s}_{\mu_2}(F) \le \mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

(2) *Assume that* $\mathcal{H}^{q,t}_{\mu_1}$ *is semifinite on* E *and* $\mathcal{P}^{q,s}_{\mu_2}$ *is semifinite on* F. *Then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F) \le \xi\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

In addition, if $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$*, then*

$$\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F) \leq \mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

Proof. We only have to prove assertion (1) (the other assertion is similar). Since $\mathcal{H}^{q,t}_{\mu_1}$ is semifinite on E and $\mathcal{H}^{q,s}_{\mu_1}$ is semifinite on F, we can find a compact set $E_1 \subseteq E$ and a compact set $F_1 \subseteq F$ such that $\mathcal{H}^{q,t}_{\mu_1}(E_1) < +\infty$ and $\mathcal{H}^{q,s}_{\mu_2}(F_1) < +\infty$. Then by Proposition 3, we have

$$\mathcal{H}^{q,t}_{\mu_1}(E_1)\mathcal{H}^{q,s}_{\mu_2}(F_1) \leq \xi^2\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E_1\times F_1) \leq \xi^2\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

Taking supremum over E_1 and F_1 we get the desired result. □

The existence of subsets of finite positive measure has been shown in certain cases, especially when $q = 0$. For example, in [7] the author proved that a closed subset E of the real line of infinite $\mathcal{H}^t$ measure has subsets of any finite measure. It was also noted in the paper that the method used in the proof of these results is easily extended to n-dimensional Euclidean space $\mathbb{R}^n$. See also Corollary 7 in [22] if E is a subset of complete separable metric space. The reader is also referred to [9,38] for these results. Now if μ satisfies the doubling condition then $\mathcal{H}^{q,t}_{\mu}$ is semifinite [29]. In addition, we strongly believe that also if μ satisfies the doubling condition then $\mathcal{P}^{q,t}_{\mu}$ is semifinite. When $q = 0$, this result can be found in [25]. As a consequence we have

$$b^q_{\mu_1}(E) + b^q_{\mu_2}(F) \leq b^q_{\mu_1\times\mu_2}(E\times F)$$

and if we assume that $\mathcal{P}^{q,s}_{\mu_2}$ is semifinite on F then

$$b^q_{\mu_1}(E) + B^q_{\mu_2}(F) \leq B^q_{\mu_1\times\mu_2}(E\times F).$$

Finally take $E = supp\,\mu_1$ and $F = supp\,\mu_2$ to get (3.1).

4. Proof of Theorem 2

Let $\mu_1 \in \mathcal{P}(\mathbb{R}^n)$, $\mu_2 \in \mathcal{P}(\mathbb{R}^k)$, $q,t,s \in \mathbb{R}$, $E \subseteq supp\,\mu_1$ and $F \subseteq supp\,\mu_2$ be Borel sets. In this section, we will prove Theorem 2. To prove the next proposition using density approach, we need to assume that the result is true in "the null cases", i.e., when one of the factors on the right is zero. The other cases will be studied in Propositions 5, 6 and 7 using elementary calculus.

Proposition 4. (1) *We have*

$$\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \leq \xi\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F).$$

provided it is true in the null cases when one of the factors on the right is zero.

(2) *We have*

$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \leq \mathcal{P}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F).$$

provided it is true in the null cases when one of the factors on the right is zero.

Proof. (1) If $\mathcal{H}^{q,t}_{\mu_1}(E) = \infty$ or $\mathcal{P}^{q,s}_{\mu_2}(F) = \infty$ there is nothing to prove, so assume they are both finite. Let ν_1 be the restriction of $\mathcal{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathcal{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \Big\{x \in E : \overline{d}^{q,t}_{\mu_1}(x, E) \geq 1\Big\}$$

and

$$\widetilde{F} = \big\{x \in F : \underline{D}^{q,s}_{\mu_2}(x, F) = 1\big\}.$$

Then, using Lemma 1, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. For $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned}\overline{d}^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y), \nu_1 \times \nu_2\Big) &= \limsup_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right]\\ &\geq \overline{d}^{q,t}_{\mu_1}\big(x,\nu_1\big)\ \ \underline{d}^{q,s}_{\mu_2}\big(y,\nu_2\big) \geq 1.\end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned}\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\leq \xi\nu_1 \times \nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \xi\nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big)\\ &= \xi\nu_1(E)\nu_2(F) = \xi\mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F).\end{aligned}$$

By the assumption for the null cases, we get the result with $E \times F$.

(2) If $\mathcal{P}^{q,t}_{\mu_1}(E) = \infty$ or $\mathcal{P}^{q,s}_{\mu_2}(F) = \infty$ there is nothing to prove, so assume they are both finite. Let ν_1 be the restriction of $\mathcal{P}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathcal{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \big\{x \in E : \underline{D}^{q,t}_{\mu_1}(x, E) = 1\big\}$$

and

$$\widetilde{F} = \big\{x \in F : \underline{D}^{q,s}_{\mu_2}(x, F) = 1\big\}.$$

Then, using Lemma 1, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. Now the product measure $\nu_1 \times \nu_2 \in \mathcal{P}(\mathbb{R}^{n+k})$. For $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned}\underline{d}^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y), \nu_1 \times \nu_2\Big) &= \liminf_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right]\\ &\geq \underline{d}^{q,t}_{\mu_1}\big(x,\nu_1\big)\ \ \underline{d}^{q,s}_{\mu_2}\big(y,\nu_2\big) = 1.\end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned}\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\leq \nu_1 \times \nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big)\\ &= \nu_1(E)\nu_2(F) = \mathcal{P}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F).\end{aligned}$$

By the assumption for the null cases, we get the result with $E \times F$. □

It is clear that if μ_1 and μ_2 satisfy the doubling condition, then $\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}(E \times F) \leq \mathcal{H}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F)$. As a consequence, we get the following result.

Corollary 4. (1) *Assume that* $0 < \mathcal{H}_{\mu_1}^{q,b_{\mu_1}^q(E)}(E) < \infty$ *and* $0 < \mathcal{P}_{\mu_2}^{q,b_{\mu_2}^q(F)}(F) < \infty$. *Then*

$$b_{\mu_1\times\mu_2}^q(E\times F) \le b_{\mu_1}^q(E) + B_{\mu_2}^q(F).$$

(2) *Assume that* $0 < \mathcal{P}_{\mu_1}^{q,B_{\mu_1}^q(E)}(E) < \infty$ *and* $0 < \mathcal{P}_{\mu_2}^{q,B_{\mu_2}^q(F)}(F) < \infty$. *Then*

$$B_{\mu_1\times\mu_2}^q(E\times F) \le B_{\mu_1}^q(E) + B_{\mu_2}^q(F).$$

Taking $E = supp\,\mu_1$ and $F = supp\,\mu_2$, in the previous corollary, we get

$$b_{\mu_1\times\mu_2} \le b_{\mu_1} + B_{\mu_2} \quad \text{and} \quad B_{\mu_1\times\mu_2} \le B_{\mu_1} + B_{\mu_2}. \tag{4.1}$$

In (4.1) we do not assume that the measures μ_1 and μ_2 satisfy the doubling condition. In addition, the assumptions in the previous corollary come from the fact that Proposition 4 does not deal with "the null case".

Now, we will study "the null case". Let $r > 0$ and $B \subset \mathbb{R}^n$. The set B is said to be r-separated set if $d(x, y) \ge r$ for all $x, y \in B$ such that $x \ne y$. B is said to be totally bounded set if for all $\epsilon > 0$ there exists a finite centered ϵ-covering of B. Therefore, if B is not totally bounded, then it has an infinite multifractal packing measure as mentioned in the next proposition.

Lemma 4. *If* $\overline{\mathcal{P}}_{\mu}^{q,t}(E) < +\infty$, *then* E *is totally bounded.*

Proof. Suppose that E is not totally bounded. There exist $r > 0$ and an infinite r-separated set $\{x_n\}$ of E. It follows that if, for all $\delta < r/2$, the family $\{(x_n, \delta)\}$ is a δ-packing of E, then

$$\overline{\mathcal{P}}_{\mu,\delta}^{q,t}(E) = +\infty \quad \text{and thus} \quad \overline{\mathcal{P}}_{\mu}^{q,t}(E) = +\infty. \qquad \square$$

Proposition 5. *Assume that* $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ *and* $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$. *If* $\mathcal{H}_{\mu_1}^{q,t}(E) < \infty$ *and* $\mathcal{P}_{\mu_2}^{q,s}(F) = 0$, *then*

$$\mathcal{H}_{\mu_1\times\mu_2}^{q,t+s}\big(E\times F\big) = 0.$$

Proof. Let $\epsilon > 0$ and fix $c > \overline{\mathcal{H}}_{\mu_1}^{q,t}(E)$. Now $\mathcal{P}_{\mu_2}^{q,s}(F) = 0$, thus there exists a sequence $\{F_n\}_n$ such that $F \subseteq \bigcup_n F_n$ and

$$\sum_n \overline{\mathcal{P}}_{\mu_2}^{q,s}(F_n) < \epsilon_1 := \epsilon/c\xi,$$

where ξ is the constant which appear in the Besicovitch covering theorem (Theorem 6). Let $p_n > \overline{\mathcal{P}}_{\mu_2}^{q,s}(F_n)$ such that $\sum_n p_n < \epsilon_1$. For fixed n there exists $\delta > 0$ such that $\overline{\mathcal{P}}_{\mu_2,\delta}^{q,s}(F_n) < p_n$. Since $\overline{\mathcal{H}}_{\mu_1,\delta}^{q,t}(E) \le \overline{\mathcal{H}}_{\mu_1}^{q,t}(E) < c$, we can find $\mathcal{B} = \{B_i = B(x_i, r_i)\}$ a δ-cover of E such that

$$\sum_{\mathcal{B}} (2r_i)^t \mu_1(B_i)^q < c.$$

Since, for all n, $\overline{\mathcal{P}}_{\mu_2}^{q,s}(F_n) < +\infty$, F_n is totally bounded by Lemma 4. Let, for each i, $\widetilde{\mathcal{B}}_i = \{B_{ij} = B(y_{ij}, r_i) : 1 \le j \le K_i\}$ be a finite centered δ-covering of F_n. By Besicovitch covering theorem, there exists ξ sub-collections of balls $A_1, \ldots, A_\xi$ contained in $\widetilde{\mathcal{B}}_i$ such that A_j, $j = 1, \ldots, \xi$, is a δ-packing of F_n and

$$F_n \subseteq \bigcup_{j=1}^{\xi} \bigcup_{B \in A_j} B.$$

Remark that

$$\sum_{B \in A_j} (2r_i)^t \mu_2(B)^q \leq \overline{\mathcal{P}}^{q,s}_{\mu_2,\delta}(F_n) < p_n$$

and then $\sum_{j=1}^{K_i} (2r_i)^t \mu_2\big(B(y_{ij}, r_i)\big)^q \leq \xi p_n$. Now

$$E \times F_n \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{K_i} B\big((x_i, y_{ij}), r_i\big).$$

Therefore,

$$\overline{\mathcal{H}}^{q,t+s}_{\mu_1 \times \mu_2,\delta}\big(E \times F_n\big) \leq \xi c p_n.$$

This is true for all small enough δ, so $\overline{\mathcal{H}}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F_n\big) \leq \xi c p_n$ and then

$$\overline{\mathcal{H}}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F\big) \leq \sum_n \overline{\mathcal{H}}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F_n\big) \leq \xi c \sum_n p_n < \epsilon.$$

Therefore, since ϵ is arbitrarily chosen, $\overline{\mathcal{H}}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F\big) = 0$ and then $\mathcal{H}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F\big) = 0$ by Proposition 1. $\square$

Proposition 6. *Assume that $\mu_1 \in \mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2 \in \mathcal{P}_D(\mathbb{R}^k)$. If $\mathcal{H}^{q,t}_{\mu_1}(E) = 0$ and $\mathcal{P}^{q,s}_{\mu_2}(F) < +\infty$, then $\mathcal{H}^{q,t+s}_{\mu_1 \times \mu_2}\big(E \times F\big) = 0$.*

Proof. Since $\mathcal{P}^{q,s}_{\mu_2}(F) < +\infty$, F is the union of countable many sets F_n such that $\overline{\mathcal{P}}^{q,s}_{\mu_2}(F_n) < +\infty$. So we may assume that $\overline{\mathcal{P}}^{q,s}_{\mu_2}(F) < +\infty$. It follows, by Lemma 4 that F is totally bounded. Let $\delta > 0$ such that $p_\delta := \overline{\mathcal{P}}^{q,s}_{\mu_2,\delta}(F) < +\infty$.

Let $\epsilon > 0$. Since $\overline{\mathcal{H}}^{q,t}_{\mu_1}(E) = 0$ we can find $\mathcal{B} := \big\{B_i = B(x_i, r_i)\big\}$ a centered δ-covering of E such that

$$\sum_i (2r_i)^t \mu_1\big(B_i\big)^q < \epsilon_1 := \epsilon / \xi p_\delta,$$

where ξ is the constant which appear in the Besicovitch covering theorem (Theorem 6). By the total boundedness of F, there exists for each i, $\widetilde{\mathcal{B}}_i = \big\{B_{ij} = B(y_{ij}, r_i) : 1 \leq j \leq K_i\big\}$ a finite δ-covering of F. By Besicovitch covering theorem, there exists ξ sub-collections of balls $A_1, \ldots, A_\xi$ contained in $\widetilde{\mathcal{B}}_i$ such that A_j is a δ-packing of F and

$$F \subseteq \bigcup_{j=1}^{\xi} \bigcup_{B \in A_j} B.$$

Remark that
$$\sum_{B\in A_j}(2r_i)^t\mu_2(B)^q \le \overline{\mathcal{P}}^{q,s}_{\mu_2,\delta}(F)=p_\delta$$
and then $\sum_{j=1}^{K_i}(2r_i)^t\mu_2\big(B(y_{ij},r_i)\big)^q \le \xi p_\delta$. Now
$$E\times F\subseteq\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{K_i}B\big((x_i,y_{ij}),r_i\big).$$
Therefore,
$$\overline{\mathcal{H}}^{q,t+s}_{\mu_1\times\mu_2,\delta}\big(E\times F\big)\le\xi\epsilon_1 p_\delta=\epsilon.$$
This is true for all small enough δ, so $\overline{\mathcal{H}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)\le\epsilon$. Since ϵ is arbitrarily, $\overline{\mathcal{H}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)=0$ and then $\mathcal{H}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)=0$ by Proposition 1. □

Proposition 7. *Assume that* $\mathcal{P}^{q,t}_{\mu_1}(E)<\infty$ *and* $\mathcal{P}^{q,s}_{\mu_2}(F)=0$. *Then,*
$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)=0.$$

Proof. Let $\epsilon>0$. Since $\mathcal{P}^{q,t}_{\mu_1}(E)<+\infty$, E is the union of countable many sets E_n such that $\overline{\mathcal{P}}^{q,t}_{\mu_1}(E_n)<+\infty$. So we may assume that $\overline{\mathcal{P}}^{q,t}_{\mu_1}(E)<+\infty$. Let $\delta_0>0$ such that $c:=\overline{\mathcal{P}}^{q,t}_{\mu_1,\delta_0}(E)<+\infty$.

Since $\mathcal{P}^{q,s}_{\mu_2}(F)=0$, thus there exist $\{F_n\}_n$ such that $F\subseteq\bigcup_n F_n$ and
$$\sum_n\overline{\mathcal{P}}^{q,s}_{\mu_2}(F_n)<\epsilon_1:=\epsilon/c.$$
Let $p_n>\overline{\mathcal{P}}^{q,s}_{\mu_2}(F_n)$ such that $\sum_n p_n<\epsilon_1$. For fixed n there exists $0<\delta<\delta_0$ such that $\overline{\mathcal{P}}^{q,s}_{\mu_2,\delta}(F_n)<p_n$.

Let $\Big\{B\big(x_i,y_i),r_i\big)\Big\}$ be a δ-packing of $E\times F_n$. Then
$$\sum_i(2r_i)^{s+t}(\mu_1\times\mu_2)\big(B(x_i,y_i),r_i\big)^q\le\overline{\mathcal{P}}^{q,t}_{\mu_1,\delta}(E)\times\overline{\mathcal{P}}^{q,s}_{\mu_2,\delta}(F_n)\le cp_n.$$
This is true for all small enough δ, so $\overline{\mathcal{P}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F_n\big)\le cp_n$ and then
$$\overline{\mathcal{P}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)\le\sum_n\overline{\mathcal{P}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F_n\big)\le c\sum_n p_n<\epsilon.$$
Therefore, since ϵ is arbitrarily, $\overline{\mathcal{P}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)=0$. Finally, we have
$$\overline{\mathcal{P}}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)\le\overline{\mathcal{P}}^{q,t}_{\mu_1}\big(E\big)\overline{\mathcal{P}}^{q,s}_{\mu_2}\big(F\big).\tag{4.2}$$
Now, let $E\subseteq\bigcup_i E_i$ and $F\subseteq\bigcup_i F_i$. By (4.2),
$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E_i\times F\big)\le\sum_j\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E_i\times F_i\big)\le\overline{\mathcal{P}}^{q,t}_{\mu_1}\big(E_i\big)\sum_j\overline{\mathcal{P}}^{q,s}_{\mu_2}\big(F_j\big).$$
Since the covering $\{F_j\}_j$ of F was arbitrary, we conclude that
$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E_i\times F\big)\le\overline{\mathcal{P}}^{q,t}_{\mu_1}\big(E_i\big)\mathcal{P}^{q,s}_{\mu_2}\big(F\big)$$

for all i hence

$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)\le\sum_i\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E_i\times F\big)\le\mathcal{P}^{q,s}_{\mu_2}\big(F\big)\sum_i\overline{\mathcal{P}}^{q,t}_{\mu_1}\big(E_i\big).$$

Since the covering $\{E_j\}_i$ of E was arbitrary we deduce that

$$\mathcal{P}^{q,t+s}_{\mu_1\times\mu_2}\big(E\times F\big)\le\mathcal{P}^{q,t}_{\mu_1}\big(E\big)\mathcal{P}^{q,s}_{\mu_2}\big(F\big).\qquad\square$$

5. Results on Hewitt-Stromberg measures

Our density approach can be applied to study the multifractal Hewitt-Stromberg measures $\mathsf{H}^{q,t}_{\mu}$ and $\mathsf{P}^{q,t}_{\mu}$ of Cartesian product sets by means of the measure of their components. In this section, we will use Corollary 1 instead of Theorem 5. But this corollary allows only partial results to be obtained. Let $q,t,s\in\mathbb{R}$, $\mu_1\in\mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2\in\mathcal{P}_D(\mathbb{R}^k)$. This section deals with sets $E\subseteq\mathbb{R}^n$ and $F\subseteq\mathbb{R}^k$ satisfying the following properties:

$$\underline{d}^{q,t}_{\mu_1}(x,\nu)=\overline{d}^{q,t}_{\mu_1}(x,\nu)$$

for all $x\in E$ and $\nu\in\big\{\mathsf{H}^{q,t}_{\mu_1\llcorner E},\mathsf{P}^{q,t}_{\mu_1\llcorner E}\big\}$ and

$$\underline{d}^{q,s}_{\mu_2}(x,\nu)=\overline{d}^{q,s}_{\mu_2}(x,\nu)$$

for all $x\in F$ and $\nu\in\big\{\mathsf{H}^{q,s}_{\mu_2\llcorner F},\mathsf{P}^{q,s}_{\mu_2\llcorner F}\big\}$.

Theorem 7. *Assume that $\mathcal{P}^{q,t}_{\mu_1}(E)<+\infty$ and $\mathcal{P}^{q,s}_{\mu_2}(F)<+\infty$. Then,*

(1) $\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{H}^{q,s}_{\mu_2}(F)\le\xi\mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F)$. *In particular, if $\mu_1\in\mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2\in\mathcal{P}_D(\mathbb{R}^k)$, then*

$$\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{H}^{q,s}_{\mu_2}(F)\le\mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

(2) $\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F)\le\xi\mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F)$. *In particular, if $\mu_1\in\mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2\in\mathcal{P}_D(\mathbb{R}^k)$, then*

$$\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F)\le\mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F).$$

Proof. (1) Let ν_1 be the restriction of $\mathsf{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathsf{H}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E}=\big\{x\in E: d^{q,t}_{\mu_1}(x,\nu_1)\le 1\big\}$$

and

$$\widetilde{F}=\big\{x\in F: d^{q,s}_{\mu_2}(x,\nu_2)\le 1\big\}.$$

Then, using Lemma 2, we have $\nu_1(E)=\nu_1(\widetilde{E})$ and $\nu_2(F)=\nu_2(\widetilde{F})$. Now, for the product measure $\nu_1\times\nu_2\in\mathcal{P}(\mathbb{R}^{n+k})$, $(x,y)\in\widetilde{E}\times\widetilde{F}$, we have

$$\begin{aligned}d^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y),\nu_1\times\nu_2\Big)&=\lim_{r\to0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right]\\&=d^{q,t}_{\mu_1}\big(x,\nu_1\big)d^{q,s}_{\mu_2}\big(y,\nu_2\big)\le 1.\end{aligned}$$

Therefore, by Lemma 3,

$$\xi \mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \geq \nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big) = \nu_1(E)\nu_2(F).$$

Hence,

$$\xi \mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \geq \xi \mathsf{H}^{t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \geq \nu_1(E)\nu_2(F) = \mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{H}^{q,s}_{\mu_2}(F).$$

(2) Let ν_1 be the restriction of $\mathsf{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathsf{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \big\{x\in E : d^{q,t}_{\mu_1}(x,\nu_1)\leq 1\big\}$$

and

$$\widetilde{F} = \big\{x\in F : d^{q,s}_{\mu_2}(x,\nu_2)\leq 1\big\}.$$

Then, using Lemma 2, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. For $(x,y)\in\widetilde{E}\times\widetilde{F}$, we have

$$\begin{aligned} d^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y),\nu_1\times\nu_2\Big) &= \lim_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\ \frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right] \\ &= d^{q,t}_{\mu_1}\big(x,\nu_1\big)\ \ d^{q,s}_{\mu_2}\big(y,\nu_2\big)\leq 1. \end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned} \xi \mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\geq \nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big) \\ &= \nu_1(E)\nu_2(F) = \mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F). \end{aligned}$$

Hence,

$$\xi \mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \geq \xi \mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) \geq \nu_1(E)\nu_2(F) = \mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F). \qquad \square$$

Next, we will study the two other inequalities.

Theorem 8. *Assume that $\mathcal{P}^{q,t}_{\mu_1}(E)<\infty$ and $\mathcal{P}^{q,s}_{\mu_2}(F)<+\infty$. Then*

(1) *We have*

$$\mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F)\leq \xi^2\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F)$$

provided it is true in the null cases when one of the factors on the right is zero.

(2) *We have*

$$\mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F)\leq \xi^2\mathsf{P}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F)$$

provided it is true in the null cases when one of the factors on the right is zero.

Proof. (1) Let ν_1 be the restriction of $\mathsf{H}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathsf{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \big\{x\in E : d^{q,t}_{\mu_1}(x,\nu_1)\geq 1/\xi\big\}$$

and

$$\widetilde{F} = \big\{x\in F : d^{q,s}_{\mu_2}(x,\nu_2)\geq 1/\xi\big\}.$$

Then, using Lemma 2, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. For $(x,y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned} d^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y),\nu_1\times\nu_2\Big) &= \lim_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\,\frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right] \\ &= d^{q,t}_{\mu_1}\big(x,\nu_1\big)d^{q,s}_{\mu_2}\big(y,\nu_2\big) \geq 1/\xi^2. \end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned} \mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\leq \xi^2\nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \xi^2\nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big) \\ &= \xi^2\nu_1(E)\nu_2(F) = \xi^2\mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F). \end{aligned}$$

By the assumption for the null cases, we get the result with $E\times F$.

(2) Let ν_1 be the restriction of $\mathsf{P}^{q,t}_{\mu_1}$ to E and ν_2 be the restriction of $\mathsf{P}^{q,s}_{\mu_2}$ to F. We set

$$\widetilde{E} = \left\{x\in E : d^{q,t}_{\mu_1}(x,\nu_1)\geq 1/\xi\right\}$$

and

$$\widetilde{F} = \left\{x\in F : d^{q,s}_{\mu_2}(x,\nu_2)\geq 1/\xi\right\}.$$

Then, using Lemma 2, we have $\nu_1(E) = \nu_1(\widetilde{E})$ and $\nu_2(F) = \nu_2(\widetilde{F})$. Now for the product measure $\nu_1\times\nu_2 \in \mathcal{P}(\mathbb{R}^{n+k})$, $(x,y)\in\widetilde{E}\times\widetilde{F}$, we have

$$\begin{aligned} d^{q,t+s}_{\mu_1\times\mu_2}\Big((x,y),\nu_1\times\nu_2\Big) &= \lim_{r\to 0}\left[\frac{\nu_1\big(B(x,r)\big)}{\mu_1\big(B(x,r)\big)^q(2r)^t}\,\frac{\nu_2\big(B(y,r)\big)}{\mu_2\big(B(y,r)\big)^q(2r)^s}\right] \\ &= d^{q,t}_{\mu_1}\big(x,\nu_1\big)d^{q,s}_{\mu_2}\big(y,\nu_2\big) \geq 1/\xi^2. \end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned} \mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}\big(\widetilde{E}\times\widetilde{F}\big) &\leq \xi^2\nu_1\times\nu_2\big(\widetilde{E}\times\widetilde{F}\big) = \xi^2\nu_1\big(\widetilde{E}\big)\nu_2\big(\widetilde{F}\big) \\ &= \xi^2\nu_1(E)\nu_2(F) = \xi^2\mathcal{P}^{q,t}_{\mu_1}(E)\mathcal{P}^{q,s}_{\mu_2}(F). \end{aligned}$$

By the assumption for the null cases, we get the result with $E\times F$. □

Remark 4. (1) It is clear that, if $\mu_1\in\mathcal{P}_D(\mathbb{R}^n)$ and $\mu_2\in\mathcal{P}_D(\mathbb{R}^k)$, then, under the hypothesis of previous theorem,

$$\mathsf{H}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \leq \mathsf{H}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F)$$

and

$$\mathsf{P}^{q,t+s}_{\mu_1\times\mu_2}(E\times F) \leq \mathsf{P}^{q,t}_{\mu_1}(E)\mathsf{P}^{q,s}_{\mu_2}(F).$$

(2) Let $t_0 = \mathsf{b}^q_{\mu_1}(E)$ and $s_0 = \mathsf{B}^q_{\mu_2}(F)$ and assume that $\mathcal{P}^{q,t_0}_{\mu_1}(E) < \infty$ and $\mathcal{P}^{q,s_0}_{\mu_2}(F) < +\infty$. If $\mathsf{H}^{q,t_0}_{\mu_1}(E) > 0$ and $\mathsf{P}^{q,s_0}_{\mu_2}(F) > 0$, then

$$\mathsf{b}^q_{\mu_1\times\mu_2}(E\times F) \leq \mathsf{b}^q_{\mu_1}(E) + \mathsf{B}^q_{\mu_2}(F).$$

(3) Let $t_0 = \mathsf{B}^q_{\mu_1}(E)$ and $s_0 = \mathsf{B}^q_{\mu_2}(F)$ and assume that $\mathcal{P}^{q,t_0}_{\mu_1}(E) < \infty$ and $\mathcal{P}^{q,s_0}_{\mu_2}(F) < +\infty$. If $\mathsf{P}^{q,t_0}_{\mu_1}(E) > 0$ and $\mathsf{P}^{q,s_0}_{\mu_2}(F) > 0$, then

$$\mathsf{B}^q_{\mu_1\times\mu_2}(E\times F) \leq \mathsf{B}^q_{\mu_1}(E) + \mathsf{B}^q_{\mu_2}(F).$$

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