Commun. Korean Math. Soc. **37** (2022), No. 4, pp. 1099–1129
https://doi.org/10.4134/CKMS.c210361
pISSN: 1225-1763 / eISSN: 2234-3024

LIOUVILLE THEOREMS FOR THE MULTIDIMENSIONAL FRACTIONAL BESSEL OPERATORS

Vanesa Galli, Sandra Molina, and Alejandro Quintero

Abstract. In this paper, we establish Liouville type theorems for the fractional powers of multidimensional Bessel operators extending the results given in [6]. In order to do this, we consider the distributional point of view of fractional Bessel operators studied in [12].

1. Introduction

Bessel operators appear in the setting of harmonic analysis related to Hankel transformations. These operators arise when we consider the Laplacian operator in polar coordinates. In order to establish Liouville type theorems for the fractional Bessel operators, we will generalize the results obtained in [12] to the n-dimensional case. The multidimensional Bessel operators in $\mathbb{R}^n_+ = (0,\infty)^n$ are given by

$$\Delta_\lambda = \sum_{i=1}^{n} \left(-\frac{\partial^2}{\partial x_i^2} + \frac{\lambda_i(\lambda_i - 1)}{x_i^2} \right) \tag{1.1}$$

and

$$B_\lambda = \sum_{i=1}^{n} \left(-\frac{\partial^2}{\partial x_i^2} - \frac{2\lambda_i}{x} \frac{\partial}{\partial x_i} \right), \tag{1.2}$$

where $\lambda \in \mathbb{R}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda_i > 0$, which are related through

$$\Delta_\lambda = x^\lambda B_\lambda x^{-\lambda}. \tag{1.3}$$

The fractional Bessel operator Δ_λ^α, $0 < \alpha < 1$, was studied in [2] in the setting of Holder spaces. In this work were studied global Holder and Schauder estimates for a fractional Bessel equation.

The fractional powers of (1.1) and (1.2) were studied in [12] for the one dimensional case using the similarity relation (1.3). This work is based on the classical theory of fractional powers initially developed by Balakrishnan in [1]. Previously in [10], the powers of the Laplace operator had been studied in the

Received October 30, 2021; Accepted January 19, 2022.
2020 *Mathematics Subject Classification.* Primary 47A60, 46F10, 46F12, 46F30.
Key words and phrases. Liouville theorem, Bessel operator, Hankel transform.

Balakrishnan setting. It should be noted that Balakrishnan representation of fractional powers is equivalent to semigroup representation, as can be seen in [9, Theorem 3.2.2].

Let X and Y be Banach spaces. Two linear operators A and B, $A : D(A) \subset X \to X$ and $B : D(B) \subset Y \to Y$ are *similar* if there exists an isomorphism $T : X \to Y$ with inverse $T^{-1} : Y \to X$ such that $D(B) = \{x \in Y : T^{-1}x \in D(A)\}$ given by

$$B = TAT^{-1}. \tag{1.4}$$

Similar operators have the same spectral properties and also that of being non-negative if one of them has this property. Thus, their powers are similar operators and verify the same similarity relation, so

$$B^\alpha = TA^\alpha T^{-1}.$$

In this work, we generalize the results obtained in [12] to the n-dimensional case obtaining the fractional powers of Bessel operators (1.1) and (1.2) in weighted Lebesgue spaces and in distributional spaces. As in [12], we first study the non-negativity of Bessel operator (1.1) in suitable weighted Lebesgue spaces. By similarity we obtain the non-negativity of (1.2) in the corresponding Lebesgue space. Analogously to the one-dimensional case, we construct a locally convex space $\mathcal{B}$ in which Δ_λ is continuous and non-negative. We considered the dual space $\mathcal{B}'$ with the strong topology and obtained non-negativity of Δ_λ in this distributional space. $\mathcal{B}'$ is contained in the distributional Zemanian space and contain the weighted Lebesgue spaces in which non-negativity was studied. Consequently, if we denote with $\Delta_{\lambda,\mathcal{B}'}$ the Bessel operator with domain $\mathcal{B}'$, we can consider the powers $\Delta^\alpha_{\lambda,\mathcal{B}'}$ with $\operatorname{Re}\alpha > 0$ and it is verified the following relation inherited from the selfadjuncture of Δ_λ

$$(\Delta^\alpha_\lambda u, \phi) = (u, \Delta^\alpha_\lambda \phi)$$

for $\phi \in \mathcal{B}$ and $u \in \mathcal{B}'$.

In [6], a Liouville-type theorem was studied for a certain general class of Bessel-type operators. This class of operators contains as a particular case the Bessel operator (1.1). The Liouville theorem applied to this operator states that: if u is a Zemanian distribution that verifies that $\Delta_\lambda u = 0$, then u is a polynomial. This property is analogous to the classical result that establishes that any harmonic tempered distribution is a polynomial. In [3], [4], [8] and [18] different versions of Liouville theorem for the fractional Laplacian were studied.

This work aims to give a proof for the following Liouville theorems for the distributional fractional Bessel operators:

Theorem 1.1. *Let $u \in \mathcal{B}'$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. If $\Delta^\alpha_{\lambda,\mathcal{B}'} u = 0$, then there exists a polynomial p such that $u = x^\lambda p[x_1^2, \dots, x_n^2]$.*

For the study of the powers of Bessel operator given by (1.2) we introduce a locally convex space $\mathcal{F}$. This space verifies that its dual space $\mathcal{F}'$ with the strong topology is a suitable distributional space for the study of fractional powers $B^{\alpha}_{\lambda,\mathcal{F}'}$ and from similarity we conclude the following result.

Theorem 1.2. *Let $u \in \mathcal{F}'$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. If $B^{\alpha}_{\lambda,\mathcal{F}'}u = 0$, then there exists a polynomial p such that $u = x^{2\lambda}p[x_1^2, \ldots, x_n^2]$.*

This paper is organized as follows. In Section 2, we summarize basic results related to harmonic analysis in the Hankel setting. Section 3 contains a brief review of non-negative operators in Banach and in locally convex spaces and properties of fractional powers of similar operators. In Sections 4, 5 and 6, we study the non-negativity of Bessel operators (1.1) and (1.2). Finally, Sections 6 and 7 contain Liouville's theorems for both fractional Bessel operators.

2. Preliminaries

In this section, we introduce the Lebesgue and distributional spaces necessary for our purposes.

We now present some notational conventions that will allow us to simplify the presentation of our results. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with $\|\cdot\|$ the Euclidean norm, and $\mathbb{R}^n_+ = (0,\infty)^n$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any n-tuple $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ we define its length to be $|k| = k_1 + \cdots + k_n$.

If $x \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$, we define

$$x^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n}. \tag{2.1}$$

In particular if $a \in \mathbb{R}$, $\beta \in \mathbb{N}_0^n$, a^{β} means

$$a^{\beta} = a^{\beta_1} \cdots a^{\beta_n} = a^{|\beta|}. \tag{2.2}$$

For $\alpha \in \mathbb{R}$, let $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha)$, then for $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$

$$a^{\boldsymbol{\alpha}} = (a^n)^{\alpha} \quad \text{and} \quad x^{\boldsymbol{\alpha}} = x_1^{\alpha} \cdots x_n^{\alpha} = x^{\alpha}. \tag{2.3}$$

If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and $\alpha \in \mathbb{R}$ means

$$\beta + \alpha = (\beta_1 + \alpha, \ldots, \beta_n + \alpha) = \beta + \boldsymbol{\alpha}. \tag{2.4}$$

If $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, as usual D^k means $D^k = D_1^{k_1} \cdots D_n^{k_n}$ with $D_j = \frac{\partial}{\partial x_j}$. We shall write

$$T^k = T_n^{k_n} T_{n-1}^{k_{n-1}} \cdots T_1^{k_1}, \tag{2.5}$$

where $T_i = \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)$ and T_i^j denotes the j-times composition of the operator T_i.

Remark 2.1. Let k be a multi-index and θ, φ differentiable functions up to order $|k|$. The following equality is valid

$$T^k\{\theta \cdot \varphi\} = \sum_{j=0}^{k} \binom{k}{j} T^{k-j}\theta \cdot T^j \varphi, \tag{2.6}$$

where " $\cdot$ " denotes the usual product of functions and $\binom{k}{j} = \binom{k_1}{j_1} \cdots \binom{k_n}{j_n}$ for $k, j \in \mathbb{N}_0^n$. For details we refer the reader to [11].

We consider two classical versions of Hankel transform given by

$$(h_\lambda f)(t) = \int_0^\infty f(x)\sqrt{xt} J_{\lambda-1/2}(xt)\, dx, \quad t \in (0,\infty) \tag{2.7}$$

and

$$(H_\lambda f)(t) = \int_0^\infty f(x)(xt)^{1/2-\lambda} J_{\lambda-1/2}(xt)\, x^{2\lambda}\, dx, \quad t \in (0,\infty) \tag{2.8}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and J_ν is the Bessel function of first kind and order ν.

S. Molina and S. Trione studied in [13] an n-dimensional generalization of (2.7), given by h_λ and defined by

$$(h_\lambda \phi)(y) = \int_{\mathbb{R}_+^n} \phi(x_1, \ldots, x_n) \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\lambda_i - 1/2}(x_i y_i)\}\, dx_1 \cdots dx_n. \tag{2.9}$$

Analogously it is possible to define an n-dimensional generalization for (2.8), given by H_λ and defined by

$$\begin{aligned}(H_\lambda \phi)(y) &\\ = \int_{\mathbb{R}_+^n} \phi(x_1, \ldots, x_n) &\left\{ \prod_{i=1}^n (x_i y_i)^{1/2-\lambda_i} J_{\lambda_i - 1/2}(x_i y_i)\, x_i^{2\lambda_i} \right\} dx_1 \cdots dx_n.\end{aligned} \tag{2.10}$$

In both (2.9) and (2.10), $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ and J_ν represents the Bessel function of first kind and order ν.

Next we define certain weighted L^p-spaces for $1 \leq p \leq \infty$. Let

$$s(x) = \frac{x^{2\lambda}}{C_\lambda}, \tag{2.11}$$

$$r(x) = x^{-\lambda}, \tag{2.12}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $x \in \mathbb{R}_+^n$, $C_\lambda = 2^{\lambda - 1/2}\, \Gamma(\lambda_1 + 1/2) \cdots \Gamma(\lambda_n + 1/2)$ and dx is the usual n-dimensional Lebesgue measure and $x^{2\lambda}$ and $x^{-\lambda}$ are given by (2.1) and $2^{\lambda-1/2}$ is given by (2.2). Let $L^p(\mathbb{R}_+^n, sr^p)$, $1 \leq p < \infty$, be the space of measurable functions f defined over $\mathbb{R}_+^n$ with norm

$$\|f\|_{L^p(sr^p)} = \left(\int_{\mathbb{R}_+^n} |f(x)|^p\, s(x) r^p(x)\, dx \right)^{1/p} \qquad 1 \leq p < \infty.$$

Moreover, $L^\infty(\mathbb{R}_+^n, r)$ is the space of measurable functions over $\mathbb{R}_+^n$ such that

$$\|f\|_{L^\infty(r)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}_+^n} |r(x) f(x)| < \infty.$$

For simplicity we write $L^p(sr^p)$ and $L^\infty(r)$ instead of $L^p(\mathbb{R}_+^n, sr^p)$ and $L^\infty(\mathbb{R}_+^n, r)$.

By $\mathcal{D}(\mathbb{R}^n_+)$ we denote the space of functions in $C^\infty(\mathbb{R}^n_+)$ with compact support in $\mathbb{R}^n_+$ with the usual topology, and by $\mathcal{D}'(\mathbb{R}^n_+)$ the space of classical distributions in $\mathbb{R}^n_+$.

Let $\lambda \in \mathbb{R}^n$. We consider the Zemanian space S_λ of the functions $\phi \in C^\infty(\mathbb{R}^n_+)$ such that

$$\gamma^\lambda_{m,k}(\phi) = \sup_{x\in\mathbb{R}^n_+} |x^m T^k\{x^{-\lambda}\phi(x)\}| < \infty, \; m, k \in \mathbb{N}^n_0 \tag{2.13}$$

endowed with the topology generated by the family of seminorms $\{\gamma^\lambda_{m,k}\}$ and the operators T^k are given by (2.5). S_λ is a Frechet space (see [13]). The dual space of S_λ is denoted by S'_λ.

Lemma 2.2. *The following inclusions hold*

$$S_\lambda \subset L^1(sr) \cap L^\infty(r) \subset L^p(sr^p), \quad 1 \le p < \infty, \tag{2.14}$$

where s and r are given by (2.11) *and* (2.12)*, respectively.*

Proof. Let $\phi \in S_\lambda$,

$$\|\phi\|_{L^\infty(r)} = \sup_{x\in\mathbb{R}^n_+} |x^{-\lambda}\phi(x)| = \gamma^\lambda_{0,0}(\phi). \tag{2.15}$$

Then $\phi \in L^\infty(r)$. Let $m \in \mathbb{N}$ such that $m > 2\lambda_i + 1$ for $i = 1, \dots, n$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} |\phi(x)| s(x) r(x)\, dx \\
&= \int_{(0,1]^n} |x^{-\lambda}\phi(x)| \,\frac{x^{2\lambda}}{C_\lambda} dx + \int_{\mathbb{R}^n_+-(0,1]^n} x^m |x^{-\lambda}\phi(x)| \,\frac{x^{2\lambda}}{C_\lambda} dx \\
&\le \gamma^\lambda_{0,0}(\phi)\, C_\lambda^{-1} \int_{(0,1]^n} x^{2\lambda} dx + \gamma^\lambda_{m,0}(\phi)\, C_\lambda^{-1} \int_{\mathbb{R}^n_+-(0,1]^n} x^{2\lambda} dx < \infty.
\end{aligned}$$

Thus

$$\|\phi\|_{L^1(sr)} \le C\{\gamma^\lambda_{0,0}(\phi) + \gamma^\lambda_{m,0}(\phi)\}, \quad \phi \in S_\lambda. \tag{2.16}$$

Now let us see that $L^1(sr) \cap L^\infty(r) \subset L^p(sr^p)$. Let $\phi \in L^1(sr) \cap L^\infty(r)$.

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |\phi(x)|^p s(x) r^p(x)\, dx &= \int_{\mathbb{R}^n_+} |\phi(x)|^{p-1} r(x)^{p-1}\, |\phi(x)| s(x) r(x)\, dx \\
&= \int_{\mathbb{R}^n_+} |r(x)\phi(x)|^{p-1}\, |\phi(x)| s(x) r(x)\, dx \\
&\le \|\phi\|^{p-1}_{L^\infty(r)} \|\phi\|_{L^1(sr)},
\end{aligned}$$

from where

$$\|\phi\|_{L^p(sr^p)} \le \|\phi\|^{\frac{p-1}{p}}_{L^\infty(r)} \|\phi\|^{\frac{1}{p}}_{L^1(sr)}. \tag{2.17}$$

From (2.15) and (2.16) we can consider that there exist constants C_1 and C_2 such that

$$\|\phi\|_{L^\infty(r)} \leq C_1\{\gamma_{0,0}^\lambda(\phi) + \gamma_{m,0}^\lambda(\phi)\}, \quad \phi \in S_\lambda, \tag{2.18}$$

$$\|\phi\|_{L^1(sr)} \leq C_2\{\gamma_{0,0}^\lambda(\phi) + \gamma_{m,0}^\lambda(\phi)\}, \quad \phi \in S_\lambda. \tag{2.19}$$

Then from (2.17), (2.18) and (2.19) we can consider a constant C_3 such that

$$\|\phi\|_{L^p(sr^p)} \leq C_3\{\gamma_{0,0}^\lambda(\phi) + \gamma_{m,0}^\lambda(\phi)\}, \quad \phi \in S_\lambda. \tag{2.20}$$

□

Remark 2.3. If $\phi \in L^1(sr)$, then the Hankel transform $h_\lambda\phi$ is well defined because the kernel $(x_iy_i)^{1/2-\lambda_i}J_{\lambda-1/2}(x_iy_i)$ is bounded for $\lambda_i > 0$, $i = 1, \dots, n$ (see [16, (1), p. 49]),

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} |\phi(x)| \prod_{i=1}^n \{(x_iy_i)^{\lambda_i} |(x_iy_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(x_iy_i)|\}\, dx \\
&\leq y^\lambda M^n \int_{\mathbb{R}_+^n} |\phi(x)|\, x^\lambda\, dx = Cy^\lambda \|\phi\|_{L^1(sr)} < \infty.
\end{aligned}$$

By Lemma 2.2, $h_\lambda\phi$ is well defined for all $\phi \in S_\lambda$ and is an automorphism of S_λ (see [13] for the n-dimensional case).

We call a function $f \in L^1_{loc}(\mathbb{R}_+^n)$ a regular element of S'_λ if the application $T_f \in S'_\lambda$, where $T_f(\phi) = \int_{\mathbb{R}_+^n} f(x)\phi(x)\, dx$ with $\phi \in S_\lambda$.

Lemma 2.4. *Let $1 \leq p < \infty$. A function in $L^p(sr^p)$ or in $L^\infty(r)$ is a regular element of S'_λ. In particular, the functions in S_λ can be considered as regular elements of S'_λ.*

Proof. Let $f \in L^\infty(r)$ and $\phi \in S_\lambda$. Since $S_\lambda \subset L^1(sr)$, $\phi \in L^1(r^{-1})$ and $(T_f, \phi) = \int_{\mathbb{R}_+^n} f(x)\phi(x)\, dx$ is well defined. So, by (2.16)

$$\begin{aligned}
|(T_f, \phi)| &\leq \|f\|_{L^\infty(r)} \|\phi\|_{L^1(r^{-1})} = C_\lambda \|f\|_{L^\infty(r)} \|\phi\|_{L^1(sr)} \\
&\leq C\, C_\lambda \|f\|_{L^\infty(r)} \{\gamma_{0,0}^\lambda(\phi) + \gamma_{m,0}^\lambda(\phi)\}.
\end{aligned}$$

Consequently, f is a regular element of S'_λ.

Now, let $f \in L^p(sr^p)$ with $1 \leq p < \infty$ and $\phi \in S_\lambda$, then

$$\begin{aligned}
|(T_f, \phi)| &\leq \int_{\mathbb{R}_+^n} |f(x)\phi(x)|\, dx \\
&= \int_{\mathbb{R}_+^n} |r(x)f(x)|\, |s^{-1}(x)\, r^{-1}(x)\, \phi(x)|\, s(x)\, dx \\
&= \int_{\mathbb{R}_+^n} |r(x)f(x)|\, C_\lambda |r(x)\phi(x)|\, s(x)\, dx.
\end{aligned} \tag{2.21}$$

Since $r|f| \in L^p(s)$ and $r|\phi| \in L^q(s)$, being q such that $\frac{1}{p} + \frac{1}{q} = 1$, then due to Hölder's inequality and (2.20) we obtain that

$$|(T_f, \phi)| \leq C_\lambda \|f\|_{L^p(sr^p)} \|\phi\|_{L^q(sr^q)} \leq C\, C_\lambda \|f\|_{L^p(sr^p)} \{\gamma^\lambda_{0,0}(\phi) + \gamma^\lambda_{m,0}(\phi)\}$$

with $m > 2\lambda_i + 1$ for $i = 1, \ldots, n$. Therefore f is a regular element of S'_λ. □

Remark 2.5. In particular if $p = 2$, $L^p(sr^p) = L^2(\mathbb{R}^n_+)$ and from the previous lemma we have that the functions in $L^2(\mathbb{R}^n_+)$ can be considered as regular elements of S'_λ.

Given f, g defined on $\mathbb{R}^n_+$, the Hankel convolution associated to the transformation h_λ is defined formally by

$$(f\sharp g)(x) = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \mathbf{D}_\lambda(x, y, z) f(y) g(z)\, dy\, dz, \tag{2.22}$$

where for every $x, y, z \in \mathbb{R}^n_+$,

$$\mathbf{D}_\lambda(x, y, z) = \prod_{i=1}^n D_{\lambda_i}(x_i, y_i, z_i), \tag{2.23}$$

and D_ν is the Delsarte kernel defined in [5], given by

$$D_\nu(u, v, w) = \frac{2^{\nu-3/2}\, (uvw)^{-\nu+1}}{\Gamma(\nu)\sqrt{\pi}} A(u, v, w)^{2\nu-2} \tag{2.24}$$

and $A(u, v, w)$ is the area of the triangle with sides $u, v, w \in \mathbb{R}_+$ and $\nu \in \mathbb{R}$, $\nu > 0$.

Note that $|u - v| < w < u + v$ is the condition for such a triangle to exist and in this case

$$\begin{aligned} &A(u, v, w) \\ &= \begin{cases} \frac{1}{4}\sqrt{[(u+v)^2 - w^2][w^2 - (u-v)^2]} & |u - v| < w < u + v, \\ 0 & 0 < w < |u - v| \text{ or } w > u + v. \end{cases} \end{aligned} \tag{2.25}$$

Remark 2.6. If u, v and w are the sides of a triangle and θ is the angle opposite the side w, then

$$A(u, v, w) = \frac{uv \sin\theta}{2}.$$

Proposition 2.7.

(i) $\boldsymbol{D}_\lambda(x, y, z) \geq 0$, $x, y, z \in \mathbb{R}^n_+$.

(ii) $\int_{\mathbb{R}^n_+} \boldsymbol{D}_\lambda(x, y, z) \prod_{i=1}^n \{\sqrt{z_i t_i} J_{\lambda_i - 1/2}(z_i t_i)\}\, dz$
$= t^{-\lambda} \prod_{i=1}^n \{\sqrt{x_i t_i} J_{\lambda_i - 1/2}(x_i t_i)\} \prod_{i=1}^n \{\sqrt{y_i t_i} J_{\lambda_i - 1/2}(y_i t_i)\}$.

(iii) $\int_{\mathbb{R}^n_+} z^\lambda \boldsymbol{D}_\lambda(x, y, z)\, dz = C_\lambda^{-1} x^\lambda y^\lambda$.

Proof. The proof follows from the one dimensional case (see [5, 12]). □

The proof of the following results are analogous to the ones proved in [12] for the one dimensional case and they will be omitted.

Lemma 2.8. *Let $f \in L^1(sr)$.*

(i) *If $g \in L^\infty(r)$, then the convolution $f\sharp g(x)$ exists for every $x \in \mathbb{R}^n_+$, $f\sharp g(x) \in L^\infty(r)$ and*

$$\|f\sharp g\|_{L^\infty(r)} \leq \|f\|_{L^1(sr)}\|g\|_{L^\infty(r)}. \tag{2.26}$$

(ii) *If $g \in L^p(sr^p)$, $1 \leq p < \infty$, then the convolution $f\sharp g(x)$ exists for almost every $x \in \mathbb{R}^n_+$, $f\sharp g(x) \in L^p(sr^p)$ and*

$$\|f\sharp g\|_{L^p(sr^p)} \leq \|f\|_{L^1(sr)}\|g\|_{L^p(sr^p)}. \tag{2.27}$$

Lemma 2.9. *Let $f, g \in L^1(sr)$. Then*

$$h_\lambda(f\sharp g) = rh_\lambda(f)h_\lambda(g). \tag{2.28}$$

Lemma 2.10. *Let $f \in L^1(sr)$. Then the Hankel transform $h_\lambda f \in L^\infty(r)$ and*

$$\|h_\lambda f\|_{L^\infty(r)} \leq \|f\|_{L^1(sr)}.$$

Remark 2.11. Given $f \in L^1(\mathbb{R}^n_+)$ we have that $h_\lambda f$ is continuous and is in $L^\infty(\mathbb{R}^n_+)$ and

$$\|h_\lambda f\|_\infty \leq C\|f\|_1.$$

Proposition 2.12. *$h_\lambda(L^1(\mathbb{R}^n_+)) \subset C_0(\mathbb{R}^n_+)$.*

Proof. First, we observe that

$$L^1(sr) \cap L^\infty(r) \subset L^1(\mathbb{R}^n_+). \tag{2.29}$$

Let the cube $Q = [0,1]^n$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |f(x)|\,dx &= \int_{\mathbb{R}^n_+} |f(x)|\,r(x)\,r^{-1}(x)\,dx \\
&= \int_{Q\cap\mathbb{R}^n_+} |f(x)|\,r(x)\,r^{-1}(x)\,dx + \int_{Q^c\cap\mathbb{R}^n_+} |f(x)|\,r(x)\,r^{-1}(x)\,dx \\
&\leq \|f\|_{L^\infty(r)} \int_{Q\cap\mathbb{R}^n_+} r^{-1}(x)\,dx + \int_{Q^c\cap\mathbb{R}^n_+} |f(x)|\,r^{-1}(x)\,dx \\
&\leq C\|f\|_{L^\infty(r)} + C_\lambda\|f\|_{L^1(sr)},
\end{aligned}$$

because $r(x) < 1$ for $\|x\| > 1$, $\lambda_i > 0$, $i = 1, \ldots, n$ and $r(x)s(x) = C_\lambda^{-1} r^{-1}(x)$.

By (2.14) and (2.29) we deduce that $S_\lambda \subset L^1(\mathbb{R}^n_+)$. Since $\mathcal{D}(\mathbb{R}^n_+) \subset S_\lambda$, S_λ is dense in $L^1(\mathbb{R}^n_+)$. Given $f \in L^1(\mathbb{R}^n_+)$ and $\{\phi_m\} \in S_\lambda$ such that $\phi_m \to f$ in $L^1(\mathbb{R}^n_+)$, then by Remark 2.11 $h_\lambda(\phi_m) \to h_\lambda(f)$ uniformly. Since $h_\lambda(\phi_m) \in C_0(\mathbb{R}^n_+)$, we have $h_\lambda(f) \in C_0(\mathbb{R}^n_+)$. ☐

We are going to consider Bessel operators in $\mathbb{R}^n_+$ given by (1.1) and (1.2) which are related through

$$\Delta_\lambda = x^\lambda B_\lambda x^{-\lambda}, \tag{2.30}$$

see Remark C.1 for a proof.

Bessel operator (1.1) and Hankel transform (2.9) were studied in the distributional setting over the Zemanian spaces S_λ and S'_λ (see [11], [13] and [17]).

Since Δ_λ is a continuous operator in S_λ and selfadjoint, so the generalized Bessel operator Δ_λ can be extended to S'_λ by transposition

$$(\Delta_\lambda f, \phi) = (f, \Delta_\lambda \phi), \quad f \in S'_\lambda, \quad \phi \in S_\lambda.$$

Analogously, generalized Hankel transform $h_\lambda f$ can be extended to S'_λ by

$$(h_\lambda f, \phi) = (f, h_\lambda \phi), \quad f \in S'_\lambda, \quad \phi \in S_\lambda$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$, $i = 1, \ldots, n$. Then h_λ is an automorphism over S_λ and S'_λ.

There exist different proofs for the inversion theorem of the Hankel transform for the 1-dimensional case. In this work, we present a proof for the inversion theorem for the n-dimensional case, in the same way as the classic versions of the results known for the inversion of the Fourier transform in Lebesgue spaces.

Theorem 2.13. *Let $f \in L^1(\mathbb{R}^n_+, x^\lambda)$ and $h_\lambda f \in L^1(\mathbb{R}^n_+, x^\lambda)$, where x^λ is given by* (2.1). *Then $f(x)$ may be redefined on a set of measure zero so that it is continuous on $\mathbb{R}^n_+$ and*

$$f(x) = h_\lambda(h_\lambda f)(x) \tag{2.31}$$

for almost every $x \in \mathbb{R}^n_+$.

Proof. For the proof of this result, we refer the reader to the Appendix. □

Remark 2.14. From Theorem 2.13 we deduce immediately the validity of equality (2.31) in S_λ and S'_λ.

For the proof of the following results, we refer the reader to [13].

Lemma 2.15. *Let $\phi \in S_\lambda$. Then*

(i) $h_\lambda \Delta_\lambda \phi = \|y\|^2 h_\lambda \phi$.
(ii) $\Delta_\lambda h_\lambda \phi = h_\lambda(\|x\|^2 \phi)$.

Lemma 2.16. *If $u \in S'_\lambda$. Then*

(i) $h_\lambda \Delta_\lambda u = \|x\|^2 h_\lambda u$.
(ii) $\Delta_\lambda h_\lambda u = h_\lambda(\|y\|^2 u)$.

Remark 2.17. According to Lemma 3.2 in [11] the functions $(t + \|x\|^2)$ for $t \geq 0$ and $(t + \|x\|^2)^{-1}$ for $t > 0$ belong to the space of multipliers of S_λ and S'_λ.

So, the next result holds.

Lemma 2.18. *The following equalities are valid in S_λ and S'_λ for $m \in \mathbb{N}$.*

If $z \in \mathbb{C}$,
(i) $(z + \Delta_\lambda)^m h_\lambda = h_\lambda(z + \|y\|^2)^m$.
If $t \in \mathbb{R}$, $t > 0$,
(ii) $h_\lambda(t + \Delta_\lambda)^{-m} = (t + \|y\|^2)^{-m} h_\lambda$.

(iii) $h_\lambda(\Delta_\lambda(t+\Delta_\lambda)^{-1})^m = \|y\|^{2m}(t+\|y\|^2)^{-m}h_\lambda$.

Proof. The proof of this result is omitted since it follows by induction. □

3. Non-negativity and fractional powers of similar operators

In this section, we include a brief review of non-negative operators in Banach spaces and in locally convex spaces.

Let X be a (real or complex) Banach space. Let A be a closed linear operator $A : D(A) \subset X \to X$ and $\rho(A)$ the resolvent set of A. We say that A is non-negative if $(-\infty, 0) \subset \rho(A)$ and

$$\sup_{t>0}\{\|t(t+A)^{-1}\|\} < \infty.$$

Now, let X be a locally convex space with a Hausdorff topology generated by a directed family of seminorms $\{\| \ \|_\alpha\}_{\alpha\in\Lambda}$. A family of linear operators $\{A_t\}_{t\in\Gamma}$, $A_t : D(A_t) \subset X \to X$, is equicontinuous if for each $\alpha \in \Lambda$ there are $\beta = \beta(\alpha) \in \Lambda$ and a constant $C = C_\alpha \geq 0$ such that for all $t \in \Gamma$

$$\|A_t\phi\|_\alpha \leq C\|\phi\|_\beta, \quad \phi \in X.$$

Under the above conditions, we say that a closed linear operator $A : D(A) \subset X \to X$ is non-negative if $(-\infty, 0) \subset \rho(A)$ and the family of operators

$$\{t(t+A)^{-1}\}_{t>0}$$

are equicontinuous.

Now, we will briefly describe the theory of fractional powers of operators. According to [9, Proposition 3.1.3], we can define the Balakrishnan operator J^α in the following way.

Let A be a non-negative operator in a Banach space or a locally convex and sequentially complete space. Let $\alpha \in \mathbb{C}$ and $0 < \operatorname{Re}\alpha < n$, $n \in \mathbb{N}$. If $\phi \in D(A^n)$ and $m \geq n$ is a positive integer, then

$$J^\alpha\phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1}\big[A(t+A)^{-1}\big]^m \phi\, dt. \tag{3.1}$$

If A is bounded, J_A^α can be considered as the fractional power of A. In other cases, we can consider the following representation for the fractional power stated in [9, Theorem 5.2.1].

Theorem 3.1. *Let A be a non-negative operator, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $z \in \rho(-A)$ and $n \in \mathbb{N}$. Then*

$$A^\alpha = (z+A)^n J_A^\alpha (z+A)^{-n}. \tag{3.2}$$

(*If $n > \operatorname{Re}\alpha$, the operator $\overline{J_A^\alpha}$ can be replaced by J_A^α in the preceding formula.*)

Similar operators have been described in the introduction. Let A and B be similar operators and T be the isomorphism that verifies (1.4). Then

$$(zId + B)^{-1} = T(zId + A)^{-1}T^{-1}$$

for z a complex number, from which we deduce immediately that A is a non-negative operator if and only if so is B.

We have the following result which holds in Banach spaces and in sequentially complete locally convex spaces.

Proposition 3.2. *Let A and B be similar non-negative operators. If $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, then*

$$J_B^\alpha = T J_A^\alpha T^{-1}, \tag{3.3}$$

and

$$B^\alpha = T A^\alpha T^{-1}, \tag{3.4}$$

where T is the isometric isomorphism that verifies $B = TAT^{-1}$.

4. Fractional powers of $\boldsymbol{\Delta_\lambda}$ in Lebesgue spaces

Let s and r be as in Section 2 and let $1 \le p < \infty$. We will denote by $\Delta_{\lambda,p}$ the part of Δ_λ in $L^p(sr^p)$, that is to say, the operator Δ_λ with domain

$$D(\Delta_{\lambda,p}) = \{f \in L^p(sr^p) : \Delta_\lambda f \in L^p(sr^p)\}$$

and given by $\Delta_{\lambda,p} f = \Delta_\lambda f$.

Analogously, with $\Delta_{\lambda,\infty}$ we will denote the part of Δ_λ in $L^\infty(r)$, $B_{\lambda,p}$ and $B_{\lambda,\infty}$ the part of B_λ in $L^p(s)$ and $L^\infty(\mathbb{R}_+^n)$, respectively.

Let L_r be the isometric isomorphism

$$L_r : L^p(sr^p) \to L^p(s) \quad \text{with} \quad 1 \le p < \infty$$

(or $L_r : L^\infty(r) \to L^\infty(\mathbb{R}_+^n)$) given by

$$L_r(f) = rf.$$

Then

$$\Delta_{\lambda,p} = L_r^{-1} B_{\lambda,p} L_r.$$

Consequently it is enough to study the operator Δ_λ in the spaces $L^p(sr^p)$ (or $L^\infty(r)$). In order to study the non-negativity of operators $\Delta_{\lambda,p}$ and $\Delta_{\lambda,\infty}$ we consider the following function given by

$$\mathcal{N}_\nu(w) = \int_0^\infty e^{-u-\frac{w^2}{4u}} \frac{du}{u^{\nu+1}} \tag{4.1}$$

which is defined for all $\nu \in \mathbb{R}$ and $w \in \mathbb{R}_+$.

Let $u \in \mathbb{R}_+$. If $\lambda = (\lambda_1, \dots, \lambda_n)$, then $u^{\lambda+1/2}$ means

$$u^{\lambda+1/2} = u^{\lambda_1+1/2} \cdots u^{\lambda_n+1/2} = u^{\lambda_1+\cdots+\lambda_n+\frac{n}{2}},$$

from which

$$\begin{aligned} \mathcal{N}_{\lambda_1+\cdots+\lambda_n+\frac{n}{2}-1}(\|x\|) &= \int_0^\infty e^{-u-\frac{\|x\|^2}{4u}} \frac{dt}{u^{\lambda_1+\cdots+\lambda_n+\frac{n}{2}-1+1}} \\ &= \int_0^\infty e^{-u-\frac{\|x\|^2}{4u}} \frac{dt}{u^{\lambda+1/2}}. \end{aligned} \tag{4.2}$$

Given $t > 0$, let us consider the function

$$N_t(x) = 2^{-\lambda-1/2} x^\lambda t^\lambda t^{\frac{n}{2}-1} \mathcal{N}_{\lambda_1+\cdots+\lambda_n+\frac{n}{2}-1}(\|\sqrt{t}x\|), \quad x \in \mathbb{R}^n_+. \tag{4.3}$$

Lemma 4.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ and $t > 0$. Then*

(a) $N_t \in L^1(sr)$ *and*

$$\|N_t\|_{L^1(sr)} = \frac{1}{t},$$

(b)

$$h_\lambda N_t(y) = \frac{y^\lambda}{t + \|y\|^2}.$$

Proof.

$$\begin{aligned}
&\|N_t\|_{L^1(sr)} \\
&= \int_{\mathbb{R}^n_+} |N_t(x)| \frac{x^\lambda}{C_\lambda} dx \\
&= \int_{\mathbb{R}^n_+} 2^{-\lambda-1/2} x^\lambda t^\lambda t^{\frac{n}{2}-1} \mathcal{N}_{\lambda_1+\cdots+\lambda_n+\frac{n}{2}-1}(\|\sqrt{t}x\|) \frac{x^\lambda}{C_\lambda} dx \\
&= 2^{-\lambda-1/2} t^\lambda t^{\frac{n}{2}-1} \frac{1}{C_\lambda} \int_{\mathbb{R}^n_+} \left\{ \int_0^\infty e^{-u-\frac{t\|x\|^2}{4u}} \frac{du}{u^{\lambda+1/2}} \right\} x^{2\lambda} dx \\
&= 2^{-\lambda-1/2} t^\lambda t^{\frac{n}{2}-1} \frac{1}{C_\lambda} \int_0^\infty \left\{ \int_{\mathbb{R}^n_+} e^{-\frac{t\|x\|^2}{4u}} x^{2\lambda} dx \right\} e^{-u} \frac{du}{u^{\lambda+1/2}} \\
&= 2^{-\lambda-1/2} t^\lambda t^{\frac{n}{2}-1} \frac{1}{C_\lambda} \int_0^\infty \prod_{i=1}^n \left\{ \int_0^\infty e^{-\frac{tx_i^2}{4u}} x_i^{2\lambda_i} dx_i \right\} e^{-u} \frac{du}{u^{\lambda+1/2}} \\
&= 2^{-\lambda-1/2} t^\lambda t^{\frac{n}{2}-1} \frac{1}{C_\lambda} \int_0^\infty \prod_{i=1}^n \left\{ 2^{\lambda_i-1/2} \Gamma(\lambda_i + 1/2) \left(\frac{2u}{t}\right)^{\lambda_i+1/2} \right\} e^{-u} \frac{du}{u^{\lambda+1/2}} \\
&= 2^{-\lambda-1/2} t^\lambda t^{\frac{n}{2}-1} \frac{1}{C_\lambda} 2^{\lambda+1/2} t^{-\lambda} t^{-\frac{n}{2}} C_\lambda \int_0^\infty u^{\lambda+1/2} e^{-u} \frac{du}{u^{\lambda+1/2}} \\
&= \frac{1}{t},
\end{aligned}$$

where we have used the formula (A.6), and thus (a) holds. To see (b),

$$\begin{aligned}
&h_\lambda N_t(y) \\
&= \int_{\mathbb{R}^n_+} N_t(x) \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\lambda_i-1/2}(x_i y_i)\} dx \\
&= \int_{\mathbb{R}^n_+} 2^{-\lambda-1/2} x^\lambda t^\lambda t^{\frac{n}{2}-1} \left\{ \int_0^\infty e^{-u-\frac{t\|x\|^2}{4u}} \frac{du}{u^{\lambda+1/2}} \right\} \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\lambda_i-1/2}(x_i y_i)\} dx
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-\lambda-1/2}y^{1/2}t^{\lambda}t^{\frac{n}{2}-1}\int_0^\infty \left\{\int_{\mathbb{R}^n_+} x^{\lambda+1/2}e^{-\frac{t\|x\|^2}{4u}}\prod_{i=1}^n\{J_{\lambda_i-1/2}(x_iy_i)\}\,dx\right\}e^{-u}\frac{du}{u^{\lambda+1/2}}\\
&= 2^{-\lambda-1/2}y^{1/2}t^{\lambda}t^{\frac{n}{2}-1}\int_0^\infty \prod_{i=1}^n\left\{\int_0^\infty x^{\lambda_i+1/2}e^{-\frac{tx_i^2}{4u}}J_{\lambda_i-1/2}(x_iy_i)\,dx_i\right\}e^{-u}\frac{du}{u^{\lambda+1/2}}\\
&= 2^{-\lambda-1/2}y^{1/2}t^{\lambda}t^{\frac{n}{2}-1}\int_0^\infty \prod_{i=1}^n\left\{\left(\frac{t}{2u}\right)^{-\lambda_i-1/2}y_i^{\lambda_i-1/2}e^{-\frac{uy_i^2}{t}}\right\}e^{-u}\frac{du}{u^{\lambda+1/2}}\\
&= 2^{-\lambda-1/2}y^{1/2}t^{\lambda}t^{\frac{n}{2}-1}2^{\lambda+1/2}t^{-\lambda-1/2}y^{\lambda-1/2}\int_0^\infty u^{\lambda+1/2}e^{-\frac{u\|y\|^2}{t}}e^{-u}\frac{du}{u^{\lambda+1/2}}\\
&= y^{\lambda}t^{\lambda}t^{\frac{n}{2}-1}t^{-\lambda}\lambda^{-\frac{n}{2}}\int_0^\infty e^{-u(1+\frac{\|y\|^2}{t})}\,du\\
&= y^{\lambda}t^{-1}\frac{t}{t+\|y\|^2}\int_0^\infty e^{-s}\,ds\\
&= \frac{y^{\lambda}}{t+\|y\|^2},
\end{aligned}
$$

where we have used (A.4). □

Lemma 4.2. *Let $1 \le p < \infty$. If $f \in L^p(sr^p)$ or $f \in L^\infty(r)$, then the following equality holds on S'_λ*

$$h_\lambda(N_t\sharp f) = \frac{1}{t+\|y\|^2}h_\lambda f. \tag{4.4}$$

Proof. Suppose that $f \in L^p(sr^p)$ and $\psi \in S_\lambda$, we claim that

$$\int_{\mathbb{R}^n_+}(N_t\sharp f)(x)\psi(x)\,dx = \int_{\mathbb{R}^n_+} f(z)(N_t\sharp\psi)(z)\,dz, \tag{4.5}$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} f(z)(N_t\sharp\psi)(z)\,dz \\
&\quad= \int_{\mathbb{R}^n_+} f(z)\left\{\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} N_t(y)\psi(x)\,\mathbf{D}_\lambda(x,y,z)\,dy\,dx\right\}dz.
\end{aligned} \tag{4.6}
$$

Let us see that $\int_{\mathbb{R}^n_+}|f(z)|\left\{\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|N_t(y)|\,|\psi(x)|\,\mathbf{D}_\lambda(x,y,z)\,dy\,dx\right\}dz$ is finite.

Let

$$G(z) = \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|N_t(y)|\,|\psi(x)|\,\mathbf{D}_\lambda(x,y,z)\,dy\,dx$$

and let q such that $\frac{1}{p}+\frac{1}{q}=1$. The function G is the convolution of $|N_t|$ and $|\psi|$. From Lemma 2.8, since $|N_t| \in L^1(sr)$ and $|\psi| \in L^q(sr^q)$ we have that for

$G \in L^q(sr^q)$,

$$\begin{aligned}\int_{\mathbb{R}^n_+} |f(z)|\,|G(z)|\,dz &= \int_{\mathbb{R}^n_+} (r|f(z)|)\,(r^{-1}s^{-1}|G(z)|)\,s\,dz\\ &= \int_{\mathbb{R}^n_+} (r|f(z)|)\,(C_\lambda r|G(z)|)\,s\,dz\\ &= C_\lambda \int_{\mathbb{R}^n_+} |rf(z)|\,|rG(z)|\,s\,dz\\ &\le C_\lambda \|rf\|_{L^p(s)}\|rG\|_{L^q(s)}\\ &= C_\lambda \|f\|_{L^p(sr^p)}\|G\|_{L^q(sr^q)}.\end{aligned}$$

Then it is possible to change the order of integration in (4.6).

$$\begin{aligned}\int_{\mathbb{R}^n_+} f(z)\,(N_t\sharp\psi)(z)\,dz &= \int_{\mathbb{R}^n_+}\left\{\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} N_t(y)\,f(z)\,\mathbf{D}_\lambda(x,y,z)\,dy\,dz\right\}\,\psi(x)\,dx\\ &= \int_{\mathbb{R}^n_+} (N_t\sharp f)(x)\,\psi(x)\,dx.\end{aligned}$$

So, we have proved (4.6).

Now let $f \in L^\infty(r)$. To see that (4.5) holds, it will be enough to see that

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\left\{\int_{\mathbb{R}^n_+}\left\{\int_{\mathbb{R}^n_+} |f(z)|\,|N_t(y)|\,|\psi(x)|\,\mathbf{D}_\lambda(x,y,z)\,dz\right\}\,dy\right\}\,dx\\ &\le \|rf\|_{L^\infty(\mathbb{R}^n_+)}\int_{\mathbb{R}^n_+}\left\{\int_{\mathbb{R}^n_+}|N_t(y)|\,|\psi(x)|\,\left\{\int_{\mathbb{R}^n_+} z^\lambda\mathbf{D}_\lambda(x,y,z)\,dz\right\}\,dy\right\}\,dx\\ &= C_\lambda\,\|f\|_{L^\infty(r)}\,\|N_t\|_{L^1(sr)}\,\|\psi\|_{L^1(sr)} < \infty.\end{aligned}$$

Let $\phi \in S_\lambda$ and $f \in L^p(sr)$ or $f \in L^\infty(r)$. From (4.5) we have that

$$\begin{aligned}(h_\lambda(N_t\sharp f),\phi) = ((N_t\sharp f),h_\lambda\phi) &= \int_{\mathbb{R}^n_+}(N_t\sharp f)(x)\,(h_\lambda\phi)(x)\,dx\\ &= \int_{\mathbb{R}^n_+} f(z)\,(N_t\sharp h_\lambda\phi)(z)\,dz.\end{aligned}\tag{4.7}$$

From Lemma 2.9, Theorem 2.13 and item (b) of Lemma 4.1 we obtain that

$$h_\lambda(N_t\sharp h_\lambda\phi)(y) = r(h_\lambda N_t)(h_\lambda(h_\lambda\phi))(y) = y^{-\lambda}\frac{y^\lambda}{t+\|y\|^2}\phi(y) = \frac{\phi(y)}{t+\|y\|^2}.$$

Then

$$N_t\sharp h_\lambda\phi = h_\lambda\left(\frac{\phi}{t+\|y\|^2}\right).\tag{4.8}$$

Finally, from (4.7) and (4.8) we obtain that for $\phi \in S_\lambda$ that

$$\begin{aligned}(h_\lambda(N_t\sharp f),\phi) &= \int_{\mathbb{R}^n_+} f(x)(N_t\sharp h_\lambda\phi)(x)\,dx\\ &= \int_{\mathbb{R}^n_+} f(x)\,h_\lambda\left(\frac{\phi}{t+\|y\|^2}\right)(x)\,dx\\ &= \int_{\mathbb{R}^n_+} \frac{1}{t+\|x\|^2} h_\lambda f(x)\,\phi(x)\,dx\\ &= \left(\frac{h_\lambda f}{t+\|x\|^2},\phi\right).\end{aligned}$$

□

Theorem 4.3. *Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ and $\lambda_i > 0$. Then $\Delta_{\lambda,p}$ and $\Delta_{\lambda,\infty}$ are closed and non-negative operators.*

Proof. Since convergence in $L^\infty(r)$ and $L^p(sr^p)$ implies convergence in $\mathcal{D}'(\mathbb{R}^n_+)$, $\Delta_{\lambda,\infty}$ and $\Delta_{\lambda,p}$ are closed.

Now let $t>0$ and $f \in D(\Delta_{\lambda,\infty})$ such that $(t+\Delta_{\lambda,\infty})f = 0$. So,

$$h_\lambda(t+\Delta_{\lambda,\infty})f = 0$$

in S'_λ. By Lemma 2.16 we obtain that

$$(t+\|y\|^2)h_\lambda f = 0$$

in S'_λ and hence by Remark 2.17

$$h_\lambda f = (t+\|y\|^2)^{-1}(t+\|y\|^2)h_\lambda f = 0.$$

Then, $f=0$ as element of S'_λ and we conclude that $f=0$ a.e. in $x\in\mathbb{R}^n_+$ and $t+\Delta_{\lambda,\infty}$ is injective.

Let $f\in L^\infty(r)$ and $g = N_t\sharp f$. Then, by Lemma 2.8 $g\in L^\infty(r)$ and

$$h_\lambda((t+\Delta_{\lambda,\infty})g) = (t+\|y\|^2)h_\lambda g = (t+\|y\|^2)h_\lambda(N_t\sharp f) = h_\lambda f.$$

By injectivity of the Hankel transform in S'_λ we obtain that

$$(t+\Delta_{\lambda,\infty})g = f,$$

so, $t+\Delta_{\lambda,\infty}$ is onto. Also

$$\begin{aligned}\|(t+\Delta_{\lambda,\infty})^{-1}f\|_{L^\infty(r)} &= \|g\|_{L^\infty(r)} = \|N_t\sharp f\|_{L^\infty(r)}\\ &\le \|N_t\|_{L^1(sr)}\|f\|_{L^\infty(r)}\\ &= \frac{1}{t}\|f\|_{L^\infty(r)},\end{aligned}$$

hence

$$\|t(t+\Delta_{\lambda,\infty})^{-1}f\|_{L^\infty(r)} \le \|f\|_{L^\infty(r)}$$

and $\Delta_{\lambda,\infty}$ is non-negative.

The proof of the non-negativity of $\Delta_{\lambda,p}$ is similar. □

Since we have proved that both $\Delta_{\lambda,p}$ and $\Delta_{\lambda,\infty}$ are non-negative we can consider the fractional powers of them. If $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ and $n > \operatorname{Re}\alpha$, then the fractional power of $\Delta_{\lambda,\infty}$ can be defined from (3.2) by:

$$(\Delta_{\lambda,\infty})^{\alpha} = (\Delta_{\lambda,\infty}+1)^n \mathcal{J}_{\infty}^{\alpha}(\Delta_{\lambda,\infty}+1)^{-n},$$

where $\mathcal{J}_{\infty}^{\alpha}$ is the Balakrishnan operator associated to $\Delta_{\lambda,\infty}$ given by:

$$\mathcal{J}_{\infty}^{\alpha}\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^{\infty} t^{\alpha-1}[\Delta_{\lambda,\infty}(t+\Delta_{\lambda,\infty})^{-1}]^n\phi\, dt$$

for $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < n$ and $\phi \in D[(\Delta_{\lambda,\infty})^n]$.

Analogously the definition of fractional powers of $\Delta_{\lambda,p}$ is defined.

5. Non-negativity of Bessel operator $\boldsymbol{\Delta_\lambda}$ in the space $\boldsymbol{\mathcal{B}}$

Remark 5.1. The operator Δ_λ is not non-negative in S_λ.

If Δ_λ were non-negative in S_λ, since Δ_λ is continuous in S_λ, given $\alpha \in \mathbb{C}$, $0 < \alpha < 1$ and according to (3.1) and (A.8), we have that fractional power Δ_λ^α would be given by

$$\Delta_\lambda^\alpha\phi = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} t^{\alpha-1}\Delta_\lambda(t+\Delta_\lambda)^{-1}\phi\, dt \tag{5.1}$$

and $D(\Delta_\lambda^\alpha) = D(\Delta_\lambda) = S_\lambda$. Applying the Hankel transform in (5.1) we obtain

$$\begin{aligned} h_\lambda\Delta_\lambda^\alpha\phi &= \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} t^{\alpha-1}h_\lambda[\Delta_\lambda(t+\Delta_\lambda)^{-1}\phi]\, dt \\ &= \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} t^{\alpha-1}\|y\|^2(t+\|y\|^2)^{-1}h_\lambda\phi(y)\, dt \\ &= (\|y\|^2)^\alpha h_\lambda\phi(y), \end{aligned}$$

where we have interchanged the Bochner integral with the Hankel transform, and then we have applied item (iii) of Lemma 2.18 and [9, Remark 3.1.1]. This would imply that $(\|y\|^2)^\alpha h_\lambda\phi(y) \in S_\lambda$ which is not true in general.

Now we consider the Banach space $Y = L^1(sr)\cap L^\infty(r)$, with norm

$$\|f\|_Y = \max\bigl\{\|f\|_{L^1(sr)}, \|f\|_{L^\infty(r)}\bigr\},$$

and the part of the Bessel operator in Y, $\Delta_{\lambda,Y}$, with domain given by

$$D[\Delta_{\lambda,Y}] = \{f \in Y : \Delta_\lambda f \in Y\}.$$

From Theorem 4.3 we have that $\Delta_{\lambda,Y}$ is closed and non-negative.

Let $k \in \mathbb{N}_0$. We will understand $\Delta_{\lambda,Y}^k$ as the iteration of the operator $\Delta_{\lambda,Y}$ k-times.

Proposition 5.2. *If $k > \frac{n}{2}$, then $D[\Delta_{\lambda,Y}^{k+1}] \subset C_0(\mathbb{R}_+^n)$.*

Proof.

$$D[\Delta_{\lambda,Y}^{k+1}] = \{\phi \in D[\Delta_{\lambda,Y}^{k}] : \Delta_{\lambda,Y}^{k}\phi \in D[\Delta_{\lambda,Y}]\}.$$

Let $f \in D[\Delta_{\lambda,Y}^{k+1}]$. Then f and $\Delta_{\lambda,Y}^{k} f$ are in $D[\Delta_{\lambda,Y}]$.

From Lemma 2.2 and (2.29) we have that

$$L^1(sr) \cap L^\infty(r) \subset L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n).$$

Then f and $\Delta_{\lambda,Y}^{k} f$ are in $L^1(\mathbb{R}_+^n)$. From Remark 2.11 we obtain that $h_\lambda f$ and $h_\lambda(\Delta_{\lambda,Y}^{k} f)$ are in $L^\infty(\mathbb{R}_+^n)$, that is to say that there exists $M > 0$ such that

$$|(1 + \|y\|^{2k})\, h_\lambda f| \leq M.$$

Since for $k > \frac{n}{2}$, $(1+\|y\|^{2k})^{-1}$ is integrable in $\mathbb{R}_+^n$, we obtain $h_\lambda f \in L^1(\mathbb{R}_+^n)$. Then, we have proved that if $f \in D[\Delta_{\lambda,Y}^{k+1}]$, f and $h_\lambda f \in L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n)$.

From Remark 2.5 we have that $L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n) \subset S'_\lambda$ and from Remark 2.14 we have

$$h_\lambda(h_\lambda f)(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R}_+^n,$$

considering f as a regular distribution in S'_λ. Since $h_\lambda f \in L^1(\mathbb{R}_+^n)$, by Proposition 2.12 we have that $f = g$ a.e. in $\mathbb{R}_+^n$ with $g \in C_0(\mathbb{R}_+^n)$. □

We now consider the following space:

$$\mathcal{B} = \{f \in Y : \Delta_\lambda^k f \in Y \ \text{ for } k = 0, 1, 2, \dots\} = \bigcap_{k=0}^{\infty} D[\Delta_{\lambda,Y}^{k}], \tag{5.2}$$

with seminorms

$$\rho_m(f) = \max_{0 \leq k \leq m} \big\{ \|\Delta_\lambda^k f\|_Y,\ m = 0, 1, 2, \dots \big\}.$$

Remark 5.3. From Proposition 5.2 is evident that $\mathcal{B} \subset C_0(\mathbb{R}_+^n)$. Moreover, from Lemma 2.2 we obtain that $\mathcal{B} \subset L^p(sr^p)$ for all $1 \leq p < \infty$, and considering that Δ_λ is a continuous operator from S_λ in itself then $S_\lambda \subset \mathcal{B}$ and the topology of S_λ induced by $\mathcal{B}$ is weaker than the usual topology generated by seminorms given by (2.13). In fact, from (2.18) and (2.19) we have that

$$\|\phi\|_Y \leq C\{\gamma_{0,0}^\lambda(\phi) + \gamma_{m,0}^\lambda(\phi)\}, \quad \phi \in S_\lambda \tag{5.3}$$

for $m > 2\lambda_i + 1$, $i = 1, \dots, n$ and by the continuity of Δ_λ in S_λ we deduce that given a seminorm ρ_m, there exist a finite set of seminorms $\{\gamma_{m_i,k_i}^\lambda\}_{i=1}^r$ and constants $c_1, \dots, c_r$ such that

$$\rho_m(\phi) \leq \sum_{i=1}^{r} c_i\, \gamma_{m_i,k_i}^\lambda(\phi), \quad \phi \in S_\lambda.$$

From the density of $\mathcal{D}(\mathbb{R}_+^n)$ in $\mathcal{B}$ we deduce the density of S_λ in $\mathcal{B}$.

We denote with $\Delta_{\lambda,\mathcal{B}}$ the part of Bessel operator Δ_λ in $\mathcal{B}$, so the domain of the operator $\Delta_{\lambda,\mathcal{B}}$ is $\mathcal{B}$ and the following result holds.

Theorem 5.4. *$\mathcal{B}$ is a Frechet space and $\Delta_{\lambda,\mathcal{B}}$ is a continuous and non-negative operator on $\mathcal{B}$.*

Proof. Let $\{\phi_k\}$ be a Cauchy sequence in $\mathcal{B}$. Then the convergence of $\{\phi_k\}$ follows considering the seminorm ρ_0 and the completeness of $L^1(sr)$ and $L^\infty(r)$.

Since $\rho_m(\Delta_\lambda\phi) = \rho_{m+1}(\phi)$, $\Delta_{\lambda,\mathcal{B}}$ is continuous. The non-negativity follows from Proposition 1.4.2 in [9]. $\square$

6. Non-negativity of Bessel operator $\boldsymbol{\Delta_\lambda}$ in the distributional space $\boldsymbol{\mathcal{B}'}$

We will study the non-negativity of Bessel operator in the topological dual space of $\mathcal{B}$ with the strong topology, that is to say, the space $\mathcal{B}'$ endowed with the topology generated by the family of seminorms $\{|\cdot|_B\}$, where the sets B are bounded sets in $\mathcal{B}$, and the seminorms are given by

$$|T|_B = \sup_{\phi\in B} |(T,\phi)|, \quad T \in \mathcal{B}'.$$

Remark 6.1. Since $\mathcal{B}$ is a Frechet space, $\mathcal{B}$ is bornological. Then from [15, Theorem 6.1, p. 148] we obtain that the strong dual $\mathcal{B}'$ is complete. Consequently $\mathcal{B}'$ is sequentially complete.

Remark 6.2. $L^p(sr^p)$ and $L^\infty(r)$ are included in $\mathcal{B}'$ $(1 \le p < \infty)$.

Let $f \in L^p(sr^p)$, $\phi \in \mathcal{B}$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\int_{\mathbb{R}^n_+} f(x)\phi(x)\,dx\right| = \left|\int_{\mathbb{R}^n_+} f(x)\phi(x)s^{-1}(x)r^{-p}(x)s(x)r^p(x)\,dx\right| \le \|f\|_{L^p(sr^p)}\|\phi\, s^{-1}r^{-p}\|_{L^q(sr^p)} \tag{6.1}$$

and

$$\begin{aligned}\|\phi\, s^{-1}r^{-p}\|_{L^q(sr^p)} &= \left\{\int_{\mathbb{R}^n_+} |\phi\, s^{-1}r^{-p}|^q sr^p\right\}^{\frac{1}{q}} \\ &= \left\{\int_{\mathbb{R}^n_+} |\phi|^q\,(C_\lambda r^2 r^{-p})^q sr^p\right\}^{\frac{1}{q}} \\ &= C_\lambda\left\{\int_{\mathbb{R}^n_+} |\phi|^q\, r^{2q-pq+p} s\right\}^{\frac{1}{q}} \\ &= C_\lambda\left\{\int_{\mathbb{R}^n_+} |\phi|^q\, sr^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{6.2}$$

Furthermore, from (2.17)

$$\|\phi\|_{L^q(sr^q)} \le \left\{\|\phi\|_{L^\infty(r)}\right\}^{\frac{q-1}{q}} \left\{\|\phi\|_{L^1(sr)}\right\}^{\frac{1}{q}},$$

from where

$$\|\phi\|_{L^q(sr^q)} \leq \rho_0(\phi). \tag{6.3}$$

Then, from (6.1), (6.2) and (6.3) we obtain that $f \in \mathcal{B}'$.

Now, let B be a bounded set in $\mathcal{B}$. Then

$$|f|_B = \sup_{\phi\in B}\left|\int_{\mathbb{R}^n_+} f\phi\right| \leq C_\lambda \|f\|_{L^p(sr^p)} \sup_{\phi\in B}\|\phi\|_{L^q(sr^q)} \leq C_\lambda \|f\|_{L^p(sr^p)} \sup_{\phi\in B} \rho_0(\phi).$$

Thus, the topology in $L^p(sr^p)$ induced by $\mathcal{B}'$ with the strong topology is weaker than the usual topology.

Remark 6.3. Since S_λ is dense in $\mathcal{B}$ and the topology of S_λ induced by $\mathcal{B}$ is weaker than the generated by seminorms given by (2.13), $\mathcal{B}' \subset S'_\lambda$. Moreover, from the continuity of the Bessel operator in $\mathcal{B}$, we can consider Δ_λ in $\mathcal{B}'$ as the adjoint operator of Δ_λ in $\mathcal{B}$, that is to say

$$(\Delta_\lambda T, \phi) = (T, \Delta_\lambda \phi), \quad T \in \mathcal{B}',\ \phi \in \mathcal{B},$$

and we denote with $\Delta_{\lambda,\mathcal{B}'}$ the part of Bessel operator in $\mathcal{B}'$.

Theorem 6.4. *The operator $\Delta_{\lambda,\mathcal{B}'}$ is continuous and non-negative considering the strong topology in $\mathcal{B}'$.*

Proof. Given a bounded set $B \subset \mathcal{B}$ and $T \in \mathcal{B}'$, then

$$|\Delta_{\lambda,\mathcal{B}'} T|_B = \sup_{\phi\in B} |(\Delta_{\lambda,\mathcal{B}'} T, \phi)| = \sup_{\phi\in B} |(T, \Delta_{\lambda,\mathcal{B}}\phi)| = |T|_E,$$

where the set $E = \{\Delta_{\lambda,\mathcal{B}}\phi : \phi \in B\}$ is also bounded. Then it follows that $\Delta_{\lambda,\mathcal{B}'}$ is continuous.

Let now $t > 0$ and $T \in \mathcal{B}'$. It is not difficult to see that the linear map $G : \psi \to (T, (t + \Delta_{\Lambda,\mathcal{B}})^{-1}\psi)$ is continuous and $(t + \Delta_{\lambda,\mathcal{B}'})G = T$. Therefore $(t + \Delta_{\lambda,\mathcal{B}'})$ is surjective.

To prove the injectivity, let $T \in \mathcal{B}'$ be such that $(t+\Delta_{\lambda,\mathcal{B}'})T = 0$. Then, for all $\phi \in \mathcal{B}$,

$$((t + \Delta_{\lambda,\mathcal{B}'})T, \phi) = (T, (t + \Delta_{\lambda,\mathcal{B}})\phi) = 0,$$

and thus $T = 0$ as $R(t+\Delta_{\lambda,\mathcal{B}}) = \mathcal{B}$, due to the fact that $\Delta_{\lambda,\mathcal{B}}$ is a non-negative operator on $\mathcal{B}$.

To see that $(t + \Delta_{\lambda,\mathcal{B}'})^{-1}$ is continuous, let $T \in \mathcal{B}'$, $B \subset \mathcal{B}$ a bounded set and let us consider the set $F = \{(t + \Delta_{\lambda,\mathcal{B}})^{-1}\phi : \phi \in B\}$, then

$$|(t + \Delta_{\lambda,\mathcal{B}'})^{-1}T|_B = |G|_B = \sup_{\psi\in B} |(G, \psi)| = \sup_{\psi\in B} |(T, (t + \Delta_{\lambda,\mathcal{B}})^{-1}\psi)|_B = |T|_F.$$

For every bounded set $B \subset \mathcal{B}$ and $T \in \mathcal{B}'$, since $\Delta_{\lambda,\mathcal{B}}$ is non-negative, the set $D = \{\eta(\eta + \Delta_{\lambda,\mathcal{B}})^{-1}\phi : \phi \in B,\ \eta > 0\}$ is also bounded and thus, for $t > 0$,

$$\begin{aligned}|t(t + \Delta_{\lambda,\mathcal{B}'})^{-1}T|_B &= \sup_{\phi\in B} |(t(t + \Delta_{\lambda,\mathcal{B}'})^{-1}T, \phi)| \\ &= \sup_{\phi\in B} |(T, t(t + \Delta_{\lambda,\mathcal{B}})^{-1}\phi)|\end{aligned}$$

$$\leq |T|_D.$$

We now conclude that the operator $\Delta_{\lambda,\mathcal{B}'}$ is non-negative. □

Remark 6.5. The operator $\Delta_{\lambda,\mathcal{B}'}$ is not injective because the function x^λ is solution of $\Delta_\lambda u = 0$ and belongs to $\mathcal{B}'$, in fact

$$|(x^\lambda, \phi)| \leq C_\lambda \|\phi\|_{L^1(sr)} \leq C_\lambda \rho_0(\phi), \quad \phi \in \mathcal{B}.$$

According to the representation of fractional powers of operators in locally convex spaces given in [9], it is applied to $\Delta^\alpha_{\lambda,\mathcal{B}'}$ by

$$\Delta^\alpha_{\lambda,\mathcal{B}'} T = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty t^{\alpha-1} [\Delta_{\lambda,\mathcal{B}'}(t + \Delta_{\lambda,\mathcal{B}'})^{-1}]^n \, T \, dt$$

for $\operatorname{Re}\alpha > 0$, $n > \operatorname{Re}\alpha$, $T \in \mathcal{B}'$.

From the general theory of fractional powers in sequentially complete locally convex spaces (see [9, p. 134]), we deduce some properties of powers such as multiplicativity.

(1) If $\operatorname{Re}\alpha > 0$, then

$$\left(\Delta^\alpha_{\lambda,\mathcal{B}}\right)^* = \left((\Delta_{\lambda,\mathcal{B}})^*\right)^\alpha. \tag{6.4}$$

Since $(\Delta_{\lambda,\mathcal{B}})^* = \Delta_{\lambda,\mathcal{B}'}$, from (6.4) we obtain the following duality formula

$$(\Delta^\alpha_{\lambda,\mathcal{B}'} T, \phi) = (T, \Delta^\alpha_{\lambda,\mathcal{B}} \phi), \quad \phi \in \mathcal{B},\ T \in \mathcal{B}'.$$

(2) Since the usual topology in $L^p(sr^p)$ is stronger than the topology induced by $\mathcal{B}'$, we can deduce that

$$(\Delta^\alpha_{\lambda,\mathcal{B}'})_{L^p(sr^p)} = \Delta^\alpha_{\lambda,p}$$

for $\operatorname{Re}\alpha > 0$ (see [9, Theorem 12.1.6, p. 284]).

This last property expresses a very desirable property in the theory of powers since it tells us that the part of the distributional power of Δ_λ to $L^p(sr^p)$ coincides with the power of Δ_λ in $L^p(sr^p)$.

7. Distributional Liouville theorem for Δ^α_λ

In this section we include the proof of Theorem 1.1. Before that, we will show the following lemma.

Lemma 7.1. *Let $\psi \in S_\lambda$ such that $\operatorname{supp}\psi \subset \mathbb{R}^n_+ \cap \{x : \|x\| \geq a\}$ with $a > 0$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. Then $\|x\|^{-2\alpha}\psi(x) \in S_\lambda$.*

Proof. It is evident that $\|x\|^{-2\alpha}\psi(x) \in C^\infty(\mathbb{R}^n_+)$. We are going to see that

$$\sup_{x \in \mathbb{R}^n_+} \left| x^m T^k \{x^{-\lambda} \|x\|^{-2\alpha} \psi(x)\} \right| < \infty,$$

with $k, m \in \mathbb{N}^n_0$. Since $\operatorname{supp}\psi \subset \mathbb{R}^n_+ \cap \{x : \|x\| \geq a\}$ with $a > 0$, we obtain

$$\sup_{x \in \mathbb{R}^n_+} \left| x^m T^k \{x^{-\lambda} \|x\|^{-2\alpha} \psi(x)\} \right| = \sup_{x \in \mathbb{R}^n_+ : \|x\| \geq a} \left| x^m T^k \{x^{-\lambda} \|x\|^{-2\alpha} \psi(x)\} \right|.$$

Since equality (2.6) holds, we have

$$\begin{aligned}
&\sup_{\|x\|\geq a}\left|x^m T^k\{x^{-\lambda}\|x\|^{-2\alpha}\psi(x)\}\right| \\
\leq\ &\sup_{\|x\|\geq a}\left|x^m \sum_{j=0}^{k}\binom{k}{j} T^{k-j}\{x^{-\lambda}\psi(x)\}\cdot T^j\|x\|^{-2\alpha}\right| \\
\leq\ &\sum_{j=0}^{k}\binom{k}{j} C(j,\alpha)\ \gamma^{\lambda}_{m,k-j}(\psi),
\end{aligned}$$

where $C(j,\alpha)$ are constants depending on α and j such that $\sup_{\|x\|\geq a} |T^j\|x\|^{-2\alpha}| \leq C(j,\alpha)$. □

7.1. Proof of Theorem 1.1

Proof. Let $u \in \mathcal{B}'$ such that $\Delta^{\alpha}_{\lambda,\mathcal{B}'}u = 0$. Then for all $\phi \in \mathcal{B}$

$$(\Delta^{\alpha}_{\lambda,\mathcal{B}'}u,\phi) = (u,\Delta^{\alpha}_{\lambda,\mathcal{B}}\phi) = 0. \tag{7.1}$$

Since Δ_λ is a continuous operator in $\mathcal{B}$ (see Theorem 5.4), $\Delta^{\alpha}_{\lambda,\mathcal{B}}\phi$ is given by the Balakrishnan operator as:

$$\Delta^{\alpha}_{\lambda,\mathcal{B}}\phi = \frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}\int_0^\infty t^{\alpha-1}[\Delta_{\lambda,\mathcal{B}}(t+\Delta_{\lambda,\mathcal{B}})^{-1}]^m\phi\ dt. \tag{7.2}$$

By definition of $\mathcal{B}$ and the fact that $L^1(sr)\cap L^\infty(r) \subset L^p(sr^p)$ for all $1 \leq p \leq \infty$ then $\mathcal{B} \subset D(\Delta_{\lambda,p})$ for all $1 \leq p \leq \infty$, in particular, $\mathcal{B} \subset D(\Delta_{\lambda,2})$. Then from Propositions 8.3 and 8.4 in [13] we obtain that:

$$\Delta^{\alpha}_{\lambda,2}\phi = \frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}\int_0^\infty t^{\alpha-1}[\Delta_{\lambda,2}(t+\Delta_{\lambda,2})^{-1}]^m\phi\ dt. \tag{7.3}$$

Since for $\phi \in \mathcal{B}$, the integrating into the expressions are equal and the fact that the convergence in $\mathcal{B}$ implies the convergence in $L^2(\mathbb{R}^n_+)$ (see Lemma 2.1 and Remark 5.3 in [12]), we obtain the equality of (7.2) and (7.3) as functions.

We conclude that

$$\Delta^{\alpha}_{\lambda,\mathcal{B}}\phi = h_\lambda\|y\|^{2\alpha}h_\lambda\phi, \quad \phi \in \mathcal{B},$$

(see [12, Proposition 8.4]). From the last equality and (7.1), we have that

$$(\Delta^{\alpha}_{\lambda,\mathcal{B}'}u,\phi) = (u,h_\lambda\|y\|^{2\alpha}h_\lambda\phi) = 0 \tag{7.4}$$

for all $\phi \in \mathcal{B}$.

Since $\mathcal{B}' \subset S'_\lambda$ (see [12, Remark 6.2]), we can consider the Hankel transform in $\mathcal{B}'$. We are going to see that the following affirmation holds:

"*If $u \in \mathcal{B}'$ is such that (7.4) is verified, then $(h_\lambda u,\psi) = 0$ for all $\psi \in S_\lambda$ such that* $\operatorname{supp}\psi \subset \mathbb{R}^n_+ \cap \{x : \|x\| \geq a\}$ *with $a > 0$.*"

Let $u \in \mathcal{B}'$ such that (7.4) is valid and $\psi \in S_\lambda$ such that $\operatorname{supp}\psi \subset \mathbb{R}^n_+ \cap \{x : \|x\| \geq a\}$ with $a > 0$. Then, by Lemma 7.1, $\|x\|^{-2\alpha}\psi(x) \in S_\lambda$ and since the Hankel transform is an isomorphism in S_λ, there exists $\phi \in S_\lambda$ such that $h_\lambda\phi = \|x\|^{-2\alpha}\psi(x)$. So,

$$(h_\lambda u, \psi) = (h_\lambda u, \|x\|^{2\alpha}\|x\|^{-2\alpha}\psi) = (h_\lambda u, \|x\|^{2\alpha}h_\lambda\phi) = (u, h_\lambda\|x\|^{2\alpha}h_\lambda\phi).$$

Consequently, from (7.4) we conclude that $(h_\lambda u, \psi) = 0$, then the assertion is valid. Thus by [6, Theorem 4.1], there exist $N \in \mathbb{N}_0$ and scalars c_k with $|k| < N$ such that $h_\lambda u = \sum_{|k|<N} c_k S^k \delta_\lambda$ where δ_λ is given by [6, Equation (2.3)] for $k = 0$. Then,

$$u = x^\lambda \sum_{|k|\leq N} c_k(-1)^{|k|}\|x\|^{2k}.$$

□

Remark 7.2 (Regular distributions in $\mathcal{B}'$). If $f \in L^1_{loc}(\mathbb{R}^n_+)$ and $f = O(x^\lambda)$, then f is a regular distribution in $\mathcal{B}'$ given by

$$(f, \phi) = \int_{\mathbb{R}^n_+} f(x)\phi(x)\,dx, \quad \phi \in \mathcal{B},$$

and

$$\begin{aligned}
|(f,\phi)| &= \Big|\int_{\mathbb{R}^n_+} f(x)\phi(x)\,dx\Big| \\
&\leq \Big|\int_{\|x\|\leq M} f(x)\phi(x)dx\Big| + \Big|\int_{\|x\|\geq M} f(x)\phi(x)dx\Big| \\
&\leq \int_{\|x\|\leq M} |r^{-1}(x)f(x)|dx\, \|\phi\|_{L^\infty(r)} + \int_{\|x\|\geq M} c\, x^\lambda|\phi(x)|dx \\
&= C\|\phi\|_{L^\infty(r)} + c\, C_\lambda\|\phi\|_{L^1(rs)} \leq C'\rho_0(\phi).
\end{aligned}$$

Corollary 7.3. *If $f \in L^1_{loc}(\mathbb{R}^n_+)$, $f = O(x^\lambda)$ and $\Delta^\alpha_{\lambda,\mathcal{B}'} f = 0$, then $f = C\,x^\lambda$.*

8. Distributional Liouville theorem for B^α_λ

From the theory of similar operators given in [12], by the similarity of Δ_λ and B_λ, and by the non-negativity of the part of Δ_λ in $L^1(sr)$ and $L^\infty(r)$ we deduce the non-negativity of the part of B_λ in $L^1(s)$ and $L^\infty(\mathbb{R}^n_+)$. Consequently, we infer the non-negativity of the part of B_λ in the Banach space $Z = L^1(s) \cap L^\infty(\mathbb{R}^n_+)$ with norm

$$\|f\|_Z = \max\left\{\|f\|_{L^1(s)}, \|f\|_{L^\infty(\mathbb{R}^n_+)}\right\}.$$

Thus, if we consider Y as in Section 5 and $L_r : Y \to Z$ given by $L_r f = rf$ then

$$\begin{aligned}
\|rf\|_Z &= \max\left\{\|rf\|_{L^1(s)}, \|rf\|_{L^\infty(\mathbb{R}^n_+)}\right\} \\
&= \max\left\{\|f\|_{L^1(rs)}, \|f\|_{L^\infty(r)}\right\}
\end{aligned}$$

$$= \|f\|_Y ,$$

so, L_r is an isometric isomorphism.

Moreover, we can consider the locally convex space $\mathcal{F}$ given by:

$$\mathcal{F} = \{f \in Z : B_\lambda^k f \in Z \quad \text{for } k = 0, 1, 2, \dots\} = \bigcap_{k=0}^{\infty} D\big[B_{\lambda,Z}^k\big],$$

where with $B_{\lambda,Z}$ we denote the part of B_λ in Z. The space $\mathcal{F}$ is endowed with the topology generated by the family of seminorms given by

$$\gamma_m(f) = \max_{0\le k\le m}\{\|B_\lambda^k f\|_Z\}, \quad m = 0, 1, 2, \dots.$$

Thus, the space $\mathcal{F}$ verifies that is a Frechet space and from Remarks 5.3 and 6.2 we deduce that $\mathcal{F} \subset C_0(\mathbb{R}_+^n)$, $\mathcal{F} \subset L^p(s)$ for all $1 \le p < \infty$, $\mathcal{F} \subset L^\infty(\mathbb{R}_+^n)$, $S_\lambda \subset \mathcal{F}$ and the topology of S_λ induced by $\mathcal{F}$ is weaker than the usual topology in S_λ. Moreover, the operator B_λ verifies that

$$\gamma_m(B_\lambda f) = \max_{0\le k\le m}\{\|B_\lambda^{k+1} f\|_Z\} = \gamma_{m+1}(f)$$

for all $f \in \mathcal{F}$. Then $B_{\lambda,\mathcal{F}}$, the part of B_λ in $\mathcal{F}$, is a continuous

$$B_{\lambda,\mathcal{F}} : \mathcal{F} \to \mathcal{F}.$$

If $f \in \mathcal{B}$, (see (5.2)), then $rf \in \mathcal{F}$ and

$$\begin{aligned}\gamma_m(rf) &= \max_{0\le k\le m}\{\|B_\lambda^k rf\|_Z\} = \max_{0\le k\le m}\{\|r\ \Delta_\lambda^k r^{-1} rf\|_Z\}\\ &= \max_{0\le k\le m}\{\|r\ \Delta_\lambda^k f\|_Z\} = \max_{0\le k\le m}\{\|\Delta_\lambda^k f\|_Y\}\\ &= \rho_m(f),\end{aligned}$$

where we have consider (2.30). So, the application $L_r : \mathcal{B} \to \mathcal{F}$ given by $L_r f = rf$ is an isomorphism of locally convex spaces with inverse given by $L_{r^{-1}} : \mathcal{F} \to \mathcal{B}$.

Remark 8.1. Since $\mathcal{B}$ and $\mathcal{F}$ are isomorphic, we can deduce that $\mathcal{F}'$ is sequentially complete as $\mathcal{B}'$ is also sequentially complete (see Remark 6.1).

So, if we consider the continuous operator $\Delta_{\lambda,\mathcal{B}} : \mathcal{B} \to \mathcal{B}$, then by (2.30) we obtain the similarity relation,

$$B_{\lambda,\mathcal{F}} = L_r\ \Delta_{\lambda,\mathcal{B}}\ L_{r^{-1}}. \tag{8.1}$$

We deduce by (8.1) the non-negativity of $B_{\lambda,\mathcal{F}}$ and by [12, Proposition 1.1], for $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, we have that

$$B_{\lambda,\mathcal{F}}^\alpha = L_r\ \Delta_{\lambda,\mathcal{B}}^\alpha\ L_{r^{-1}}. \tag{8.2}$$

Consequently,

$$B_{\lambda,\mathcal{F}'}^\alpha = ((B_{\lambda,\mathcal{F}})^*)^\alpha = (B_{\lambda,\mathcal{F}}^\alpha)^*,$$

where we have considered in the second equality that $\mathcal{F}$ is a Frechet space (see [9, p. 134]). Thus,

$$B^{\alpha}_{\lambda,\mathcal{F}'} = (L_{r^{-1}})^* \, (\Delta^{\alpha}_{\lambda,\mathcal{B}})^* \, (L_r)^* = (L_{r^{-1}})^* \, \Delta^{\alpha}_{\lambda,\mathcal{B}'} \, (L_r)^*, \tag{8.3}$$

and for $T \in \mathcal{F}'$, $\phi \in \mathcal{F}$

$$\begin{aligned}(B^{\alpha}_{\lambda,\mathcal{F}'}T, \phi) &= ((L_{r^{-1}})^* \, \Delta^{\alpha}_{\lambda,\mathcal{B}'} \, (L_r)^* T, \phi) \\ &= (T, L_r \Delta^{\alpha}_{\lambda,\mathcal{B}} L_{r^{-1}} \phi).\end{aligned} \tag{8.4}$$

From now on, we will use the notation:

$$B_{\lambda,\mathcal{F}} = x^{-\lambda} \, \Delta_{\lambda,\mathcal{B}} \, x^{\lambda},$$

$$B^{\alpha}_{\lambda,\mathcal{F}} = x^{-\lambda} \, \Delta^{\alpha}_{\lambda,\mathcal{B}} \, x^{\lambda}$$

and

$$B^{\alpha}_{\lambda,\mathcal{F}'} = x^{\lambda} \, \Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda}$$

to refer to (8.1), (8.2) and (8.3). In the last equation, the operators x^{λ} and $x^{-\lambda}$ represent $(L_{r^{-1}})^*$ and $(L_r)^*$, so,

$$x^{\lambda} : \mathcal{B}' \to \mathcal{F}',$$

$$x^{-\lambda} : \mathcal{F}' \to \mathcal{B}',$$

are given by

$$(x^{\lambda} T_1, \phi) = (T_1, x^{\lambda} \phi), \qquad (T_1 \in \mathcal{B}'), (\phi \in \mathcal{F})$$

$$(x^{-\lambda} T_2, \psi) = (T_2, x^{-\lambda} \psi), \qquad (T_2 \in \mathcal{F}'), (\psi \in \mathcal{B})$$

Now we are able to give the proof of Theorem 1.2:

8.1. Proof of Theorem 1.2

Proof. Let $u \in \mathcal{F}'$ such that $B^{\alpha}_{\lambda,\mathcal{F}'} u = 0$. Then

$$(B^{\alpha}_{\lambda,\mathcal{F}'} u, \phi) = (x^{\lambda} \, \Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda} u, \phi) = 0 \tag{8.5}$$

for all $\phi \in \mathcal{F}$. Since $\Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda} u \in \mathcal{B}'$, then given $\psi \in \mathcal{B}$ and considering (8.5), we obtain that

$$(\Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda} u, \psi) = (\Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda} u, x^{\lambda} x^{-\lambda} \psi) = (x^{\lambda} \Delta^{\alpha}_{\lambda,\mathcal{B}'} \, x^{-\lambda} u, x^{-\lambda} \psi) = 0.$$

By Theorem 1.1 we deduce that there exists a polynomial p such that $x^{-\lambda} u = x^{\lambda} p(\|x\|^2)$ and consequently $u = x^{2\lambda} p(\|x\|^2)$. □

Corollary 8.2. *If $f \in L^1_{loc}(\mathbb{R}^n_+)$, $f = O(x^{2\lambda})$ and $B^{\alpha}_{\lambda,\mathcal{F}'} f = 0$, then $f = C \, x^{2\lambda}$.*

Appendix A. Special functions

In this appendix we summarize properties of the Bessel function of the first kind and order ν given by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(\nu+n+1)}. \tag{A.1}$$

According to [7, p. 310], for $\nu \in \mathbb{R}$, $\nu > -\frac{1}{2}$, the Bessel function verifies that

$$|2^\nu \Gamma(\nu+1)\, z^{-\nu}\, J_\nu(z)| \leq 1. \tag{A.2}$$

The following equalities are also verified:

$$\begin{aligned} &\int_0^\pi \frac{J_\nu\left(\sqrt{y^2+z^2-2yz\cos\phi}\right)}{(y^2+z^2-2yz\cos\phi)^{\frac{1}{2}\nu}} \sin^{2\nu}\phi\, d\phi \\ &\quad = 2^\nu\, \Gamma(\nu+1/2)\, \Gamma(1/2)\, \frac{J_\nu(y)}{y^\nu}\, \frac{J_\nu(z)}{z^\nu} \end{aligned} \tag{A.3}$$

for $\nu > -\frac{1}{2}$, see [16, p. 367] and

$$\int_0^\infty e^{\frac{-ay^2}{2}}\, J_\nu(ry)\, y^{\nu+1}\, dy = r^\nu\, a^{-\nu-1}\, e^{-\frac{r^2}{2a}} \tag{A.4}$$

for $\nu > -1$ and $a > 0$, see [14, p. 46].

The following equalities are valid for integrals that involve the Gamma function:

Let $a > 0$ and $\lambda > -\frac{1}{2}$. Then the following equalities are valid

$$\int_0^\infty e^{-\frac{x^2}{2}}\, x^{2\lambda}\, dx = 2^{\lambda-1/2}\, \Gamma(\lambda+1/2), \tag{A.5}$$

$$\int_0^\infty e^{-\frac{x^2}{2a}}\, x^{2\lambda}\, dx = 2^{\lambda-1/2}\, \Gamma(\lambda+1/2)\, a^{\lambda+1/2}, \tag{A.6}$$

$$\int_0^{\pi/2} \sin^{2\lambda}\theta\, d\theta = \frac{\Gamma(1/2)\,\Gamma(\lambda+1/2)}{2\Gamma(\lambda+1)} = \frac{\sqrt{\pi}\,\Gamma(\lambda+1/2)}{2\Gamma(\lambda+1)}. \tag{A.7}$$

Another important equation is the Euler Complements Formula

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\pi\nu} \quad (0 < \operatorname{Re}\nu < 1). \tag{A.8}$$

Appendix B. Some results on Hankel transforms, convolution and the Inversion theorem

Hirschman defined in [7] for the 1-dimensional case, a kernel $\mathfrak{D}_\nu$ which is defined for $u, v, w \in \mathbb{R}_+$, $\nu > 0$, by

$$\mathfrak{D}_\nu(u,v,w) = \frac{2^{3\nu-5/2}\Gamma^2(\nu+1/2)}{\Gamma(\nu)\sqrt{\pi}}(uvw)^{-2\nu+1} A(u,v,w)^{2\nu-2}, \tag{B.1}$$

where $A(u,v,w)$ is the area of a triangle of sides $u,v,w \in \mathbb{R}_+$ defined by (2.25).

For the n-dimensional case, let $x,y,z \in \mathbb{R}^n_+$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i > 0$ for all $i = 1, \ldots, n$. We define

$$\boldsymbol{\mathfrak{D}}_\lambda(x,y,z) = \prod_{i=1}^{n} \mathfrak{D}_{\lambda_i}(x,y,z), \tag{B.2}$$

where $\mathfrak{D}_{\lambda_i}$ is given by (B.1).

A convolution operation associated to the n-dimensional Hankel transform H_λ can be defined. Given f,g defined on $\mathbb{R}^n_+$, the Hankel convolution associated to the transformation H_λ is defined formally by

$$f \# g(x) = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} f(y)\, g(z)\, \boldsymbol{\mathfrak{D}}_\lambda(x,y,z)\, s(y)\, s(z)\, dy\, dz, \tag{B.3}$$

where $x,y,z \in \mathbb{R}^n_+$.

Remark B.1 (Relation between $\mathbf{D}_\lambda$ and $\boldsymbol{\mathfrak{D}}_\lambda$).

$$\mathbf{D}_\lambda(x,y,z) = C_\lambda^{-2}(xyz)^{1-\lambda}\boldsymbol{\mathfrak{D}}_\lambda(x,y,z), \tag{B.4}$$

where $\boldsymbol{\mathfrak{D}}_\lambda(x,y,z)$ is given by (B.2) and $\mathbf{D}_\lambda(x,y,z)$ is given by (2.23).

Proposition B.2. *In this proposition we summarize some properties of the kernel* $\boldsymbol{\mathfrak{D}}_\lambda(x,y,z)$ *given by* (B.2).

(i) $\boldsymbol{\mathfrak{D}}_\lambda(x,y,z) > 0$,

(ii) $\int_{\mathbb{R}^n_+} \boldsymbol{\mathfrak{D}}_\lambda(x,y,z) \prod_{i=1}^{n} \left\{ (z_i t_i)^{\frac{1}{2}-\lambda_i} J_{\lambda_i - \frac{1}{2}}(z_i t_i) \right\} s(z)\, dz$
$= C_\lambda \prod_{i=1}^{n} \left\{ (x_i t_i)^{\frac{1}{2}-\lambda_i} J_{\lambda_i - \frac{1}{2}}(x_i t_i) \right\} \prod_{i=1}^{n} \left\{ (y_i t_i)^{\frac{1}{2}-\lambda_i} J_{\lambda_i - \frac{1}{2}}(y_i t_i) \right\}$,

(iii) $\int_{\mathbb{R}^n_+} \boldsymbol{\mathfrak{D}}_\lambda(x,y,z)\, s(z)\, dz = 1$,

where $x,y,z,t \in \mathbb{R}^n_+$ *and* J_ν *denotes the well known Bessel function of the first kind and order* ν *given by* (A.1).

Theorem B.3. *Let* $\{\phi_m\} \subset L^1(s)$ *be a sequence of functions such that*:

(1) $\phi_m(x) \geq 0$ *in* $\mathbb{R}^n_+$,
(2) $\int_{\mathbb{R}^n_+} \phi_m(x)\, s(x)\, dx = 1$ *for all* $m \in \mathbb{N}$,
(3) *For all* $\eta > 0$, $\lim_{m\to\infty} \int_{\|x\|>\eta} \phi_m(x)\, s(x)\, dx = 0$.

If $f \in L^1(s)$, *then* $\lim_{n\to\infty} \|f \# \phi_m - f\|_{L^1(s)} = 0$.

Proof. This result is an n-dimensional generalization of [7, Corollary 2c], relative to approximate identities. ☐

Lemma B.4. *Let* f,g *be functions in* $L^1(s)$. *Then*

$$\int_{\mathbb{R}^n_+} H_\lambda f(t)\, g(t)\, s(t)\, dt = \int_{\mathbb{R}^n_+} f(t)\, H_\lambda g(t)\, s(t)\, dt.$$

Theorem B.5. *If $f(x) \in L^1(s)$ and $H_\lambda f(t) \in L^1(s)$, then $f(x)$ may be redefined on a set of measure zero so that it is continuous in $x \in \mathbb{R}^n_+$, and then*

$$f(x) = \int_{\mathbb{R}^n_+} H_\lambda f(t) \left\{ \prod_{i=1}^n (x_i t_i)^{1/2-\lambda_i} J_{\lambda_i - 1/2}(x_i t_i)\, t_i^{2\lambda_i} \right\} dt \tag{B.5}$$

for almost every $x \in \mathbb{R}^n_+$.

Proof. We consider the sequence $\{\phi_m\}_{m\in\mathbb{N}}$ defined by

$$\phi_m(x) = m^{|\lambda+1/2|}\, e^{-\frac{\|x\|^2 m}{2}}. \tag{B.6}$$

This sequence verifies conditions (1), (2) and (3) of Theorem B.3, then if $f \in L^1(s)$,

$$\lim_{n\to\infty} \|f \# \phi_m - f\|_{L^1(s)} = 0.$$

Let us show that:

$$\phi_m \# f(x) = \int_{\mathbb{R}^n_+} H_\lambda(f)(z)\, e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(x_i z_i) \right\} z^{2\lambda}\, dz. \tag{B.7}$$

To see this we define

$$G_x(z) = e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(x_i z_i) \right\}. \tag{B.8}$$

Clearly, $G_x(z) \in L^1(s)$ and from Lemma B.4 we have

$$\begin{aligned} &\int_{\mathbb{R}^n_+} H_\lambda(f)(z)\, e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(x_i z_i) \right\} z^{2\lambda}\, dz \\ &= \int_{\mathbb{R}^n_+} H_\lambda f(z)\, G_x(z)\, z^{2\lambda}\, dz \\ &= \int_{\mathbb{R}^n_+} f(t)\, H_\lambda(G_x(z))(t)\, t^{2\lambda}\, dt. \end{aligned} \tag{B.9}$$

Moreover,

(B.10)

$$\begin{aligned} &H_\lambda(G_x(z))(t) \\ &= \int_{\mathbb{R}^n_+} G_x(z) \left\{ \prod_{i=1}^n (z_i t_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(z_i t_i)\, z_i^{2\lambda_i} \right\} dz \\ &= \int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (x_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(x_i z_i) \right\} \left\{ \prod_{i=1}^n (z_i t_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(z_i t_i) \right\} z^{2\lambda}\, dz \\ &= \int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \int_{\mathbb{R}^n_+} \mathfrak{D}_\lambda(x,t,\xi) \left\{ \prod_{i=1}^n (\xi_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(\xi_i z_i) \right\} s(\xi)\, d\xi \right\} s(z)\, dz. \end{aligned}$$

Since

$$\int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \int_{\mathbb{R}^n_+} \mathfrak{D}_\lambda(x,t,\xi) \left\{ \prod_{i=1}^n |(\xi_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(\xi_i z_i)| \right\} s(\xi)\, d\xi \right\} s(z)\, dz$$
$$\leq C_\lambda^{-1} \int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \int_{\mathbb{R}^n_+} \mathfrak{D}_\lambda(x,t,\xi)\, s(\xi)\, d\xi \right\} s(z)\, dz < \infty,$$

it is possible to change the order of integration in (B.10), then

$$\begin{aligned}
&H_\lambda(G_x(z))(t) \\
&= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (\xi_i z_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(\xi_i z_i) \right\} s(z)\, dz \right\} \mathfrak{D}_\lambda(x,t,\xi) s(\xi)\, d\xi \\
&= C_\lambda^{-2} \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n J_{\lambda_i-1/2}(\xi_i z_i)\, z_i^{\lambda_i+1/2} \right\} dz \right\} \mathfrak{D}_\lambda(x,t,\xi)\, \xi^{\lambda+1/2}\, d\xi \\
&= C_\lambda^{-2} \int_{\mathbb{R}^n_+} \xi^{\lambda-1/2}\, m^{|\lambda+1/2|}\, e^{-\frac{\|\xi\|^2 m}{2}} \mathfrak{D}_\lambda(x,t,\xi)\, \xi^{\lambda+1/2}\, d\xi \\
&= C_\lambda^{-1} \int_{\mathbb{R}^n_+} m^{|\lambda+1/2|}\, e^{-\frac{\|\xi\|^2 m}{2}} \mathfrak{D}_\lambda(x,t,\xi)\, s(\xi)\, d\xi,
\end{aligned}$$

where we have used (A.4) with $a = 1/m$. From the last equality we obtain that

$$\text{(B.11)} \quad \begin{aligned}
&\int_{\mathbb{R}^n_+} H_\lambda(G_x(z))(t)\, f(t)\, t^{2\lambda}\, dt \\
&= \int_{\mathbb{R}^n_+} \left\{ C_\lambda^{-1} \int_{\mathbb{R}^n_+} m^{|\lambda+1/2|}\, e^{-\frac{\|\xi\|^2 m}{2}} \mathfrak{D}_\lambda(x,t,\xi)\, s(\xi)\, d\xi \right\} f(t)\, t^{2\lambda}\, dt \\
&= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} m^{|\lambda+1/2|}\, e^{-\frac{\|\xi\|^2 m}{2}} \mathfrak{D}_\lambda(x,t,\xi)\, s(\xi)\, d\xi \right\} f(t)\, s(t)\, dt \\
&= \phi_m \# f(x).
\end{aligned}$$

From (B.9) and (B.11) we obtain (B.7). We may now take limit in (B.7), and considering that $H_\lambda f \in L^1(s)$ and

$$\left| e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (z_i x_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(z_i x_i) \right\} H_\lambda(f)(z) \right|$$
$$\leq C_\lambda^{-1}\, e^{-\frac{\|z\|^2}{2m}}\, |H_\lambda(f)(z)| \leq C\, |H_\lambda(f)(z)|\,,$$

then we obtain in the right side of (B.7) by the dominated convergence theorem that

$$\text{(B.12)} \quad \lim_{m\to\infty} \int_{\mathbb{R}^n_+} H_\lambda(f)(z)\, e^{-\frac{\|z\|^2}{2m}} \left\{ \prod_{i=1}^n (z_i x_i)^{1/2-\lambda_i} J_{\lambda_i-1/2}(z_i x_i) \right\} z^{2\lambda}\, dz$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n_+} H_\lambda(f)(z) \left\{ \prod_{i=1}^n (z_i x_i)^{1/2-\lambda_i} J_{\lambda_i - 1/2}(z_i x_i) \right\} z^{2\lambda}\, dz \\
&= H_\lambda(H_\lambda f)(x).
\end{aligned}
$$

So,

$$
\lim_{m\to\infty} \phi_m \# f(x) = H_\lambda(H_\lambda f)(x). \tag{B.13}
$$

On the other hand, by (B.3) there exists a subsequence $\{\phi_{m_k} \# f\}_{k\in\mathbb{N}}$ such that

$$
\lim_{k\to\infty} \phi_{m_k} \# f(x) = f(x) \tag{B.14}
$$

for almost every $x \in \mathbb{R}^n_+$, and so this completes the proof. □

From Theorem B.5 we can obtain the inversion theorem for the Hankel transform h_λ given by (2.9)

B.1. Proof of Theorem 2.13

Proof. If $f \in L^1(\mathbb{R}^n_+, x^\lambda)$, then $x^{-\lambda} f \in L^1(s)$.

Since

$$
x^{-\lambda} h_\lambda(f) = H_\lambda(x^{-\lambda} f)
$$

and for hypothesis

$$
h_\lambda f \in L^1(\mathbb{R}^n_+, x^\lambda),
$$

we have $H_\lambda(x^{-\lambda} f) \in L^1(s)$.

Then the result continues to apply Theorem B.5 to $x^{-\lambda} f$ and obtain (2.31). □

Lemma B.6. *Let f, g be functions in $L^1(sr)$. Then*

$$
\int_{\mathbb{R}^n_+} h_\lambda f(t)\, g(t)\, dt = \int_{\mathbb{R}^n_+} f(t)\, h_\lambda g(t)\, dt.
$$

Appendix C. Similarity of Bessel operators

Remark C.1. The operators B_λ and Δ_λ given by (1.2) and (1.1), respectively, are related through

$$
\Delta_\lambda = x^\lambda B_\lambda x^{-\lambda}. \tag{C.1}
$$

Let $\phi \in C^2(\mathbb{R}^n_+)$. Then

$$
\begin{aligned}
x^{-\lambda}(-\Delta_\lambda) x^\lambda \phi(x) &= x^{-\lambda} \left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \frac{\lambda_i(\lambda_i - 1)}{x_i^2} \right) x^\lambda \phi(x) \\
&= \sum_{i=1}^n x^{-\lambda} \frac{\partial^2}{\partial x_i^2} x^\lambda \phi(x) - \sum_{i=1}^n \frac{\lambda_i(\lambda_i - 1)}{x_i^2} \phi(x)
\end{aligned}
$$

and also

$$\begin{aligned} x^{-\lambda}\frac{\partial^2}{\partial x_i^2}\{x^\lambda\phi(x)\} &= x^{-\lambda}x^\lambda x_i^{-\lambda_i}\frac{\partial^2}{\partial x_i^2}\{x_i^{\lambda_i}\phi(x)\} = x_i^{-\lambda_i}\frac{\partial^2}{\partial x_i^2}\{x_i^{\lambda_i}\phi(x)\} \\ &= x_i^{-\lambda_i}\frac{\partial}{\partial x_i}\{\lambda_i\, x_i^{\lambda_i-1}\phi(x) + x_i^{\lambda_i}\frac{\partial}{\partial x_i}\phi(x)\} \\ &= x_i^{-\lambda_i}\{\lambda_i(\lambda_i-1)\, x_i^{\lambda_i-2}\phi(x) + \lambda_i\, x_i^{\lambda_i-1}\frac{\partial}{\partial x_i}\phi(x) \\ &\quad + \lambda_i\, x_i^{\lambda_i-1}\frac{\partial}{\partial x_i}\phi(x) + x_i^{\lambda_i}\frac{\partial^2}{\partial x_i^2}\phi(x)\} \\ &= \lambda_i(\lambda_i-1)\, x_i^{-2}\phi(x) + 2\lambda_i\, x_i^{-1}\frac{\partial}{\partial x_i}\phi(x) + \frac{\partial^2}{\partial x_i^2}\phi(x), \end{aligned}$$

from where

$$\begin{aligned} x^{-\lambda}(-\Delta_\lambda)x^\lambda\phi(x) &= \sum_{i=1}^n x^{-\lambda}\frac{\partial^2}{\partial x_i^2}x^\lambda\phi(x) - \sum_{i=1}^n \frac{\lambda_i(\lambda_i-1)}{x_i^2}\phi(x) \\ &= \sum_{i=1}^n \lambda_i(\lambda_i-1)\, x_i^{-2}\phi(x) + 2\lambda_i\, x_i^{-1}\frac{\partial}{\partial x_i}\phi(x) + \frac{\partial^2}{\partial x_i^2}\phi(x) \\ &\quad - \sum_{i=1}^n \frac{\lambda_i(\lambda_i-1)}{x_i^2}\phi(x) \\ &= \sum_{i=1}^n 2\lambda_i\, x_i^{-1}\frac{\partial}{\partial x_i}\phi(x) + \frac{\partial^2}{\partial x_i^2}\phi(x) = (-B_\lambda)\phi(x). \end{aligned}$$

References

[1] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math. **10** (1960), 419–437. http://projecteuclid.org/euclid.pjm/1103038401

[2] J. J. Betancor, A. J. Castro, and P. R. Stinga, *The fractional Bessel equation in Hölder spaces*, J. Approx. Theory **184** (2014), 55–99. https://doi.org/10.1016/j.jat.2014.05.003

[3] K. Bogdan, T. Kulczycki, and A. Nowak, *Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes*, Illinois J. Math. **46** (2002), no. 2, 541–556. http://projecteuclid.org/euclid.ijm/1258136210

[4] W. Chen, L. D'Ambrosio, and Y. Li, *Some Liouville theorems for the fractional Laplacian*, Nonlinear Anal. **121** (2015), 370–381. https://doi.org/10.1016/j.na.2014.11.003

[5] J. Delsarte, *Sur une extension de la formule de Taylor*, Journal de Mathematiques Pures et Appliquees. Neuvieme Serie **17** (1938), 213–231.

[6] V. Galli, S. Molina, and A. Quintero, *A Liouville theorem for some Bessel generalized operators*, Integral Transforms Spec. Funct. **29** (2018), no. 5, 367–383. https://doi.org/10.1080/10652469.2018.1441295

[7] I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Analyse Math. **8** (1960/61), 307–336. https://doi.org/10.1007/BF02786854

[8] Y. Y. Li, *Local gradient estimates of solutions to some conformally invariant fully nonlinear equations*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 4, 249–252. https://doi.org/10.1016/j.crma.2006.06.008

[9] C. Martínez and M. Sanz, *The theory of fractional powers of operators*, North-Holland Mathematics Studies, 187, North-Holland Publishing Co., Amsterdam, 2001.

[10] C. Martínez, M. Sanz, and F. Periago, *Distributional fractional powers of the Laplacean. Riesz potentials*, Studia Math. **135** (1999), no. 3, 253–271.

[11] S. Molina, *A generalization of the spaces H_μ, H'_μ and the space of multipliers*, in Proceedings of the Seventh "Dr. Antonio A. R. Monteiro" Congress of Mathematics (Spanish), 49–56, Univ. Nac. del Sur, Bahía Blanca, 2003.

[12] S. M. Molina, *Distributional fractional powers of similar operators with applications to the Bessel operators*, Commun. Korean Math. Soc. **33** (2018), no. 4, 1249–1269. https://doi.org/10.4134/CKMS.c170440

[13] S. Molina and S. E. Trione, *n-dimensional Hankel transform and complex powers of Bessel operator*, Integral Transforms Spec. Funct. **18** (2007), no. 11-12, 897–911. https://doi.org/10.1080/10652460701511244

[14] F. Oberhettinger, *Tables of Bessel Transforms*, Springer-Verlag, New York, 1972.

[15] H. H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Mathematics, Vol. 3, Springer-Verlag, New York, 1971.

[16] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, reprint of the second (1944) edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995.

[17] A. H. Zemanian, *A distributional Hankel transformation*, SIAM J. Appl. Math. **14** (1966), 561–576. https://doi.org/10.1137/0114049

[18] R. Zhuo, W. Chen, X. Cui, and Z. Yuan, *A Liouville theorem for the fractional Laplacian*, arXiv:1401.7402 (2014)

Vanesa Galli
CEMIM - Departamento de Matematica - FCEyN
Universidad Nacional de Mar del Plata
Funes 3350 (7600) Mar del Plata, Argentina
Email address: vggalli@mdp.edu.ar

Sandra Molina
CEMIM - Departamento de Matematica - FCEyN
Universidad Nacional de Mar del Plata
Funes 3350 (7600) Mar del Plata, Argentina
Email address: smolina@mdp.edu.ar

Alejandro Quintero
CEMIM - Departamento de Matematica - FCEyN
Universidad Nacional de Mar del Plata
Funes 3350 (7600) Mar del Plata, Argentina
Email address: aquinter@mdp.edu.ar