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GOLDEN PARA-CONTACT METRIC MANIFOLDS

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Abstract. The purpose of the present paper is to introduce a new class of almost para-contact metric manifolds namely, Golden para-contact metric manifolds. Then, we are particularly interested in a more special type called Golden para-Sasakian manifolds, where we will study their fundamental properties and we present many examples which justify their study.

1. Introduction

The Golden section or Golden mean ϕ is the positive root of the polynomial equation $x^2 - x - 1 = 0$; i.e., $\phi = \frac{1+\sqrt{5}}{2}$. The negative root of the previous equation, usually denoted by ϕ^*, satisfies $\phi^* = \frac{1-\sqrt{5}}{2} = 1 - \phi$. In the last years the Golden mean can be found in many areas of mathematical and physical research.

In [3], Crasmareanu and Hretcanu introduced and studied the Golden structures and they gives relationships between it and other structures (almost product, almost tangent and almost complex). As a generalization of the Golden mean appear the metallic means (see [7]), which are the positive root of the equation $x^2 - px - q = 0$, where p, q are positive integers.

Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds and relations between them have been studied widely in differential geometry. Recently, the author [1] introduced the notion of Golden Riemannian manifolds of type (r, s) and starting from a Golden Riemannian structure, we have established many well-known structures on a Riemannian manifold. Also, he defined a new class of Golden manifolds [2].

The notion of almost para-contact manifolds (respectively, almost para-contact Riemannian manifolds) as analogue of almost contact manifolds (respectively, almost contact Riemannian manifolds) was introduced by Sato in [5]

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and [6]. Remarkable that an almost contact manifold is always odd dimensional but an almost para-contact manifold could be even dimensional as well.

Here we show that there is a correspondence between the Golden Riemannian structures and the almost para-contact metric structures. This text is organized in the following way: Section 2 is devoted to the background of the structures which will be used in the sequel. In Section 3, we will give the relationship between Golden Riemannian structures and almost para-contact metric structures and then employ it to extract a new class that we will call *Golden para-contact metric* structures where we will study their fundamental properties. Section 4 is intended to study a more special type in this class by giving their geometric properties with concrete examples.

2. Review of needed notions

In this section, we give a brief information for Golden Riemannian manifolds and almost para-contact metric manifolds.

2.1. Golden Riemannian manifold

Let (M, g) be a Riemannian manifold. A Golden structure on (M, g) is a non-null tensor field Φ of type $(1, 1)$ which satisfies the equation

$$\Phi^2 = \Phi + I, \tag{1}$$

where I is the identity transformation.

We say that the metric g is Φ compatible if

$$g(\Phi X, Y) = g(X, \Phi Y) \tag{2}$$

for all vector fields X, Y on M.

If we substitute ΦX into X in (2), equation (2) may also written as

$$g(\Phi X, \Phi Y) = g(\Phi^2 X, Y) = g((\Phi + I)X, Y) = g(\Phi X, Y) + g(X, Y).$$

The Riemannian metric (2) is called Φ-compatible and (M, Φ, g) is named a Golden Riemannian manifold [3].

Here, it's the occasion to show that each Golden structure on a Riemannian manifold generates a family of Riemannian metrics.

Proposition 2.1. *Any Golden structure Φ on a paracompact manifold M admits a Riemannian metric Φ-compatible.*

Proof. Let h be any Riemannian metric on M and define g by

$$g(X, Y) = h(\Phi X, \Phi Y) + h(X, Y) = h(\Phi X, Y) + 2h(X, Y),$$

and check the details. □

It is known that a Golden structure Φ is integrable if the Nijenhuis tensor N_Φ vanishes [3], i.e.,

$$N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] = 0.$$

We know that the integrability of Φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla\Phi = 0$ holds [4].

Recently, the first author [2] have defined a class of almost Golden Riemannian manifolds namely s-Golden manifolds, by:

Definition 2.2. Let M be a differentiable manifold of dimension $n+s$. An almost s-Golden structure on M is the data $\big(\Phi, (\xi_\alpha, \eta_\alpha)_{\alpha=1}^s, g\big)$, where

(i) ξ_α is a global vector field, (called Golden vector field).
(ii) η_β is a differential 1-form on M such that $\eta_\beta(\xi_\alpha) = \delta_{\alpha\beta}$, where $\alpha, \beta \in \{1, \dots, s\}$.
(iii) g is a Riemannian metric such that $g(X, \xi_\alpha) = \eta_\alpha(X)$.
(iv) Φ is a tensor field of type $(1,1)$ satisfying

$$\Phi = \phi^* I + \sqrt{5}\sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha \tag{3}$$

for all vector field X on M.

In addition, if Φ is integrable, then $\big(\Phi, (\xi_\alpha, \eta_\alpha)_{\alpha=1}^s, g\big)$ is an s-Golden structure and $\big(M, \Phi, (\xi_\alpha, \eta_\alpha)_{\alpha=1}^s, g\big)$ is called an s-Golden manifold.

In this class of manifolds and for $s = 1$, there is a remarkable type called generalized $\mathcal{G}$-Golden manifolds satisfying:

$$(\nabla_X \Phi)Y = \sigma\sqrt{5}\big(g(X,Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi\big) \tag{4}$$

for all vector fields X and Y on M and σ is a function on M. For $\sigma = 1$ we obtain $\mathcal{G}$-Golden manifold and from (4), it follows that

$$\nabla_X \xi = \sigma\big(X - \eta(Y)\xi\big). \tag{5}$$

2.2. Almost para-contact metric manifold

An n-dimensional Riemannian manifold (M^n, g) is said to be an almost para-contact metric manifold if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = X - \eta(X)\xi \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X,Y on M. In particular, in an almost para-contact metric manifold we also have

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0 \quad \text{and} \quad \operatorname{rank}\varphi = n-1.$$

The fundamental $(0,2)$ symmetric tensor of the almost para-contact metric structure is defined by

$$\Omega(X,Y) = g(X, \varphi Y)$$

for any vector fields X,Y on M. Such a manifold is said to be a para-contact metric manifold if [8]

$$2\Omega(X,Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X = (\mathcal{L}_\xi g)(X,Y), \tag{6}$$

where $\mathcal{L}$ is the operator of Lie differentiation.

On the other hand, the almost para-contact metric structure of M is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) - 2d\eta\ (X,Y)\xi = 0 \tag{7}$$

for any X, Y, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y].$$

An almost para-contact metric manifold (M,φ,ξ,η,g) on M is said to be a para-Sasakian manifold if

$$(\nabla_X\varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

For more background on almost para-contact metric manifolds, we recommend the reference [5], [6] and [8].

3. Golden para-contact metric manifold

In this section, we will associate and combine the structures Golden with the structures para-contact defined on the same manifold M and extract a new structure we will call it Golden para-contact metric structure.

Theorem 3.1. *Every almost para-contact metric structure (φ,ξ,η,g) on a Riemannian manifold (M,g) induces only two Golden Riemannian structures on (M,g), given as follows*:

$$\Phi = \frac{1}{2}I + \frac{\sqrt{5}}{2}\big(\varepsilon\varphi + \eta\otimes\xi\big), \tag{8}$$

where ξ is the unique eigenvector of Φ associated with ϕ and $\varepsilon = \pm 1$.

Proof. We try to write the Golden structure Φ defined on (M,g), using almost para-contact metric structure (φ,ξ,η,g), in the form $\Phi = a\varphi + bI + c\eta\otimes\xi$, where $a,b,c\in\mathbb{R}^*$. Thus

$$\Phi^2 = 2ab\varphi + (a^2+b^2)I + (c^2 - a^2 + 2bc)\eta\otimes\xi,$$

and using formula (1) we obtain the formulas (8). Moreover, we have

$$g(\Phi X,Y) = g(X,\Phi Y) \Leftrightarrow g(\varphi X,Y) = g(X,\varphi Y)$$

for every tangent vector fields X and Y on M, which completes the demonstration. □

Regarding expressions (8) and (3) with $s=1$, one can ask if it's possible to get an explicit and direct expression for φ without Φ? The answer is positive and this gives the following definition:

Definition 3.2. An almost Golden almost para-contact metric manifold is the quintuple (M,φ,ξ,η,g) which satisfies:

$$\eta(\xi) = 1, \qquad \eta(X) = g(\xi,X), \quad \varphi X = \varepsilon\big(-X + \eta(X)\xi\big) \tag{9}$$

for all vector field X on M and $\varepsilon = \pm 1$.

Remark 3.3. We can easily check that any almost Golden almost para-contact metric manifold is an almost para-contact metric manifold.

Proposition 3.4. *Let $(M, \varphi, \xi, \eta, g)$ be an almost Golden almost para-contact metric manifold and the set $\{\xi, e_i\}_{1 \leq i \leq n-1}$ of vector fields where $\varphi e_i = -\varepsilon e_i$. Then we may easily check that $\{\xi, e_i\}$ is an orthonormal basis on M.*

We refer to this basis as G-basis.

Lemma 3.5. *Let $(M, \varphi, \xi, \eta, g)$ be an almost Golden almost para-contact metric manifold. If ∇ is the Levi-Cevita connection, then for all vector fields X and Y on M we have*

$$(\nabla_X \varphi)Y = \varepsilon\big(g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi\big).$$

Proof. Knowing that

$$(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y,$$

and using formulas (9), the proof is direct. □

Proposition 3.6. *Let $(M, \varphi, \xi, \eta, g)$ be an almost Golden almost para-contact metric manifold. If ∇ is the Levi-Cevita connection, then*

$$(\nabla_X \varphi)Y = 0 \Leftrightarrow \nabla_X \xi = 0$$

for all vector fields X and Y on M.

Proof. Suppose that $\nabla\varphi = 0$, from Lemma 3.5 we have

$$g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi = 0,$$

taking $Y = \xi$ we obtain $\nabla_X \xi = 0$. The inverse is direct. □

Now, for the tensor N_φ, using (7) and (9), we can check that is very simply as follows:

$$N_\varphi(X, Y) = -2\mathrm{d}\eta(\varphi^2 X, \varphi^2 Y)\xi - 2\mathrm{d}\eta(X, Y)\xi, \tag{10}$$

witch give the following proposition:

Proposition 3.7. *The almost Golden almost para-contact metric structure (φ, ξ, η, g) is normal if and only if η is closed.*

Proof. Suppose that $N_\varphi = 0$, from formula (10), we get

$$2d\eta(X, Y) - \eta(Y)d\eta(X, \xi) - \eta(X)d\eta(\xi, Y) = 0 \tag{11}$$

for $Y = \xi$ we obtain

$$d\eta(X, \xi) = 0, \tag{12}$$

using (12) in (11) we get $d\eta = 0$. The inverse is direct. □

Remark 3.8. An almost Golden almost para-contact metric structure (φ, ξ, η, g) is called an almost Golden para-contact metric structure if it is normal.

In addition, we say that (M,φ,ξ,η,g) is an almost Golden para-contact metric manifold if the condition (6) is satisfies. Based on these facts, we give the following definition:

Definition 3.9. We say that (φ,ξ,η,g) is a Golden para-contact metric structure if and only if

$$d\eta = 0 \quad \text{and} \quad 2\Omega = \mathcal{L}_\xi g. \tag{13}$$

In this case (M,φ,ξ,η,g) is a Golden para-contact metric manifold.

Theorem 3.10. *Every almost Golden almost para-contact metric structure (φ,ξ,η,g) is a Golden para-contact metric structure if and only if*

$$\nabla_X \xi = \varphi X.$$

Proof. The proof is direct, just use the formulas (13). □

4. Golden α-para-Sasakian manifolds

In this section, a generalization of para-Sasakian manifolds is included. First of all, we're going to give the concept of Golden para-Sasakian manifolds.

Theorem 4.1. *Every Golden para-contact metric manifold (M,φ,ξ,η,g) is a para-Sasakian manifold.*

Proof. Suppose that (M,φ,ξ,η,g) is a Golden para-contact metric manifold. Thus we have

$$\varphi X = \varepsilon\big(-X + \eta(X)\xi\big) \quad \text{and} \quad \nabla_X\xi = \varphi X.$$

Knowing that $\nabla_X \varphi Y = (\nabla_X\varphi)Y + \varphi\nabla_X Y$, then

$$\begin{aligned}(\nabla_X\varphi)Y &= \varepsilon\big((\nabla_X\eta)(Y)\xi + \eta(Y)\nabla_X\xi\big)\\ &= \varepsilon\big(g(\nabla_X\xi, Y)\xi + \eta(Y)\nabla_X\xi\big)\\ &= -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.\end{aligned}$$

Which completes the proof. □

The notion of α-para-Sasakian manifolds as analogue of α-Sasakian manifolds [9]. It gives its definition as follows:

Definition 4.2. An α-para-Sasakian manifold is an almost para-contact metric manifold (M,φ,ξ,η,g) which satisfies:

$$(\nabla_X\varphi)Y = \alpha\big(-g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi\big), \tag{14}$$

where α is a function on M.

Example 4.3. We denote the Cartesian coordinates in a 3-dimensional Euclidean space E^3 by (x,y,z) and define a symmetric tensor field g by

$$g = \begin{pmatrix} f\mathrm{e}^{2z} & 0 & 0 \\ 0 & f\mathrm{e}^{-2z} & 0 \\ 0 & 0 & f^2 \end{pmatrix},$$

where $f = f(z)$ is a positive function on E^3.

Further, we define an almost para-contact metric structure (φ, ξ, η) on E^3 by

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{f} \end{pmatrix}, \quad \eta = (0, 0, f).$$

Using Koszul's formula for the metric g for the local orthonormal basis $\{\partial_i = \frac{\partial}{\partial x_i}\}$,

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_i, \partial_k) - \partial_k g(\partial_i, \partial_i),$$

we get

$$\nabla_{\partial x}\partial x = -\frac{\mathrm{e}^{2z}}{2f^2}\big(f' + 2f\big)\partial z, \qquad \nabla_{\partial x}\partial y = 0, \qquad \nabla_{\partial x}\partial z = \frac{1}{2f}\big(f' + 2f\big)\partial x,$$

$$\nabla_{\partial y}\partial x = 0, \qquad \nabla_{\partial y}\partial y = -\frac{\mathrm{e}^{-2z}}{2f^2}\big(f' - 2f\big)\partial z, \qquad \nabla_{\partial y}\partial z = \frac{1}{2f}\big(f' - 2f\big)\partial y,$$

$$\nabla_{\partial z}\partial x = \frac{1}{2f}\big(f' + 2f\big)\partial x, \qquad \nabla_{\partial z}\partial y = \frac{1}{2f^2}\big(f' - 2f\big)\partial z, \qquad \nabla_{\partial z}\partial z = \frac{f'}{f}\partial z,$$

where $f' = \frac{\partial f}{\partial z}$.

Knowing that

$$\big(\nabla_{\partial_i}\varphi\big)\partial_j = \nabla_{\partial_i}\varphi\partial_j - \varphi\nabla_{\partial_i}\partial_j,$$

one can easily check that

$$\big(\nabla_{\partial_i}\varphi\big)\partial_j = \frac{2f^2}{f' + 2f}\big(- g(\partial_i, \partial_j)\xi - \eta(\partial_j)\partial_i + 2\eta(\partial_i)\eta(\partial_j)\xi\big),$$

with $f \neq \mathrm{e}^{-2z}$, i.e., $(E^3, \varphi, \xi, \eta)$ is an α-para-Sasakian manifold, where $\alpha = \frac{2f^2}{f'+2f}$.

Proposition 4.4. *For every α-para-Sasakian manifold (M, Φ, ξ, η, g) we have*

$$\nabla_X \xi = \alpha\varphi X \tag{15}$$

for all vector field X on M.

Proof. Putting $Y = \xi$ in formula (14), we get

$$\begin{aligned} \big(\nabla_X\varphi\big)\xi = \alpha(-X + \eta(X)\xi) &\Leftrightarrow \varphi\nabla_X\xi = \alpha(X - \eta(X)\xi) \\ &\Leftrightarrow \nabla_X\xi = \alpha\varphi X. \end{aligned}$$

□

The famous Eq. (15) gives important informations about the curvature properties of α-para-Sasakian manifold. We start with the first proposition:

Proposition 4.5. *Let (M, Φ, ξ, η, g) be an n-dimensional α-para-Sasakian manifold. Then we have*

$$\begin{aligned} R(X, Y)\xi &= \alpha^2\big(\eta(X)Y - \eta(Y)X\big) + X(\alpha)\varphi Y - Y(\alpha)\varphi X, \\ R(X, \xi)Y &= \alpha^2\big(g(X, Y)\xi - \eta(Y)X\big) + g(X, \varphi Y)\mathrm{grad}\alpha - Y(\alpha)\varphi X, \end{aligned} \tag{16}$$

$$S(X,\xi)=\alpha^2(1-n)\eta(X)+\varphi X(\alpha)-X(\alpha)tr_g\varphi \tag{17}$$

for all vector fields X and Y on M and S denotes the Ricci curvature and R is the curvature tensor defined by:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \tag{18}$$

and

$$S(X,Y)=\sum_{i=1}^{n} g\big(R(X,e_i)e_i,Y\big),$$

where $\{e_i\}_{1\leq i\leq n}$ is an orthonormal basis on M.

Proof. The relation (16) follows from (18) and (15) with $Z=\xi$. For the second relation, we have for all vector fields X,Y,Z on M,

$$g\big(R(X,\xi)Y,Z\big)=-g\big(R(Y,Z)\xi,X\big),$$

and using (16). Finally, knowing that

$$\begin{aligned}S(X,\xi)&=\sum_{i=1}^{n} g\big(R(X,e_i)e_i,\xi\big)\\&=-\sum_{i=1}^{n} g\big(R(X,e_i)\xi,e_i\big),\end{aligned}$$

and using (16), we obtain (17). This completes the proof of the proposition. □

Definition 4.6. Let (M,φ,ξ,η,g) be an almost Golden almost para-contact metric manifold. If

$$\mathrm{d}\eta=0 \quad\text{and}\quad 2\alpha\Omega=\mathcal{L}_\xi g, \tag{19}$$

then, M is called a Golden α-para-Sasakian manifold. For $\alpha=1$, we get a Golden para-Sasakian manifold.

Theorem 4.7. *Let (M,φ,ξ,η,g) be an almost Golden almost para-contact metric manifold. Then M is a Golden α-para-Sasakian manifold if and only if*

$$\nabla_X\xi=\alpha\varphi X$$

for all vector field X on M.

Proof. The proof is direct. □

Theorem 4.8. *Every generalized $\mathcal{G}$-Golden manifold (M,Φ,ξ,η,g) induces an Golden α-para-Sasaki manifold (M,φ,ξ,η,g) and conversely.*

Proof. From formula (8), we obtain

$$(\nabla_X\Phi)Y=\frac{\sqrt{5}}{2}\Big(\varepsilon(\nabla_X\varphi)Y+g(\nabla_X\xi,Y)\xi+\eta(Y)\nabla_X\xi\Big). \tag{20}$$

Suppose that (M,Φ,ξ,η,g) is a generalized $\mathcal{G}$-Golden manifold. Using formulas (4) and (5), we get

$$(\nabla_X\varphi)Y=-\varepsilon\sigma\big(-g(X,Y)\xi-\eta(Y)X+2\eta(X)\eta(Y)\xi\big),$$

i.e., $(M, \varphi, \xi, \eta, g)$ is a Golden α-para-Sasaki manifold with $\alpha = -\varepsilon\sigma$.

Conversely, suppose that $(M, \varphi, \xi, \eta, g)$ is a Golden α-para-Sasaki manifold. From formulas (20), (14) and (15), we obtain

$$(\nabla_X \Phi)Y = -\varepsilon\alpha\sqrt{5}\big(g(X,Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi\big),$$

i.e., (M, Φ, ξ, η, g) is a generalized $\mathcal{G}$-Golden manifold with $\sigma = -\varepsilon\alpha$. □

5. A class of examples

Let $\mathbb{R}^{n+1}$ be an Euclidean space with Cartesian coordinates $\{x_1, \dots, x_n, z\}$. We put

$$\xi = \frac{\partial}{\partial z},$$
$$\eta = dz - \tau dx_1,$$
$$\varphi = \varepsilon\big(-I + \eta \otimes \xi\big),$$
$$g = \eta \otimes \eta + \rho^2 \sum_{i=1}^{n} dx_i^2,$$

where ρ and τ are two functions on $\mathbb{R}^{n+1}$.

It's clear that (φ, ξ, η, g) is an almost Golden almost para-contact metric structure.

In addition,

$$\mathrm{d}\eta = \sum_{k=2}^{n} \tau_k dx_1 \wedge dx_k \quad \text{with} \quad \tau_k = \frac{\partial \tau}{\partial x_k}.$$

So, if $\tau_k = 0$ for all $k \in \{2, \dots, n\}$ (i.e., $\tau = \tau(x_1)$) then, (φ, ξ, η, g) is a Golden almost para-contact metric structure.

Now, for the metric g, form the G-basis $\{\mathrm{e}_1, \mathrm{e}_i, \xi\}$ where $i \in \{2, \dots, n\}$ as follows:

$$e_1 = \frac{1}{\rho}\Big(\frac{\partial}{\partial x_1} + \tau\xi\Big), \quad e_i = \frac{1}{\rho}\frac{\partial}{\partial x_i}.$$

Taking into account the above condition, i.e., $\tau = \tau(x_1)$, one can find:

$$[e_1, e_i] = \frac{\rho_i}{\rho^2} e_1 - \frac{\rho_1 + \tau\xi(\rho)}{\rho^2} e_i, \qquad [e_1, \xi] = \frac{\rho'}{\rho} e_1,$$

$$[e_i, e_j] = \frac{\rho_j}{\rho^2} e_i - \frac{\rho_i}{\rho^2} e_j, \qquad [e_i, \xi] = \frac{\rho'}{\rho} e_i,$$

where $\rho' = \xi(\rho) = \frac{\partial \rho}{\partial z}$ and $\rho_i = \frac{\partial \rho}{\partial x_i}$.

Using Koszul's formula for the metric g,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y], \end{aligned}$$

we obtain:

$$\nabla_{e_1} e_1 = -\frac{\rho_i}{\rho^2} e_i - \frac{\rho'}{\rho}\xi, \quad \nabla_{e_1} e_i = \frac{\rho_i}{\rho^2} e_1, \quad \nabla_{e_1}\xi = \frac{\rho'}{\rho} e_1,$$

$$\nabla_{e_i} e_j = -\frac{\rho_1 + \tau\xi(\rho)}{\rho^2}\delta_{ij}\psi_1 + \frac{1}{\rho^2}\sum_{k=2}^{n}(\rho_j\delta_{ik} - \rho_k\delta_{ij})e_k - \frac{\rho'}{\rho}\delta_{ij}\xi,$$

$$\nabla_{e_i} e_1 = \frac{\rho_1 + \tau\xi(\rho)}{\rho^2}e_i, \quad \nabla_{e_i}\xi = \frac{\rho'}{\rho}e_i, \quad \nabla_\xi e_1 = \nabla_\xi e_i = \nabla_\xi \xi = 0.$$

On the other hand, with a simple calculation, we can find

$$\Omega = -\varepsilon \sum_{i=1}^{n} \mathrm{d}x_i \otimes \mathrm{d}x_i,$$

and

$$\mathcal{L}_\xi g = \frac{2\rho'}{\rho}\sum_{i=1}^{n} \mathrm{d}x_i \otimes \mathrm{d}x_i,$$

using formulas (19) or Theorem 4.7, we have immediately that, $(\mathbb{R}^{n+1}, \varphi, \xi, \eta, g)$ is a Golden α-para-Sasakian manifold if and only if $\tau = \tau(x_1)$ and $\alpha = -\varepsilon\frac{\rho'}{\rho}$. For $\alpha = 1$, $(\mathbb{R}^{n+1}, \varphi, \xi, \eta, g)$ is a Golden para-Sasakian manifold if and only if $\tau = \tau(x_1)$ and $\rho = c\mathrm{e}^{-\varepsilon z}$ where $c \in \mathbb{R}$.

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