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GROUND STATE SIGN-CHANGING SOLUTIONS FOR NONLINEAR SCHRÖDINGER-POISSON SYSTEM WITH INDEFINITE POTENTIALS

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Abstract. This paper is concerned with the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $4 < p < 6$. For the case that K is nonnegative, V and a are indefinite, we prove the above problem possesses one ground state sign-changing solution with exactly two nodal domains by constraint variational method and quantitative deformation lemma. Moreover, we show that the energy of sign-changing solutions is larger than that of the ground state solutions. The novelty of this paper is that the potential a is indefinite and allowed to vanish at infinity. In this sense, we complement the existing results obtained by Batista and Furtado [5].

1. Introduction

In recent years, the following Schrödinger-Poisson system

$$(1.1) \qquad \begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

has attached considerable attention both in physics and mathematics. From the physical point of view, system (1.1) has a great importance in the study of standing wave solutions $e^{-i\omega t}u(x)$ of the time-dependent Schrödinger-Poisson system. Here, the potential $V(x)$ represents the perturbation of the particle at point $x \in \mathbb{R}^3$, the function $K(x)$ is a measurable function representing a charge corrector to the density u^2, and the local nonlinearity $f(x,u)$ simulates the interaction effect among many particles. As is known to all, problem (1.1)

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is a nonlocal problem due to the appearance of the term $K(x)\phi u$, which causes many mathematical difficulties and makes the study of system (1.1) particularly interesting. For more details on the mathematical and physical background of system (1.1), we refer the reader to the papers [1, 2, 21, 22] and the references listed therein.

As far as system (1.1) is concerned, the studies related to it mainly focus on the existence of positive solutions, sign-changing (nodal) solutions, radial solutions and semiclassical states under variant assumptions on $V(x)$, $K(x)$ and $f(x,u)$ via variational methods, see, for instance, [3, 4, 8–12, 16–19, 23–27, 31] and the references therein. Motivated by the above results, especially [5] and [10], in present paper, we continue to discuss the sign-changing solutions for system (1.1). To present our hypotheses conveniently, we give brief description to those mentioned in [5] and [10]. Explicitly, Chen and Tang [10] studied the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = a(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where V, a are positive continuous potentials, f is a continuous function and λ is a positive parameter. By developing a direct approach, the authors obtained the existence results on the ground state sign-changing solutions. As just mentioned, the results in [10] depend heavily on the positiveness of V and a. Hence, it is natural to ask whether there admits ground-state sign-changing solutions to system (1.1) if the potentials are allowed to change sign. For this topic, to our best knowledge, only the recent paper [5] is related to it. Indeed, Batista and Furtado [5] investigated the following system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $4 < p < 6$, the potentials V and a can be indefinite, K is nonnegative. Explicitly, the potential functions V, K and a are supposed to satisfy the following hypothesis:

(V_0) $V^- \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} |V^-(x)|^{\frac{3}{2}} dx < S^{\frac{3}{2}}$;

(V_1) there exist $\gamma > 0$ and $C_V > 0$ such that

$$V(x) \le V_\infty - C_V e^{-\gamma|x|}$$

for a.e. $x \in \mathbb{R}^3$, where $V^-(x) := \min\{V(x), 0\}$, $V^+(x) := \max\{V(x), 0\}$, $0 < V_\infty^+ := \lim_{|x|\to\infty} V(x)$ and S is the best Sobolev constant of the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$;

(K_0) $K \in L^2(\mathbb{R}^3)$ is nonnegative;

(K_1) there exist $\mu > 0$ and $C_K > 0$ such that

$$0 \le K(x) \le C_K e^{-\mu|x|}$$

for a.e. $x \in \mathbb{R}^3$;

(a_0) $a \in L^\infty(\mathbb{R}^3)$;

(a_1) there exist $\theta > 0$ and $C_a > 0$ such that

$$a(x) \geq a_\infty - C_a e^{-\theta|x|}$$

for a.e. $x \in \mathbb{R}^3$, where $a_\infty := \lim_{|x|\to\infty} a(x) > 0$.

Based on the above assumptions, the authors in [5] proved that problem (1.2) possesses one sign-changing minimal solution (ground state sign-changing solution) by using some comparison argument of the minimax level of the energy functional and that of the limit problem corresponding to system (1.2).

From the comparison techniques used in [5], we observe that the existence of positive ground state solutions to the limit problem corresponding to system (1.2) plays an essential role, namely, the following limit equation

$$-\Delta u + V_\infty u = a_\infty |u|^{p-1} u. \tag{1.3}$$

It is obvious that if $a_\infty = 0$, then Eq. (1.3) has only trivial solution. Therefore, the method applied in [5] is no longer applicable to this situation. Inspired by the above observation, the purpose of this paper is to consider the existence of ground state sign-changing solutions to system (1.2) for the case that the potential a is allowed to vanish at infinity. Explicitly, in this paper the assumptions related to V, K and a are presented as below:

(V) $V^- \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} |V^-(x)|^{\frac{3}{2}} dx < S^{\frac{3}{2}}$ and $0 < V_\infty^+ := \lim_{|x|\to+\infty} V(x)$;

(K) $K \in L^2(\mathbb{R}^3)$ is nonnegative;

(a) $a \in C(\mathbb{R}^3) \cap L^{\frac{6}{6-p}}(\mathbb{R}^3)$ is indefinite.

Theorem 1.1. *Suppose that $4 < p < 6$ and the potentials satisfy (V), (K) and (a). Then problem (1.2) possesses one ground state sign-changing solution, which has precisely two nodal domains.*

Remark 1.2. Obviously, (V) is weaker than (V_0) and (V_1), since it does not need the exponent decay at infinity as in (V_1); (K) and (K_0) are the same; the condition (a) evidently implies that it could not be $\lim_{|x|\to\infty} a(x) = a_\infty > 0$. In fact, if $\lim_{|x|\to\infty} a(x)$ exists, then it must be $a_\infty = 0$. Thus, our Theorem 1.1 complements the results obtained in [5].

Next, we introduce some notations and establish the variational framework to deal with problem (1.2). For any $2 \leq q \leq \infty$, $\|\cdot\|_q$ denotes the norm of usual Lebesgue space $L^q(\mathbb{R}^3)$, and let $\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}$ be the norm of $D^{1,2}(\mathbb{R}^3)$. Define

$$H := \left\{ u \in W^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 < \infty \right\},$$

then, according to [13, Lemma 2.1] and [5, Section 2], the assumption (V) guarantees that the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x) u^2 \right)^{1/2}$$

is well defined and is equivalent to $\|\cdot\|_{W^{1,2}(\mathbb{R}^3)}$. Therefore, the embedding of $H \hookrightarrow L^q(\mathbb{R}^3)$ is continuous for $q \in [2, 6]$.

To deal with problem (1.2) applying for variational methods, the key observation is that it can be transformed into a Schrödinger equation with a nonlocal term. Effectively, for $K \in L^2(\mathbb{R}^3)$, in view of the Lax-Milgram Theorem, for each $u \in H$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi = K(x)u^2$, and ϕ_u is of the form

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}dy,$$

see [21]. Inserting this ϕ_u into system (1.2), we can rewrite it into the following equivalent scalar equation

$$-\Delta u + V(x)u + K(x)\phi_u u = a(x)|u|^{p-2}u.$$

Therefore, for problem (1.2), the associated energy functional is defined on H as follows

$$I(u) := \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx.$$

Obviously, $I \in C^1(H, \mathbb{R})$ and

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla\varphi + V(x)u\varphi)dx + \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx - \int_{\mathbb{R}^3} a(x)|u|^{p-2}u\varphi dx$$

for all $u, \varphi \in H$. Clearly, the critical points of I correspond to the weak solutions of system (1.2). Furthermore, if $u \in H$ is a solution of system (1.2) with $u^{\pm} \neq 0$, then u is a sign-changing solution, where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$.

Remark 1.3. To obtain the existence of ground state sign-changing solutions of system (1.2), the usual strategy is to deal with the minimizing problem as follows

$$m := \inf_{u\in\mathcal{M}} I(u), \tag{1.4}$$

where $\mathcal{M}$ is the sign-changing Nehari manifold defined by

$$\mathcal{M} = \{u \in H : u^{\pm} \neq 0, \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\}.$$

This is also the scheme we intent to employ to our Theorem 1.1. However, for our situation, due to the facts that the potentials V and a are indefinite and the corresponding limit equation could not give us any help, the techniques involving the potential a in existing literature cannot be applied directly to our case. Having this in our mind and observing the key steps for sign-changing solutions, the first step we need to do is to check that $\mathcal{M} \neq \emptyset$. To this end, we introduce the set $\mathcal{A}$ (see (2.1) for its definition). Once $\mathcal{A} \neq \emptyset$ is verified, the argument in Lemma 2.4 below guarantees that $\mathcal{M} \neq \emptyset$. Meanwhile, the condition (a) ensures that $m = \inf_{u\in\mathcal{M}} I(u)$ is achieved by some $u_0 \in \mathcal{M}$.

Finally, applying quantitative deformation lemma, we show that u_0 is a ground state sign-changing solution.

Another aim of this paper is to show that the energy of ground state sign-changing solutions of system (1.2) is larger than two times of the least energy. This property is called *energy doubling* by Weth in [28] and has not been considered for Schrödinger-Poisson system with indefinite potentials. For this purpose, we need to consider the following minimizing problem

$$c := \inf_{u \in \mathcal{N}} I(u), \tag{1.5}$$

where $\mathcal{N}$ is the Nehari manifold corresponding to system (1.2) defined by

$$\mathcal{N} := \{u \in H \backslash \{0\} : \langle I'(u), u \rangle = 0\}. \tag{1.6}$$

Theorem 1.4. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then*

$$I(u_0) \geq 2c,$$

where u_0 is the ground state sign-changing solution obtained in Theorem 1.1. In particular, c is achieved by a nonnegative function.

This paper is organized as follows. In Section 2, some useful preliminary lemmas are presented to pave the way for ground state sign-changing solutions. Then, Section 3 is devoted to finish the proofs of our main results.

2. Preliminaries

We begin this section by listing some properties related to ϕ_u (see [6, Lemma 2.1], [14, Lemma 2.2] and [15, Lemma 2.3]).

Lemma 2.1. *For any $u \in H$, ϕ_u has following properties*:

(i) $\phi_u \geq 0$ *for a.e.* $x \in \mathbb{R}^3$ *and* $\phi_{tu} = t^2 \phi_u$, $\forall t > 0$;

(ii) *there exists* $C > 0$ *independent of* u *such that* $\int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx \leq C \|u\|_6^2$;

(iii) *if* $u_n \rightharpoonup u$ *in* H *and* $u_n \to u$ *a.e. in* $\mathbb{R}^3$, *then*

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} K(x) \phi_u u^2 dx,$$
$$\lim_{n \to \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n} (u_n^{\pm})^2 dx = \int_{\mathbb{R}^3} K(x) \phi_u (u^{\pm})^2 dx,$$
$$\lim_{n \to \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n \varphi dx = \int_{\mathbb{R}^3} K(x) \phi_u u \varphi dx, \ \forall \varphi \in H.$$

Hereafter, we use the letter C to denote a positive constant whose value may change from line to line. Next, we recall one convergence conclusion, which plays a crucial role in checking the reachability of m. For its proof, we refer the reader to [30, Lemma 2.1] or [7, Lemma 2.3].

Lemma 2.2. *Assume that* (a) *holds and* $u_n \rightharpoonup u$ *in* H, $u_n \to u$ *a.e. in* $\mathbb{R}^3$. *Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} a(x)|u_n|^p dx = \int_{\mathbb{R}^3} a(x)|u|^p dx$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} a(x)|u_n|^{p-2}u_n\varphi dx = \int_{\mathbb{R}^3} a(x)|u|^{p-2}u\varphi dx,\ \forall \varphi \in H.$$

In the following, we check that the sign-changing Nehari manifold $\mathcal{M}$ is non-empty. To this end, define $\mathcal{A}$ as follows:

$$\mathcal{A} := \left\{ u \in H : u^\pm \neq 0, \int_{\mathbb{R}^3} a(x)|u^\pm|^p dx > 0 \right\}. \tag{2.1}$$

Lemma 2.3. *Assume that* (V), (K) *and* (a) *hold. Then* $\mathcal{A} \neq \emptyset$ *and* $\mathcal{M} \subset \mathcal{A}$.

Proof. Clearly, $\mathcal{A} \neq \emptyset$. To see it, we pick some $u \in H$ with $\mathrm{supp}(u) \subset a_+$ where $a_+ := \{x \in \mathbb{R}^3 : a(x) > 0\}$. Then, $u \in \mathcal{A}$, namely, $\mathcal{A} \neq \emptyset$. Therefore, to reach the conclusion of Lemma 2.3, it only needs to verify $\mathcal{M} \subset \mathcal{A}$. In fact, for $u \in \mathcal{M}$, that is, $u^\pm \neq 0$ and $\langle I'(u), u^\pm\rangle = 0$, it gives that

$$\|u^\pm\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u (u^\pm)^2 dx = \int_{\mathbb{R}^3} a(x)|u^\pm|^p dx.$$

The positiveness of the left-hand side of the above equality yields that $u \in \mathcal{A}$, that is to say, $\mathcal{M} \subset \mathcal{A}$. □

Now, with the help of the elements in $\mathcal{A}$, it can be checked that $\mathcal{M} \neq \emptyset$. To demonstrate this point and discuss some properties of elements in $\mathcal{M}$ conveniently, define $\psi_u : \mathbb{R}^2_+ \to \mathbb{R}$ by $\psi_u(s,t) = I(su^+ + tu^-)$ for $u \in H$ with $u^\pm \neq 0$.

Lemma 2.4. *Assume that* (V), (K) *and* (a) *hold. Then, for any* $u \in \mathcal{A}$, ψ_u *has the following properties*:

1. *the pair* (s,t) *is a critical point of* ψ_u *with* $s,t > 0$ *if and only if* $su^+ + tu^- \in \mathcal{M}$;
2. *the map* ψ_u *has a unique critical point* (s_u, t_u) *on* $(0,\infty)\times(0,\infty)$, *which is also the unique maximum point of* ψ_u *on* $[0,\infty)\times[0,\infty)$; *furthermore, if* $\langle I'(u), u^\pm\rangle \le 0$, *then* $0 < s_u, t_u \le 1$.

Proof. Note that

$$\begin{aligned}\nabla\psi_u(s,t) &= (\langle I'(su^+ + tu^-), u^+\rangle, \langle I'(su^+ + tu^-), u^-\rangle)\\ &= \left(\frac{1}{s}\langle I'(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle I'(su^+ + tu^-), tu^-\rangle\right),\end{aligned}$$

it is obvious that (1) holds.

Next, we prove (2). Firstly, we show the existence of s_u and t_u, namely, $\mathcal{M} \neq \emptyset$. From (a) and the Hölder's inequality, we obtain that

$$\int_{\mathbb{R}^3} a(x)|u^\pm|^p dx \le \|a\|_{\frac{6}{6-p}} \|u^\pm\|_6^p. \tag{2.2}$$

Then, Sobolev embedding theorem and (2.2) deduce that

$$\begin{aligned}\langle I'(su^+ + tu^-), su^+\rangle &\geq s^2\|u^+\|^2 - s^p\|a\|_{\frac{6}{6-p}}\|u^+\|_6^p \\ &\geq s^2\|u^+\|^2 - Cs^p\|u^+\|^p.\end{aligned}$$

Since $4 < p < 6$, the above inequality implies that $\langle I'(su^+ + tu^-), su^+\rangle > 0$ for s small enough and all $t > 0$. Similarly, we obtain that $\langle I'(su^+ + tu^-), tu^-\rangle > 0$ for t small enough and all $s > 0$. Therefore, there exists $\delta_1 > 0$ such that

$$\langle I'(\delta_1 u^+ + tu^-), \delta_1 u^+\rangle > 0 \text{ and } \langle I'(su^+ + \delta_1 u^-), \delta_1 u^-\rangle > 0 \text{ for all } s, t > 0. \tag{2.3}$$

On the other hand, based on $u \in \mathcal{A}$ and $4 < p < 6$, we can choose $s = \delta_2' > \delta_1$ large enough to guarantee that

$$\begin{aligned}\langle I'(\delta_2' u^+ + tu^-), \delta_2' u^+\rangle &\leq (\delta_2')^2\|u^+\|^2 + (\delta_2')^4\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^+)^2 dx \\ &\quad + (\delta_2')^4\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^+)^2 dx - (\delta_2')^p\int_{\mathbb{R}^3} a(x)|u^+|^p dx \\ &< 0 \quad \text{for } t \in [\delta_1, \delta_2'].\end{aligned}$$

Similarly, we can take $t = \delta_2'' > \delta_1$ large enough to ensure that

$$\begin{aligned}\langle I'(su^+ + \delta_2'' u^-), \delta_2'' u^-\rangle &\leq (\delta_2'')^2\|u^-\|^2 + (\delta_2'')^4\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^-)^2 dx \\ &\quad + (\delta_2'')^4\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^-)^2 dx - (\delta_2'')^p\int_{\mathbb{R}^3} a(x)|u^-|^p dx \\ &< 0 \quad \text{for } s \in [\delta_1, \delta_2''].\end{aligned}$$

For $\delta_2 = \max\{\delta_2', \delta_2''\}$, it deduces that

$$\langle I'(\delta_2 u^+ + tu^-), \delta_2 u^+\rangle < 0 \text{ and } \langle I'(su^+ + \delta_2 u^-), \delta_2 u^-\rangle < 0 \tag{2.4}$$

for all $s, t \in [\delta_1, \delta_2]$.

Combining (2.3), (2.4) with Miranda's Theorem ([20]), there exists $(s_u, t_u) \in (0, \infty) \times (0, \infty)$ such that $\langle I'(su^+ + tu^-), su^+\rangle = \langle I'(su^+ + tu^-), tu^-\rangle = 0$, i.e., $s_u u^+ + t_u u^- \in \mathcal{M}$.

In the following, we prove the uniqueness of the pair (s_u, t_u).

Case 1. $u \in \mathcal{M}$. It is sufficient to show that $(s_u, t_u) = (1, 1)$ is the unique pair of numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$. Let (s_0, t_0) be a pair of numbers such that $s_0 u^+ + t_0 u^- \in \mathcal{M}$. Then, one has

$$\begin{aligned}&s_0^2\|u^+\|^2 + s_0^4\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^+)^2 dx + s_0^2 t_0^2\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^+)^2 dx \\ &= s_0^p\int_{\mathbb{R}^3} a(x)|u^+|^p dx\end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
&t_0^2\|u^-\|^2 + t_0^4\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^-)^2dx + s_0^2t_0^2\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^-)^2dx \\
&= t_0^p\int_{\mathbb{R}^3} a(x)|u^-|^pdx.
\end{aligned}
\tag{2.6}
$$

Assume that $0 < s_0 \le t_0$, it holds from (2.6) that

$$
\begin{aligned}
&\frac{\|u^-\|^2}{t_0^2} + \int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^-)^2dx + \int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^-)^2dx \\
&\ge t_0^{p-4}\int_{\mathbb{R}^3} a(x)|u^-|^pdx.
\end{aligned}
\tag{2.7}
$$

In addition, since $u \in \mathcal{M}$, it means that

$$
\|u^\pm\|^2+\int_{\mathbb{R}^3} K(x)\phi_{u^\pm}(u^\pm)^2dx+\int_{\mathbb{R}^3} K(x)\phi_{u^\mp}(u^\pm)^2dx=\int_{\mathbb{R}^3} a(x)|u^\pm|^pdx.
\tag{2.8}
$$

Therefore, together with (2.7) and (2.8), it readily yields that

$$
\left(\frac{1}{t_0^2}-1\right)\|u^-\|^2 \ge (t_0^{p-4}-1)\int_{\mathbb{R}^3} a(x)|u^-|^pdx.
$$

If $t_0 > 1$, the left-hand side of the above inequality is negative, which is absurd because the right-hand side is positive due to $4 < p < 6$. Therefore, it indicates that $0 < s_0 \le t_0 \le 1$. Similarly, in light of (2.5) and $0 < s_0 \le t_0$, we can conclude that $s_0 \ge 1$. Consequently, $s_0 = t_0 = 1$.

Case 2. $u \notin \mathcal{M}$. Suppose that there exist (s_1, t_1) and (s_2, t_2) such that $u_1 = s_1u^+ + t_1u^- \in \mathcal{M}$ and $u_2 = s_2u^+ + t_2u^- \in \mathcal{M}$, respectively. Observe that

$$
u_2 = \left(\frac{s_2}{s_1}\right)s_1u^+ + \left(\frac{t_2}{t_1}\right)t_1u^- = \left(\frac{s_2}{s_1}\right)u_1^+ + \left(\frac{t_2}{t_1}\right)u_1^- \in \mathcal{M}.
$$

then it follows from **Case 1** that $\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1$, thanks to $u_1 \in \mathcal{M}$.

Now, it turns to prove that (s_u, t_u) is the unique maximum point of ψ_u on $[0,\infty)\times[0,\infty)$. In fact, according to the following expression

$$
\begin{aligned}
\psi_u(s,t) &= I(su^+ + tu^-) \\
&= \frac{s^2}{2}\|u^+\|^2 + \frac{t^2}{2}\|u^-\|^2 + \frac{s^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^+)^2dx \\
&\quad + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^-)^2dx + \frac{s^2t^2}{4}\int_{\mathbb{R}^3} K(x)\phi_{u^-}(u^+)^2dx \\
&\quad + \frac{s^2t^2}{4}\int_{\mathbb{R}^3} K(x)\phi_{u^+}(u^-)^2dx - \frac{s^p}{p}\int_{\mathbb{R}^3} a(x)|u^+|^pdx \\
&\quad - \frac{t^p}{p}\int_{\mathbb{R}^3} a(x)|u^-|^pdx,
\end{aligned}
$$

it is evident that $\lim_{|(s,t)|\to\infty}\psi_u(s,t) = -\infty$. Since we already know that (s_u, t_u) is the unique critical point of ψ_u on $(0,\infty)\times(0,\infty)$, it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. Without loss of generality, we may assume that $(0,t_0)$ is a maximum point of ψ_u. Then, for $s>0$ small enough, it is obvious that

$$(\psi_u)'_s(s,t_0)=s\|u^+\|^2+s^3\int_{\mathbb{R}^3}K(x)\phi_{u^+}(u^+)^2dx+\frac{st_0^2}{2}\int_{\mathbb{R}^3}K(x)\phi_{u^-}(u^+)^2dx$$
$$+\frac{st_0^2}{2}\int_{\mathbb{R}^3}K(x)\phi_{u^+}(u^-)^2dx-s^{p-1}\int_{\mathbb{R}^3}a(x)|u^+|^pdx>0,$$

which indicates that ψ_u is increasing with respect to s when s is small enough. In other words, ψ_u can not achieve its global maximum on $(0,t_0)$ with $t_0>0$.

To accomplish this lemma, it remains to show that $0<s_u,t_u\le 1$ if $\langle I'(u),u^\pm\rangle\le 0$. Suppose $s_u\ge t_u>0$, since $s_uu^++t_uu^-\in\mathcal{M}$, it follows that

$$\begin{aligned}&s_u^2\|u^+\|^2+s_u^4\int_{\mathbb{R}^3}K(x)\phi_{u^+}(u^+)^2dx+s_u^4\int_{\mathbb{R}^3}K(x)\phi_{u^-}(u^+)^2dx\\ &\ge s_u^2\|u^+\|^2+s_u^4\int_{\mathbb{R}^3}K(x)\phi_{u^+}(u^+)^2dx+s_u^2t_u^2\int_{\mathbb{R}^3}K(x)\phi_{u^-}(u^+)^2dx\\ &=s_u^p\int_{\mathbb{R}^3}a(x)|u^+|^pdx.\end{aligned}\tag{2.9}$$

Besides, the assumption $\langle I'(u),u^+\rangle\le 0$ gives that

$$\|u^+\|^2+\int_{\mathbb{R}^3}K(x)\phi_{u^+}(u^+)^2dx+\int_{\mathbb{R}^3}K(x)\phi_{u^-}(u^+)^2dx\le\int_{\mathbb{R}^3}a(x)|u^+|^pdx.\tag{2.10}$$

So, by virtue of $u\in\mathcal{A}$, (2.9) and (2.10), the same argument in **Case 1** brings that $s_u\le 1$. Thus, we have $0<t_u\le s_u\le 1$. □

Lemma 2.5. *Assume that (V), (K) and (a) hold. Then $m>0$ is achieved.*

Proof. Taking into account that $\langle I'(u),u^\pm\rangle=0$, $\forall u\in\mathcal{M}$, (2.2), Lemma 2.1(i), and Sobolev embedding theorem, we obtain that

$$\begin{aligned}\|u^\pm\|^2&\le\|u^\pm\|^2+\int_{\mathbb{R}^3}K(x)\phi_u(u^\pm)^2dx\\ &=\int_{\mathbb{R}^3}a(x)|u^\pm|^pdx\le\|a\|_{\frac{6}{6-p}}\|u^\pm\|_6^p\le C\|u^\pm\|^p,\end{aligned}$$

which leads to

$$\|u^\pm\|^2\ge\left(\frac{1}{C}\right)^{2/(p-2)}:=\alpha>0.\tag{2.11}$$

In addition, for all $u\in\mathcal{M}\subset\mathcal{N}$, one has

$$I(u)=I(u)-\frac{1}{4}\langle I'(u),u\rangle=\frac{1}{4}\|u\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\mathbb{R}^3}a(x)|u|^pdx\ge\frac{1}{4}\|u\|^2.\tag{2.12}$$

Consequently, (2.11) and (2.12) bring that $m\ge\frac{\alpha}{2}>0$.

Let $\{u_n\} \subset \mathcal{M}$ be a sequence such that $I(u_n) \to m$. It is easy to see that $\{u_n\}$ is bounded in H. Therefore, there exists $u_0 \in H$ such that, passing to a subsequence if necessary,

$$u_n^{\pm} \rightharpoonup u_0^{\pm} \text{ in } H \quad \text{and} \quad u_n^{\pm} \to u_0^{\pm} \text{ a.e. in } \mathbb{R}^3. \tag{2.13}$$

Meanwhile, since $u_n \in \mathcal{M}$, then same to (2.11) it also holds that $\|u_n^{\pm}\|^2 \geq \alpha$. Thus, on the observation of

$$\|u_n^{\pm}\|^2 \leq \|u_n^{\pm}\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^{\pm})^2 dx = \int_{\mathbb{R}^3} a(x)|u_n^{\pm}|^p dx,$$

it follows from (2.13) and Lemma 2.2 that

$$\int_{\mathbb{R}^3} a(x)|u_0^{\pm}|^p dx \geq \alpha > 0, \tag{2.14}$$

which means that $u_0^{\pm} \neq 0$. Furthermore, since $\{u_n\} \subset \mathcal{M}$, applying Lemma 2.1, Lemma 2.2 and the weakly lower semicontinuity of norm, we have

$$\begin{aligned}
\|u_0^{\pm}\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_0}(u_0^{\pm})^2 dx &\leq \liminf_{n\to\infty}\left[\|u_n^{\pm}\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^{\pm})^2 dx\right] \\
&= \int_{\mathbb{R}^3} a(x)|u_0^{\pm}|^p dx,
\end{aligned}$$

which yields that

$$\langle I'(u_0), u_0\rangle \leq 0.$$

At this point, using Lemma 2.4(2), we know that there exists $(s_0, t_0) \in (0,1] \times (0,1]$ such that $s_0 u_0^+ + t_0 u_0^- \in \mathcal{M}$. Hence, in view of (2.14) and the weakly lower semicontinuity of norm, we infer that

$$\begin{aligned}
m &\leq I(s_0 u_0^+ + t_0 u_0^-) \\
&= I(s_0 u_0^+ + t_0 u_0^-) - \frac{1}{4}\langle I'(s_0 u_0^+ + t_0 u_0^-), s_0 u_0^+ + t_0 u_0^-\rangle \\
&= \frac{s_0^2}{4}\|u_0^+\|^2 + \frac{t_0^2}{4}\|u_0^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) s_0^p \int_{\mathbb{R}^3} a(x)|u_0^+|^p dx \\
&\quad + t_0^p \left(\frac{1}{4} - \frac{1}{p}\right) t_0^p \int_{\mathbb{R}^3} a(x)|u_0^-|^p dx \\
&\leq \frac{1}{4}\|u_0\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \int_{\mathbb{R}^3} a(x)|u_0|^p dx \\
&\leq \liminf_{n\to\infty}\left[I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right] = m.
\end{aligned}$$

Then, we conclude that $s_0 = t_0 = 1$. Thus, $u_0 = u_0^+ + u_0^- \in \mathcal{M}$ and $I(u_0) = m$. $\square$

3. Proofs of the main results

In this section we focus our attention to complete the proofs of Theorems 1.1 and 1.4. We first prove that the minimizer u_0 obtained in Lemma 2.5 for minimization problem (1.4) is indeed a sign-changing solution of system (1.2), that is, $I'(u_0) = 0$. Although this procedure is standard, we still give the details for the reader's convenience as well as the completeness of our article.

Proof of Theorem 1.1. Arguing by contradiction, if $I'(u_0) \neq 0$, by the continuity of I', there exist $\gamma, \delta > 0$ such that

$$\|I'(u)\| \geq \gamma, \ \forall u \in H \text{ with } \|u - u_0\| \leq 3\delta. \tag{3.1}$$

Choose $\sigma \in (0, \min\{\frac{1}{2}, \frac{\delta}{\sqrt{2}\|u_0\|}\})$. Let $D := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and define the function $g : D \to H$ by $g(s,t) = su_0^+ + tu_0^-$. Since $u_0 \in \mathcal{M}$, it follows from Lemma 2.4 that

$$\bar{m} := \max_{(s,t)\in\partial D} I(g(s,t)) < I(g(1,1)) = m. \tag{3.2}$$

For $0 < \varepsilon < \min\{\frac{m-\bar{m}}{2}, \frac{\gamma\delta}{8}\}$ and $S = \{u \in H : \|u - u_0\| \leq \delta\}$, by (3.1) we see that

$$\|I'(u)\| \geq \frac{8\varepsilon}{\delta}, \ u \in I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}, \tag{3.3}$$

where $S_{2\delta}$ is the set $\{u \in H : \mathrm{dist}(u, S) \leq 2\delta\}$. In view of (3.3) and applying [29, Lemma 2.3], there exists a deformation $\eta \in C([0,1] \times H, H)$ such that

(i) $\eta(1,u) = u$ if $u \notin I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;
(ii) $\eta(1, I^{m+\varepsilon} \cap S) \subset I^{m-\varepsilon}$;
(iii) $I(\eta(1,u)) \leq I(u)$ for all $u \in H$.

For this deformation, with the help of Lemma 2.4, it is clear that

$$\max_{(s,t)\in\bar{D}} I(\eta(1, g(s,t))) < m. \tag{3.4}$$

In fact, on the one had, due to $I(g(s,t)) \leq m < m+\varepsilon$, one has $g(s,t) \in I^{m+\varepsilon}$. On the other hand, from the definition of σ, we have

$$\begin{aligned}\|g(s,t) - u_0\|^2 &= \|(s-1)u_0^+ + (t-1)u_0^-\|^2 \\ &\leq 2[(s-1)^2\|u_0^+\|^2 + (t-1)^2\|u_0^-\|^2] \leq 2\sigma^2\|u_0\|^2 < \delta^2,\end{aligned}$$

which shows that $g(s,t) \in S$ for all $(s,t) \in \bar{D}$. Therefore, according to (ii) for η, (3.4) holds.

Next, we claim that $\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset$, which contradicts to the definition of m. To do this, let us define $h(s,t) := \eta(1, g(s,t))$ and

$$\Phi_0 := (\langle I'(su_0^+ + tu_0^-), u_0^+\rangle, \langle I'(su_0^+ + tu_0^-), u_0^-\rangle) := (\tau_{u_0}^1(s,t), \tau_{u_0}^2(s,t)),$$

$$\Phi_1 := (\frac{1}{s}\langle I'(h(s,t)), h^+(s,t)\rangle, \frac{1}{t}\langle I'(h(s,t)), h^-(s,t)\rangle).$$

By a direct calculation, we deduce that

$$\frac{\partial \tau^1_{u_0}(s,t)}{\partial s}|(1,1) = \|u_0^+\|^2 + 3\int_{\mathbb{R}^3} K(x)\phi_{u_0^+}(u_0^+)^2dx + \int_{\mathbb{R}^3} K(x)\phi_{u_0^-}(u_0^+)^2dx - (p-1)\int_{\mathbb{R}^3} a(x)|u_0^+|^p dx,$$

$$\frac{\partial \tau^2_{u_0}(s,t)}{\partial t}|(1,1) = \|u_0^-\|^2 + 3\int_{\mathbb{R}^3} K(x)\phi_{u_0^-}(u_0^-)^2dx + \int_{\mathbb{R}^3} K(x)\phi_{u_0^+}(u_0^-)^2dx - (p-1)\int_{\mathbb{R}^3} a(x)|u_0^-|^p dx,$$

$$\frac{\partial \tau^1_{u_0}(s,t)}{\partial t}|(1,1) = 2\int_{\mathbb{R}^3} K(x)\phi_{u_0^-}(u_0^+)^2dx,\ \frac{\partial \tau^2_{u_0}(s,t)}{\partial s}|(1,1) = 2\int_{\mathbb{R}^3} K(x)\phi_{u_0^+}(u_0^-)^2dx.$$

Since $u_0 \in \mathcal{M}$, equivalently we have

$$\frac{\partial \tau^1_{u_0}(s,t)}{\partial s}|(1,1) = -2\|u_0^+\|^2 - 2\int_{\mathbb{R}^3} K(x)\phi_{u_0^-}(u_0^+)^2dx - (p-4)\int_{\mathbb{R}^3} a(x)|u_0^+|^p dx$$

and

$$\frac{\partial \tau^2_{u_0}(s,t)}{\partial t}|(1,1) = -2\|u_0^-\|^2 - 2\int_{\mathbb{R}^3} K(x)\phi_{u_0^+}(u_0^-)^2dx - (p-4)\int_{\mathbb{R}^3} a(x)|u_0^-|^p dx.$$

Then, we readily derive that the determinant of Hessian matrix of Φ_0 at $(1,1)$ is positive, namely,

$$\det M = \begin{vmatrix} \frac{\partial \tau^1_{u_0}(s,t)}{\partial s}|(1,1) & \frac{\partial \tau^1_{u_0}(s,t)}{\partial t}|(1,1) \\ \frac{\partial \tau^2_{u_0}(s,t)}{\partial s}|(1,1) & \frac{\partial \tau^2_{u_0}(s,t)}{\partial t}|(1,1) \end{vmatrix} > 0.$$

Observing that Φ_0 is continuously differentiable and $(1,1)$ is the unique isolated zero point, we deduce that $\deg(\Phi_0, D, 0) = 1$ by using the degree theory. At this point, taking into account that $\bar{m} < m - 2\varepsilon$, (3.2) and (i) corresponding to η, we infer that $g = h$ on ∂D. Thus, it concludes that $\deg(\Phi_1, D, 0) = \deg(\Phi_0, D, 0) = 1$, which brings that there exists a pair $(s_0, t_0) \in D$ such that $\Phi_1(s_0, t_0) = 0$. That is to say, $\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{M}$, which is a contradiction with (3.4). Therefore, we derive that u_0 is a nontrivial ground state sign-changing solution for system (1.2).

Now, we show that u_0 has exactly two nodal domains. To this end, we assume by contradiction that $u_0 = u_1 + u_2 + u_3$ with

$$u_i \neq 0,\ u_1 \geq 0,\ u_2 \leq 0 \text{ and } \mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset \text{ for } i \neq j\ (i,j = 1,2,3).$$

Note that $I'(u_0) = 0$, it is obvious that

$$\langle I'(u_0), u_i\rangle = 0 \text{ for } i = 1,2,3.$$

Choose $v := u_1 + u_2$, it holds that $v^+ = u_1 \neq 0$ and $v^- = u_2 \neq 0$. Moreover, due to the facts

$$I'(v)v^+ = I'(v)u_1 = I'(u_0)u_1 - \int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_1)^2 dx \leq 0$$

and

$$I'(v)v^- = I'(v)u_2 = I'(u_0)u_2 - \int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_2)^2 dx \leq 0,$$

it is easy to see

$$\int_{\mathbb{R}^3} a(x)|u_1|^p dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^3} a(x)|u_2|^p dx > 0.$$

That is, $v \in \mathcal{A}$. Then, by Lemma 2.4, there is a unique pair $(s_v, t_v) \in (0,1] \times (0,1]$ such that $s_v v^+ + t_v v^- \in \mathcal{M}$, or equivalently, $s_v u_1 + t_v u_2 \in \mathcal{M}$. In addition, a direct calculation shows that

$$\begin{aligned} 0 &= \frac{1}{4}\langle I'(u_0), u_3\rangle \\ &= \frac{1}{4}\|u_3\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_3)^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_3)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_3)^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} a(x)|u_3|^p dx \\ &< I(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_3)^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_3)^2 dx \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} I(s_v u_1 + t_v u_2) &= I(s_v u_1) + I(t_v u_2) + \frac{s_v^2 t_v^2}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_2)^2 dx \\ &\quad + \frac{s_v^2 t_v^2}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_1)^2 dx \\ &\leq \frac{1}{4}\|u_1\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} a(x)|u_1|^p dx + \frac{1}{4}\|u_2\|^2 \\ &\quad + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} a(x)|u_2|^p dx \\ &= I(u_1) + I(u_2) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_1)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_1)^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_2)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_2)^2 dx. \end{aligned} \tag{3.6}$$

Hence, from (3.5) and (3.6), we have

$$\begin{aligned} m &\leq I(s_v u_1 + t_v u_2) \\ &< I(u_1) + I(u_2) + I(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_1)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_1)^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_2)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_3}(u_2)^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_1}(u_3)^2 dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_2}(u_3)^2 dx \\ &= I(u_0) = m, \end{aligned}$$

which reaches to a contradiction. In this way, $u_3 = 0$ and u_0 has exactly two nodal domains. □

As the end of this paper, we are devoted to finish the proof of Theorem 1.4.

Proof of Theorem 1.4. Let c and $\mathcal{N}$ be given by (1.5) and (1.6), respectively. Then, similar to the proof of Lemma 2.5, we can deduce that there exists $v \in \mathcal{N}$ such that $I(v) = c > 0$. Consider $\bar{v} = |v|$, it is obvious that $\bar{v} \in \mathcal{N}$ and $I(\bar{v}) = I(v) = c$. Namely, $\bar{v}$ is a nonnegative ground state solution of system (1.2).

Since u_0 is a sign-changing solution with exactly two nodal domains, it can be supposed that $u_0 = u_0^+ + u_0^-$. Because $u_0 \in \mathcal{M} \subset \mathcal{A}$, it is easy to check that there is a unique $\tilde{s}_{u_0^+} > 0$ such that $\tilde{s}_{u_0^+} u_0^+ \in \mathcal{N}$. Observing that

$$\langle I'(u_0^+), u_0^+\rangle \leq \langle I'(u_0^+), u_0^+\rangle + \int_{\mathbb{R}^3} K(x)\phi_{u_0^-}(u_0^+)^2 dx = \langle I'(u_0), u_0^+\rangle = 0,$$

we infer that $\tilde{s}_{u_0^+} \in (0, 1]$. The same argument deduces that there is a unique $\tilde{t}_{u_0^-} \in (0, 1]$ such that $\tilde{t}_{u_0^-} u_0^- \in \mathcal{N}$. As a result, using Lemma 2.4, we have

$$2c \leq I(\tilde{s}_{u_0^+} u_0^+) + I(\tilde{t}_{u_0^-} u_0^-) \leq I(\tilde{s}_{u_0^+} u_0^+ + \tilde{t}_{u_0^-} u_0^-) \leq I(u_0^+ + u_0^-) = m,$$

that is, $I(u_0) \geq 2c$. □

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