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RIEMANNIAN SUBMERSIONS WHOSE TOTAL SPACE IS ENDOWED WITH A TORSE-FORMING VECTOR FIELD

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Abstract. In the present paper, a Riemannian submersion π between Riemannian manifolds such that the total space of π endowed with a torse-forming vector field ν is studied. Some remarkable results of such a submersion whose total space is Ricci soliton are given. Moreover, some characterizations about any fiber of π or the base manifold B to be an almost quasi-Einstein are obtained.

1. Introduction

A differentiable map $\pi : (M, g) \to (B, g^{'})$ between Riemannian manifolds is called a Riemannian submersion if the derivative map π_* is onto and satisfies

$$g_p(X, Y) = g^{'}_{\pi(p)}(\pi_{*p}X, \pi_{*p}Y)$$

for any vector fields X, Y which are tangent to M. The theory of Riemannian submersions were studied by O'Neill and Gray, independently. The basic features of such theory have been formulated and developed in the last three decades. Since then, it has been studied different types of submersion whose total space was equipped with some geometric structures (see [6, 9]).

The notion of Riemannian submersion is very important in not only in Differential Geometry but also Physics and Mechanics, since they have many applications in Kaluza-Klein theory, Yang-Mills theory, general relativity and modelling and controlling of certain types of redundant robotic chains. Hence, the Riemannian submersions have been studied for different kinds of spaces by many authors (for details, see [1, 5]).

In 1982, Hamilton defined the concept of Ricci flow and showed that Ricci solitons are self similar solutions of Ricci flows. According to his definition, a vector field V on a Riemannian manifold (M, g) is called to define a Ricci

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soliton if it holds

$$\frac{1}{2}\mathscr{L}_V g + Ric + \lambda g = 0, \tag{1}$$

where $\mathscr{L}_V g$ is the Lie-derivative of the metric tensor of g with respect to V, Ric is the Ricci tensor of M, V is a vector field (the potential field) and λ is a constant on M (see [7]). A Ricci soliton is denoted by (M, g, V, λ). Also, the Ricci soliton (M, g, V, λ) is called shrinking, steady or expanding according to $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. Also, the Ricci soliton is said to be an Einstein, if the potential field V of Ricci soliton is zero or Killing (that is the Lie-derivative of g with respect to V is zero). Recently, the geometry of Ricci solitons in literature have been studying by many mathematicians after Grigori Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in [11] (for other papers, see [3, 8]).

Furthermore, Pigola and et al. introduced a new class of Ricci solitons by taking a function rather than a constant λ. In such case, (M, g, V, λ) is called an almost Ricci soliton. Particularly, if the function λ is a constant, then (M, g, V, λ) becomes a Ricci soliton (see [12]).

On the other hand, a vector field ν on a Riemannian manifold M is said to be a torse-forming if it satisfies

$$\nabla_X \nu = fX + \phi(X)\nu, \tag{2}$$

where f is a function, ϕ is a 1-form and ∇ is a the Levi-Civita connection of M. In above (2), the vector field ν is called recurrent, if the function f is zero; ν is called concircular if the 1-form ϕ is zero. Also, the concircular vector field ν is called concurrent, if $f = 1$ is satisfied (see [4]).

In the present paper, our goal is to study a Riemannian submersion π between Riemannian manifolds such that the total space M of π is endowed with a torse-forming vector field ν. In particular, we show that if the horizontal distribution $\mathcal{H}$ of such a submersion is parallel, then the tangential part of ν is also torse-forming on the vertical distribution $\mathcal{V}$. Also, we consider the total space M of π admits an almost Ricci soliton and give some characterizations for which any fiber of π or the base manifold B are almost quasi-Einstein.

2. Notes on Riemannian submersions

In this section, we recall some necessary notions for Riemannian submersions for Riemannian submersions between Riemannian manifolds from [5]:

Let (M, g) and $(B, g^{'})$ be Riemannian manifolds of dimension m and n, respectively. A Riemannian submersion $\pi : (M, g) \to (B, g^{'})$ is a map from M onto B which satisfies the following conditions:

(i) π has a maximal rank,
(ii) The differential π_{*p} preserves the length of the horizontal vector fields at each point of M.

Let $\pi : (M, g) \to (B, g^{'})$ be a Riemannian submersion. If X, Y are the basic vector fields, π-related to $X^{'}, Y^{'}$, one has the followings:

(i) $g(X, Y) = g^{'}(X^{'}, Y^{'}) \circ \pi$,
(ii) $h[X, Y]$ is the basic vector field π-related to $[X^{'}, Y^{'}]$,
(iii) $h(\nabla_X Y)$ is the basic vector field π-related to $\nabla^{'}_{X^{'}} Y^{'}$,
(iv) for any vertical vector field V, $[X, V]$ is the vertical,

where ∇ and $\nabla^{'}$ denote the Levi-Civita connections of M and B, respectively.

A Riemannian submersion π determines two tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M, which are called the fundamental tensor fields of π. Such tensor fields are defined by

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$

$$\mathcal{A}(E, F) = \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,$$

where v and h are the vertical and horizontal projections, respectively. Also, the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M satisfy

$$g(\mathcal{T}_E F, G) = -g(\mathcal{T}_E G, F), \tag{3}$$

$$g(\mathcal{A}_E F, G) = -g(\mathcal{A}_E G, F), \tag{4}$$

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{5}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} v[X, Y], \tag{6}$$

for any $V, W \in \Gamma(VM)$, $X, Y \in \Gamma(HM)$ and $E, F, G \in \Gamma(TM)$.

We here note that the vanishing of the tensor field $\mathcal{A}$ means the horizontal distribution $\mathcal{H}$ is integrable and the vanishing of the tensor field $\mathcal{T}$ means any fibre of Riemannian submersion π is a totally geodesic submanifold of M. Using fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, one can see that

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{7}$$

$$\nabla_V X = h\nabla_V X + \mathcal{T}_V X, \tag{8}$$

$$\nabla_X V = \mathcal{A}_X V + v\nabla_X V, \tag{9}$$

$$\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y, \tag{10}$$

where ∇ and $\hat{\nabla}$ are the Levi-Civita connections of M and any fiber of π respectively, for any $V, W \in \Gamma(VM)$ and $X, Y \in \Gamma(HM)$.

On the other hand, the Gauss-Codazzi type equations for Riemannian submersions are given by

$$R(U, V, F, W) = \hat{R}(U, V, F, W) - g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F), \tag{11}$$

$$\begin{aligned} R(X, Y, Z, H) = {} & R^{'}(X^{'}, Y^{'}, Z^{'}, H^{'}) \circ \pi + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ & - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H), \end{aligned} \tag{12}$$

where R, $R^{'}$ and $\hat{R}$ denote the Riemannian curvature tensors for the manifolds M, B and any fiber of π, respectively, for any $U, V, W, F \in \Gamma(VM)$ and $X, Y, Z, H \in \Gamma(HM)$.

On the other hand, the mean curvature vector field H on any fiber of Riemannian submersion π is given by

$$N = rH,$$

such that

$$N = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j \tag{13}$$

and r denotes the dimension of any fibre of π and $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis on vertical distribution. Using the equality (13), we get

$$g(\nabla_E N, X) = \sum_{j=1}^{r} g((\nabla_E \mathcal{T})(U_j, U_j), X)$$

for any $E \in \Gamma(TM)$ and $X \in \Gamma(HM)$.

From (11)-(13), one has the following:

Proposition 2.1 ([6], [9]). *For the Riemannian submersion $\pi : (M, g) \rightarrow (B, g^{'})$, the Ricci tensor Ric satisfies*

$$Ric(U, W) = \hat{Ric}(U, W) + g(N, \mathcal{T}_U W) - \sum_{i=1}^{n} g((\nabla_{X_i} \mathcal{T})(U, W), X_i) - g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W), \tag{14}$$

$$Ric(X, Y) = Ric^{'}(X^{'}, Y^{'}) \circ \pi - \frac{1}{2}\{g(\nabla_X N, Y) + g(\nabla_Y N, X)\} + 2\sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y), \tag{15}$$

where $\{X_i\}$ and $\{U_j\}$ are the orthonormal basis of $\mathcal{H}$ and $\mathcal{V}$, respectively, for any $U, V \in \Gamma(VM)$ and $X, Y \in \Gamma(HM)$.

Note that $Ric(U, W)$ and $\hat{Ric}(U, W)$ are the extrinsic and intrinsic Ricci tensors of any fiber of π, respectively. Similarly, $Ric(X, Y)$ and $Ric^{'}(X^{'}, Y^{'}) \circ \pi$ are the extrinsic and intrinsic Ricci tensors of the horizontal distribution $\mathcal{H}$, respectively.

3. Riemannian submersions whose total spaces admit an almost Ricci soliton

A non-flat Riemannian manifold (M, g) is called an almost quasi-Einstein, if its Ricci tensor Ric of type-$(0, 2)$ satisfies

$$Ric = ag + b(\phi \otimes \alpha + \alpha \otimes \phi) \tag{16}$$

for some functions a, b and 1-forms ϕ, α. In particular case, one can see that if the function b is zero, then M becomes an Einstein. For more details, we refer to readers [4, 10].

We begin to this section with the following:

Lemma 3.1. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds. The horizontal distribution $\mathcal{H}$ is parallel with respect to ∇ if and only if the fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ vanish, identically (see [8]).*

For a Riemannian submersion $\pi : (M, g) \to (B, g^{'})$ between Riemannian manifolds, we denote by $\nu^{\top}$ and $\nu^{\perp}$ the tangential and normal parts of the vector field ν on any fiber of π, respectively.

Theorem 3.2. *Let $\pi : (M, g) \to (B, g^{'})$ be a Riemannian submersion such that M is endowed with a concircular vector field ν and let $(M, g, \nu^{\top}, \lambda)$ be an almost Ricci soliton. If the horizontal distribution $\mathcal{H}$ is parallel, then any fiber of π is an Einstein.*

Proof. Since the total space M admits an almost Ricci soliton with the potential field $\nu^{\top}$, from (1), one has

$$\frac{1}{2}(\mathscr{L}_{\nu^{\top}} g)(U, W) + Ric(U, W) + \lambda g(U, W) = 0 \tag{17}$$

for any $U, W \in \Gamma(VM)$. Because ν is a concircular vector field on M and using the equalities (7) in (17), we get

$$\frac{1}{2}\{g(fU - \mathcal{T}_U \nu^{\perp}, W) + g(fW - \mathcal{T}_W \nu^{\perp}, U)\} + Ric(U, W) + \lambda g(U, W) = 0. \tag{18}$$

Since the horizontal distribution $\mathcal{H}$ is parallel and considering (5), (3) and (14) in (18), we obtain

$$\hat{Ric} + (\lambda + f)g = 0, \tag{19}$$

which gives any fiber of π is an Einstein. □

Example 3.3. We consider a Riemannian submersion

$$\pi : (M^n = I \times_\sigma N^{n-1}, g) \to (B^m, g^{'}),$$

where M is a warped product manifold with the metric $g = dt^2 + \sigma^2 g_N$, I is an open interval and (N, g_N) and $(B, g^{'})$ are Einstein and Riemannian manifolds, respectively, $n \geq 3$.

Let (M, g, ν, λ) be a Ricci soliton with vertical potential field ν. Then, according to Theorem 5.1 in [2], B.-Y. Chen showed that a Ricci soliton (M, g, ν, λ) on a Riemannian manifold M is equipped with a concircular vector field if and only if the function f is equal to λ, which is a nonzero constant and M is an open portion a warped product manifold $I \times_\sigma N$.

Motivated from Chen's paper in [2], we can say that the total space M of π has a concircular vector field ν such that

$$\nabla_E \nu = \lambda E, \tag{20}$$

where λ is a constant in (1), for any vector field E on M.

On the other hand, if the horizontal distribution $\mathcal{H}$ of π is parallel, then the direct computations give that any fiber of π is an Einstein.

From now on, we make the following:

Assumption. The total space of a Riemannian submersion of π is endowed with a torse-forming vector field ν.

Theorem 3.4. *Let $\pi : (M, g) \to (B, g^{'})$ be a Riemannian submersion between Riemannian manifolds and $(M, g, \nu^{\top}, \lambda)$ be an almost Ricci soliton. The extrinsic vertical Ricci tensor $Ric\,|_{\mathcal{V}}$ satisfies*

$$Ric\,|_{\mathcal{V}} = -(\lambda + f)g - \frac{1}{2}(\phi \otimes \alpha + \alpha \otimes \phi) + g(\mathcal{T}_U W, \nu^{\perp}), \tag{21}$$

where λ and f are functions, ϕ and α are 1-forms, for any $U, W \in \Gamma(VM)$.

Proof. Since the total space M of π is endowed with a torse-forming vector field ν, from (2) one has

$$\nabla_U \nu = fU + \phi(U)\nu, \tag{22}$$

where f is a function and ϕ is a 1-form, for any $U \in \Gamma(VM)$.

Using the equations (7)-(8) in (22) and by comparing the tangential parts, it gives

$$\hat{\nabla}_U \nu^{\top} + \mathcal{T}_U \nu^{\perp} = fU + \phi(U)\nu^{\top}. \tag{23}$$

On the other hand, if the total space M of π admits an almost Ricci soliton with potential field $\nu^{\top}$ and using the definition of Lie derivative and (7), we have

$$\frac{1}{2}\{g(\hat{\nabla}_U \nu^{\top}, W) + (\hat{\nabla}_W \nu^{\top}, U)\} + Ric(U, W) + \lambda g(U, W) = 0. \tag{24}$$

Using (23) in (24), it follows

$$\frac{1}{2}\big\{g(fU + \phi(U)\nu^{\top} - \mathcal{T}_U \nu^{\perp}, W) + g(fW + \phi(W)\nu^{\top} - \mathcal{T}_W \nu^{\perp}, U)\big\}$$
$$+ Ric(U, W) + \lambda g(U, W) = 0.$$

Hence, the last equation is equivalent to

$$fg(U, W) + \frac{1}{2}\phi(U)g(\nu^{\top}, W) - \frac{1}{2}g(\mathcal{T}_U \nu^{\perp}, W) + \frac{1}{2}\phi(W)g(\nu^{\top}, U)$$
$$- \frac{1}{2}g(\mathcal{T}_W \nu^{\perp}, U) + Ric(U, W) + \lambda g(U, W) = 0. \tag{25}$$

Using (5) and (3) in (25), it follows

$$fg(U,W) - g(\mathcal{T}_U W, \nu^{\perp}) + \frac{1}{2}\phi(U)g(\nu^{\top}, W) + \frac{1}{2}\phi(W)g(\nu^{\top}, U)$$
$$+ Ric(U,W) + \lambda g(U,W) = 0. \tag{26}$$

If we denote the dual 1-form of $\nu^{\top}$ by α, then (26) yields

$$Ric\mid_{\mathcal{V}} = -(\lambda + f)g - \frac{1}{2}(\phi \otimes \alpha + \alpha \otimes \phi) + g(\mathcal{T}_U W, \nu^{\perp}). \tag{27}$$

□

From Theorem 3.4 and the equality (14), we obtain the following:

Theorem 3.5. *Let $\pi : (M,g) \to (B,g^{'})$ be a Riemannian submersion between Riemannian manifolds and $(M,g,\nu^{\top},\lambda)$ be an almost Ricci soliton. If the horizontal distribution $\mathcal{H}$ is parallel, then any fiber of π is an almost quasi-Einstein manifold.*

Theorem 3.6. *Let $\pi : (M,g) \to (B,g^{'})$ be a Riemannian submersion between Riemannian manifolds and $(M,g,\nu^{\perp},\lambda)$ be an almost Ricci soliton. Then, the extrinsic horizontal Ricci tensor $Ric\mid_{\mathcal{H}}$ satisfies*

$$Ric\mid_{\mathcal{H}} = -(\lambda + f)g - \frac{1}{2}(\phi \otimes \varphi + \varphi \otimes \phi) \tag{28}$$

for constant λ, f and 1-forms ϕ, φ.

Proof. Since ν is a torse-forming vector field on the total space M, from the equalities (3.1), (9) and (10), we get

$$fX + \phi(X)\nu^{\top} + \phi(X)\nu^{\perp} = h(\nabla_X \nu^{\perp}) + \mathcal{A}_X \nu^{\top} + v(\nabla_X \nu^{\top}) + \mathcal{A}_X \nu^{\perp} \tag{29}$$

for any $X \in \Gamma(HM)$. By comparing normal parts of (29), one has

$$h(\nabla_X \nu^{\perp}) + \mathcal{A}_X \nu^{\top} = fX + \phi(X)\nu^{\perp}. \tag{30}$$

From the definition of Lie-derivative and above equality (30), it follows

$$\begin{aligned}(\mathscr{L}_{\nu^{\perp}} g)(X,Y) &= g(h(\nabla_X \nu^{\perp}), Y) + g(h(\nabla_Y \nu^{\perp}), X)\\ &= g(fX + \phi(X)\nu^{\perp} - \mathcal{A}_X \nu^{\top}, Y) + g(fY + \phi(Y)\nu^{\perp}\\ &\quad - \mathcal{A}_Y \nu^{\top}, X)\\ &= 2fg(X,Y) + \phi(X)g(\nu^{\perp}, Y) - g(\mathcal{A}_X \nu^{\top}, Y)\\ &\quad + \phi(Y)g(\nu^{\perp}, X) - g(\mathcal{A}_Y \nu^{\top}, X).\end{aligned} \tag{31}$$

Using (6) and (4) in (31), we obtain

$$(\mathscr{L}_{\nu^{\perp}} g)(X,Y) = 2fg(X,Y) + \phi(X)g(\nu^{\perp}, Y) + \phi(Y)g(\nu^{\perp}, X) \tag{32}$$

for any $X, Y \in \Gamma(HM)$. Since M is an almost Ricci soliton with potential field $\nu^{\perp}$, using (32), we have

$$fg(X,Y) + \frac{1}{2}\phi(X)g(\nu^{\perp}, Y) + \frac{1}{2}\phi(Y)g(\nu^{\perp}, X) + Ric(X,Y) + \lambda g(X,Y) = 0. \tag{33}$$

Denoting the dual 1-form of $\nu^{\perp}$ by φ, (33) yields

$$Ric\,|_{\mathcal{H}}= -(\lambda+f)g - \frac{1}{2}(\phi\otimes\varphi+\varphi\otimes\phi). \tag{34}$$

Hence, the proof is completed. □

Considering Theorem 3.6 and the equality (15), we obtain the following:

Theorem 3.7. *Let $\pi:(M,g)\rightarrow(B,g^{'})$ be a Riemannian submersion between Riemannian manifolds and let $(M,g,\nu^{\perp},\lambda)$ be an almost Ricci soliton. If the horizontal distribution $\mathcal{H}$ is parallel, then the base manifold $(B,g^{'})$ is an almost quasi-Einstein.*

Proof. Since $(M,g,\nu^{\perp},\lambda)$ is an almost Ricci soliton, using the equalities (15) and (34), we have

$$\begin{aligned}
&Ric^{'}(X^{'},Y^{'})\circ\pi - \frac{1}{2}\big\{g(\nabla_X N,Y)+g(\nabla_Y N,X)\big\} + 2\sum_{i=1}^{n} g(\mathcal{A}_X X_i,\mathcal{A}_Y X_i)\\
&\qquad + \sum_{j=1}^{r} g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}Y) + (\lambda+f)g(X,Y) - \frac{1}{2}\big\{\phi(X)g(Y,\nu^{\perp})\\
&\qquad + g(X,\nu^{\perp})\phi(Y)\big\} = 0.
\end{aligned} \tag{35}$$

Because the horizontal distribution $\mathcal{H}$ is parallel, the equation (35) follows

$$Ric^{'}(X^{'},Y^{'})\circ\pi + (\lambda+f)g(X,Y) - \frac{1}{2}\big\{\phi(X)g(Y,\nu^{\perp}) + g(X,\nu^{\perp})\phi(Y)\big\} = 0.$$

If we denote the dual 1-form of $\nu^{\perp}$ by φ, then it gives

$$Ric^{'}(X^{'},Y^{'})\circ\pi + (\lambda+f)g(X,Y) - \frac{1}{2}\big\{\phi(X)\varphi(Y)+\varphi(X)\phi(Y)\big\} = 0. \tag{36}$$

Also, we note that the horizontal vector fields X,Y are π-related to $X^{'},Y^{'}$ on $M^{'}$, respectively. Then, Eq. (36) is equivalent to

$$Ric^{'} = -(\lambda+f)g^{'} + \frac{1}{2}(\phi^{'}\otimes\varphi^{'}+\varphi^{'}\otimes\phi^{'}),$$

where $\phi^{'}$ and $\varphi^{'}$ are 1-forms on $M^{'}$, such that for any horizontal vector field X, $g(X,\nu^{\perp}) = \varphi(X) = \varphi^{'}(X^{'})\circ\pi$ and $\phi(X) = \phi^{'}(X^{'})\circ\pi$ are satisfied. Therefore, the proof is completed. □

Considering Theorem 3.7, in particular case, we notice the followings:

Corollary 3.8. 1. *If the torse-forming vector field ν is a recurrent (that is, f vanishes identically in (22)) on M, then the base manifold B is an almost quasi-Einstein.*

2. *If the almost Ricci soliton $(M,g,\nu^{\perp},\lambda)$ is a Ricci soliton (that is, the function λ is a constant) and the torse-forming vector field ν is a concurrent (that is, $f=1$ and 1-form ϕ vanishes identically in (2)), then the Riemannian manifold B becomes an Einstein.*

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