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JORDAN $\mathcal{G}_n$-DERIVATIONS ON PATH ALGEBRAS

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Abstract. Recently, Brešar's Jordan $\{g, h\}$-derivations have been investigated on triangular algebras. As a first aim of this paper, we extend this study to an interesting general context. Namely, we introduce the notion of Jordan $\mathcal{G}_n$-derivations, with $n \geq 2$, which is a natural generalization of Jordan $\{g, h\}$-derivations. Then, we study this notion on path algebras. We prove that, when $n > 2$, every Jordan $\mathcal{G}_n$-derivation on a path algebra is a $\{g, h\}$-derivation. However, when $n = 2$, we give an example showing that this implication does not hold true in general. So, we characterize when it holds. As a second aim, we give a positive answer to a variant of Lvov-Kaplansky conjecture on path algebras. Namely, we show that the set of values of a multi-linear polynomial on a path algebra KE is either $\{0\}$, KE or the space spanned by paths of a length greater than or equal to 1.

1. Introduction and definitions

Through this paper, K will denote a field with characteristic zero, A will be a K-algebra with the center $Z(A)$. For $x, y \in A$, we use $x \circ y$ (resp., $[x, y]$) to denote the Jordan product $xy + yx$ (resp., the Lie product $xy - yx$) of x and y.

In [6], Brešar introduced the notion of Jordan $\{g, h\}$-derivations as follows: Let $g : A \to A$ and $h : A \to A$ be linear maps. A linear map $f : A \to A$ is said to be a Jordan $\{g, h\}$-derivation if

$$f(x \circ y) = g(x) \circ y + x \circ h(y) \qquad (x, y \in A).$$

For $g = f$, a Jordan $\{g, h\}$-derivation is just a Jordan generalized derivation, and for $g = h = f$, it is nothing but the classical Jordan derivation. Several authors have been interested in investigating when Jordan derivations are derivations on various algebra constructions (see for instance [3,4,9,12,13,17]). In order to extend this classical question to the introduced context, Brešar, in the same paper [6], introduced the notion of $\{g, h\}$-derivation as follows: Let $g : A \to A$ and $h : A \to A$ be linear maps. A linear map $f : A \to A$ is said to

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be a $\{g, h\}$-derivation if

$$f(xy) = g(x)y + xh(y) = h(x)y + xg(y) \qquad (x, y \in A).$$

But, it turned out from the discussion at the beginning of [6, Section 2], that $\{g, h\}$-derivations are maps of the from:

$$f(x) = \lambda x + d(x)$$

for some $\lambda \in Z(A)$ and a derivation $d : A \to A$. These kind of maps are in fact a particular case of generalized derivations. Recall that a linear map $D : A \to A$ is said to be a generalized d-derivation, for some derivation $d : A \to A$, if it satisfies

$$D(xy) = D(x)y + xd(y) \qquad (x, y \in A).$$

The main aim in [6] was to investigate when a Jordan $\{g, h\}$-derivation is a $\{g, h\}$-derivation on tensor algebras. In [11], Kong and Zhang investigated the same question on triangular algebras.

Inspired by the context above, one can naturally continue the way started by Brešar and introduce a rather general case of Jordan $\{g, h\}$-derivations using the following notations: Denote $x_1 \circ x_2$ by $\circ_2 x_i$ for all $x_1, x_2 \in A$ and $(\circ_{n-1} x_i) \circ x_n$ by $\circ_n x_i$ for all $x_1, \ldots, x_n \in A$ with $n \geq 2$. By convention, we set by $\circ_0 x_i = \frac{1}{2}$ and $\circ_1 x_i = x_1$ for all $x_1 \in A$. Whence, the generalization of Jordan $\{g, h\}$-derivations is stated as follows: Let $\mathcal{G}_n = \{g_i\}_{1 \leq i \leq n}$ be a finite family of linear maps on A with $n \geq 2$. We say that a linear map $f : A \to A$ is a Jordan $\mathcal{G}_n$-derivation, if for every n-tuple $(x_1, \ldots, x_n) \in A^n$

$$f(\circ_n x_i) = \sum_{j=1}^{n} (((\circ_{j-1} x_i \circ g_{\sigma(j)}(x_j)) \circ x_{j+1}) \cdots) \circ x_n, \qquad \forall \sigma \in S_n,$$

where S_n is the symmetric group of degree n. Also, following Brešar's approach, we consider the following notion: we say that a linear map $f : A \to A$ is a $\mathcal{G}_n$-derivation on A, if for every $(x_1, \ldots, x_n) \in A^n$,

$$f(\prod_{i=1}^{n} x_i) = \sum_{i=1}^{n} x_1 \cdots x_{i-1} g_{\sigma(i)}(x_i) x_{i+1} \cdots x_n, \qquad \forall \sigma \in S_n. \tag{1.1}$$

However, following similar argument was done by Brešar, we deduce that $\mathcal{G}_n$-derivations and $\{g, h\}$-derivations are the same. In fact, let f be a $\mathcal{G}_n$-derivation with $n \geq 2$. Then, by taking $x_2 = \cdots = x_n = 1$ in (1.1), we obtain

$$f(x_1) = g_{\sigma(1)}(x_1) + x_1 g_{\sigma(2)}(1) + x_1 (\sum_{i=3}^{n} g_{\sigma(i)}(1)), \quad \forall \sigma \in S_n. \tag{1.2}$$

And taking $x_1 = x_3 = \cdots = x_n = 1$ in (1.1), we obtain

$$f(x_2) = g_{\sigma(1)}(1) x_2 + g_{\sigma(2)}(x_2) + x_2 (\sum_{i=3}^{n} g_{\sigma(i)}(1)), \quad \forall \sigma \in S_n.$$

Comparing both expressions, we see that every $g_i(1)$ lies in $Z(A)$. Setting $\lambda = f(1) = \sum_{i=1}^{n} g_i(1)$, we then infer from (1.1) and (1.2) that, for all $i \in \{1, \ldots, n\}$, $f(x) - \lambda x = g_i(x) - g_i(1)x$. If we set $d(x) = f(x) - \lambda x$ for all $x \in A$, then d is a derivation. Thus, every $\mathcal{G}_n$-derivation f can be written as

$$f(x) = \lambda x + d(x). \tag{1.3}$$

Therefore, every $\mathcal{G}_n$-derivation can be viewed as a generalized d-derivation on A. Conversely, if a linear map f has the form as in (1.3) and $\lambda = \sum_i^n \lambda_i$, where $\lambda_i \in Z(A)$. Then, f is a $\mathcal{G}_n$-derivation on A where each g_i is defined by

$$g_i(x) = \lambda_i x + d(x) \quad (x \in A).$$

In this context, it is natural to ask whether a Jordan $\mathcal{G}_n$-derivation is nothing but a Jordan $\mathcal{G}_2$-derivation.

In order to answer this question, we investigate Jordan $\mathcal{G}_n$-derivations on path algebras associated with a finite acyclic quiver. Thus, we assume some familiarity with basic notions of path algebras (for more details, see [15]).

In the sequel, $E = (E^0, E^1, s, t)$ designates a finite acyclic quiver, where E^0 and E^1 are sets of vertices and edges of E, respectively, and the maps s and t from E^1 into E^0 determine the edges of E. We denote by KE the path algebra over K associated with E.

In Section 2, we give our main results. The first one, Theorem 2.3, shows that a Jordan $\mathcal{G}_2$-derivation on KE is a $\mathcal{G}_2$-derivation if and only if $g_1(1) \in Z(KE)$ or $g_2(1) \in Z(KE)$. The second main result, Theorem 2.4, shows that for every $n > 2$, any Jordan $\mathcal{G}_n$-derivation on KE is a $\mathcal{G}_n$-derivation. Now, using these two theorems one can answer the above question. Namely, for every $n > 2$, Jordan $\mathcal{G}_n$-derivations on KE are $\mathcal{G}_n$-derivations. Unlike the case $n = 2$, there exist some Jordan $\mathcal{G}_2$-derivations which are not $\mathcal{G}_2$-derivations as will be shown in Example 2.1, which yields that Jordan $\mathcal{G}_n$-derivations generalize naturally Jordan $\mathcal{G}_2$-derivations (i.e., Jordan $\{g, h\}$-derivations).

Section 3 presents our investigations on a variant of Lvov-Kaplansky conjecture. Recall the following question known as Lvov-Kaplansky conjecture (see [7]):

Question 1.1. *Let $\zeta(x_1, \ldots, x_n)$ be a multi-linear polynomial over a field $\mathbb{F}$. Is the set of values of ζ on the matrix algebra $M_m(\mathbb{F})$ a vector space?*

The reader is referred to [10] for more information about recent and important results on this subject. Our investigation is motivated by the work done in [8, 14, 16] on particular upper triangular matrix algebras. In fact, since upper triangular matrix algebras are path algebras associated with line quivers (see [5]), we will push the question further in another direction and ask:

Question 1.2. *Let $\zeta(x_1, \ldots, x_n)$ be a multi-linear polynomial over K. Is the set of values of ζ on KE a vector space?*

Theorem 3.1 answers Question 1.2 positively and so it generalizes the work done for upper triangular matrix algebras. We give also some examples which apply Theorem 3.1 on some particular important cases.

2. Main results

In this section, the set $\mathcal{G}_n$ will be a fixed family $\{g_i\}_{1\leq i\leq n}$ of linear maps on KE, where $n \geq 2$. We show when every Jordan $\mathcal{G}_n$-derivation on path algebras is a $\mathcal{G}_n$-derivation. We will see that for every $n > 2$ this implication holds, however for the case $n = 2$, it does not as shown by the following example.

Example 2.1. Let E be the following quiver: $v_2 \xleftarrow{e_1} v_1 \xrightarrow{e_2} v_3$ and let f be a Jordan $\mathcal{G}_2$-derivation on KE defined by:

$$\begin{array}{lll} f(v_1) = 2v_1, & g_1(v_1) = v_1 + e_1 + e_2, & g_2(v_1) = v_1 - e_1 - e_2, \\ f(v_2) = 2v_2, & g_1(v_2) = v_2 + e_1, & g_2(v_2) = v_2 - e_1, \\ f(v_3) = 2v_3, & g_1(v_3) = v_3 + e_2, & g_2(v_3) = v_3 - e_2, \\ f(e_1) = 2e_1, & g_1(e_1) = e_1, & g_2(e_1) = e_1, \\ f(e_2) = 2e_2, & g_1(e_2) = e_2, & g_2(e_2) = e_2. \end{array}$$

By elementary calculations, we have $g_1(v_1)v_1 + v_1g_2(v_1) \neq f(v_1^2)$, hence f is not a $\mathcal{G}_2$-derivation on KE.

To prove the main results, we need the following lemma.

Lemma 2.2. *For every Jordan $\mathcal{G}_n$-derivation f on KE with $n \geq 2$, $f(1)$ is in $Z(KE)$. Moreover, if $n > 2$, then $g_i(1)$ is in $Z(KE)$ for all $i \in \{1, \ldots, n\}$.*

Proof. Assume f to be a Jordan $\mathcal{G}_n$-derivation on KE with $n \geq 2$. Let z be a non-trivial idempotent in KE. Then, we have

$$\begin{aligned} 0 &= f((((z \circ (1-z)) \circ 1) \cdots) \circ 1) \\ &= (((g_1(z) \circ (1-z)) \circ 1) \cdots) \circ 1 + (((z \circ g_2(1-z)) \circ 1) \cdots) \circ 1 + 0 \\ &= 2^{n-2}(g_1(z) \circ (1-z) + z \circ g_2(1-z)) \\ &= g_1(z) \circ (1-z) + z \circ g_2(1-z) \\ &= 2g_1(z) - g_1(z)z - zg_1(z) + zg_2(1) - zg_2(z) + g_2(1)z - g_2(z)z. \qquad (2.1) \end{aligned}$$

Multiplying (2.1) by z from the left, we obtain

$$0 = zg_1(z) - zg_1(z)z + zg_2(1) - zg_2(z) + zg_2(1)z - zg_2(z)z. \qquad (2.2)$$

Multiplying (2.1) by z from the right, we obtain

$$0 = g_1(z)z - zg_1(z)z + zg_2(1)z - zg_2(z)z + g_2(1)z - g_2(z)z. \qquad (2.3)$$

By comparing the equalities (2.2) and (2.3), we get

$$zg_1(z) + zg_2(1) - zg_2(z) = g_1(z)z + g_2(1)z - g_2(z)z. \qquad (2.4)$$

Similarly, by the definition of Jordan $\mathcal{G}_n$-derivations, we obtain

$$zg_{\sigma(1)}(z) + zg_{\sigma(2)}(1) - zg_{\sigma(2)}(z) = g_{\sigma(1)}(z)z + g_{\sigma(2)}(1)z - g_{\sigma(2)}(z)z$$

for every $\sigma \in S_n$. Therefore, we have

$$zg_2(z) + zg_1(1) - zg_1(z) = g_2(z)z + g_1(1)z - g_1(z)z. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$z(g_1(1) + g_2(1)) = (g_1(1) + g_2(1))z.$$

Since every element $s(p) + p$ is a non-trivial idempotent in KE with p is a non-trivial path in E, $g_1(1) + g_2(1)$ commutes with all paths in KE. Thus, $g_1(1) + g_2(1) \in Z(KE)$. Hence by the definition of Jordan $\mathcal{G}_n$-derivations, we conclude that $g_{\sigma(1)}(1) + g_{\sigma(2)}(1) \in Z(KE)$ for all $\sigma \in S_n$. Now, assume that $n > 2$, then it follows that $g_1(1) + g_2(1)$, $g_3(1) + g_2(1)$ and $g_1(1) + g_3(1)$ are in $Z(KE)$. Since $Z(KE)$ is a group, we have $g_1(1)+g_2(1)-g_3(1)-g_2(1) = g_1(1)-g_3(1) \in Z(KE)$. Therefore, $g_1(1) - g_3(1) + g_1(1) + g_3(1) = 2g_1(1) \in Z(KE)$. So, $g_1(1) \in Z(KE)$. By similar reasoning, we obtain that all $g_i(1)$ are in $Z(KE)$. □

We start with the first main result which treats the case $n = 2$.

Theorem 2.3. *Every Jordan $\mathcal{G}_2$-derivation f on KE is a $\mathcal{G}_2$-derivation if and only if $g_1(1) \in Z(KE)$ or $g_2(1) \in Z(KE)$.*

Proof. It is clear that if f is a $\mathcal{G}_2$-derivation, then $g_1(1) \in Z(KE)$ and $g_2(1) \in Z(KE)$. So, it remains to prove the converse implication. Let f be a Jordan $\mathcal{G}_2$-derivation on KE, then we have

$$f(x \circ y) = g_1(x) \circ y + x \circ g_2(y) \quad (x, y \in KE). \tag{2.6}$$

Take $y = 1$ in (2.6), then we obtain

$$f(x) = g_1(x) + x \circ g_2(\frac{1}{2}) \quad (x \in KE). \tag{2.7}$$

Similarly, take $x = 1$, then we obtain

$$f(y) = g_2(y) + y \circ g_1(\frac{1}{2}) \quad (y \in KE). \tag{2.8}$$

Without loss of generality, suppose that $g_1(1) \in Z(KE)$. It follows by Lemma 2.2, that $g_2(1) \in Z(KE)$. Therefore, the equalities (2.7) and (2.8) become $f(x) = g_1(x)+g_2(1)x$ and $f(y) = g_2(y)+g_1(1)y$ for all $x, y \in KE$, respectively. For all $x, y \in KE$, we have

$$\begin{aligned} f(x \circ y) &= g_1(x) \circ y + x \circ g_2(y) \\ &= (f(x) - g_2(1)x) \circ y + x \circ (f(y) - g_1(1)y) \\ &= f(x) \circ y + x \circ (f(y) - f(1)y). \end{aligned}$$

Hence, f is a Jordan generalized derivation on KE. Therefore, by the discussion in [1, Preliminaries] and [13, Proposition 3.7], f is a generalized derivation with $f(1) = g_1(1) + g_2(1)$. Hence, it follows that f is a $\mathcal{G}_2$-derivation. □

In the rest of this paper, $\mathcal{P}$ will denote the set of all paths in E including vertices. Note that $\mathcal{P}$ is a basis of KE as a K-vector space. Now, for the case where $n > 2$, we have the following second main result.

Theorem 2.4. *Every Jordan $\mathcal{G}_n$-derivation on KE with $n > 2$ is a $\mathcal{G}_n$-derivation.*

Proof. Let f be a Jordan $\mathcal{G}_n$-derivation on KE with $n > 2$. Then, for every path $p \in \mathcal{P}$, we have

$$\begin{aligned} f(p) &= \frac{1}{2^{n-1}} f(((p \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-1}} \big(((g_1(p) \circ 1) \cdots) \circ 1 + \cdots + ((p \circ 1) \cdots) \circ g_n(1)\big) \\ &= \frac{1}{2^{n-1}} \big(2^{n-1} g_1(p) + 2^{n-1} (\sum_{i=2}^{n} g_i(1)) p\big) \\ &= g_1(p) + (\sum_{i=2}^{n} g_i(1)) p. \end{aligned} \tag{2.9}$$

And,

$$\begin{aligned} f(p) &= \frac{1}{2^{n-1}} f(((p \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-1}} \big(((g_2(p) \circ 1) \cdots) \circ 1 + \cdots + ((p \circ 1) \cdots) \circ g_n(1)\big) \\ &= \frac{1}{2^{n-1}} \big(2^{n-1} g_2(p) + 2^{n-1} (\sum_{\substack{i=1 \\ i \neq 2}}^{n} g_i(1)) p\big) \\ &= g_2(p) + (\sum_{\substack{i=1 \\ i \neq 2}}^{n} g_i(1)) p. \end{aligned} \tag{2.10}$$

This is due to the fact that by Lemma 2.2, all $g_i(1) \in Z(KE)$. We claim that f is a Jordan generalized derivation, we only need to check it on every element in $\mathcal{P}$. Let x and y be two elements in $\mathcal{P}$. Then, we have

$$\begin{aligned} f(x \circ y) &= \frac{1}{2^{n-2}} f((((x \circ y) \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-2}} \big((((g_1(x) \circ y) \circ 1) \cdots) \circ 1 + (((x \circ g_2(y)) \circ 1) \cdots) \circ 1\big) \\ &\quad + \frac{1}{2^{n-2}} \big((((x \circ y) \circ g_3(1)) \cdots) \circ 1 + \cdots + (((x \circ y) \circ 1) \cdots) \circ g_n(1)\big) \\ &= g_1(x) \circ y + x \circ g_2(y) + (x \circ y)(\sum_{i=3}^{n} g_i(1)). \end{aligned} \tag{2.11}$$

It follows by (2.9) and (2.10) that

$$\begin{aligned}(2.11) &= (f(x) - (\sum_{i=2}^{n} g_i(1))x) \circ y + x \circ (f(y) - (\sum_{\substack{i=1\\ i\neq 2}}^{n} g_i(1))y) \\ &\quad + (x \circ y)(\sum_{i=3}^{n} g_i(1)) \\ &= f(x) \circ y + x \circ (f(y) - (\sum_{i=1}^{n} g_i(1))y).\end{aligned}$$

Hence, f is a Jordan generalized derivation on KE. Therefore, by the discussion in [1, Preliminaries] and [13, Proposition 3.7], f is a generalized derivation with $f(1) = \sum_{i=1}^{n} g_i(1)$ and $g_i(1) \in Z(KE)$. Hence, it follows that f is a $\mathcal{G}_n$-derivation. ☐

3. Application on a variant of Lvov-Kaplansky conjecture

In this section, we investigate a variant of Lvov-Kaplansky conjecture (see Question 1.2 in the introduction). Our main result is as follows.

In the proof, we denote the length of a path p in E by $\ell(p)$ (i.e., the number of edges in the path p). By convention, we set the length of vertices to zero.

Theorem 3.1. *Let $\zeta(x_1, \ldots, x_n) = \sum_{\sigma\in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, with $c_\sigma \in K$. Then, the set of values of ζ on KE is either $\{0\}$, KE or the space spanned by paths of a length greater than or equal to* 1.

Proof. We prove the result by recurrence on the length l of the longest path in E. Let V_j be the space spanned by paths in E with a length greater than or equal to $j \in \mathbb{N}$. It follows that $V_0 = KE$ and $V_{l+k+1} = \{0\}$ for all $k \in \mathbb{N}$, since there is no path with a length greater than l. Now, define I_p to be

$$(3.1) \qquad I_p = \{(x_1, \ldots, x_n) \in (\mathcal{P} \cup \{1\})^n : \exists \sigma \in S_n, \prod_{i=1}^{n} x_{\sigma(i)} = p\},$$

where $p \in \mathcal{P}$. Let $\zeta(x_1, \ldots, x_n) = \sum_{\sigma\in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, where $c_\sigma \in K$. Let d_σ be a $\mathcal{G}_n$-derivation on KE with $g_i = \frac{c_\sigma}{n} I$, where I is the identity map on KE. Then, ζ can be written as

$$\zeta(x_1, \ldots, x_n) = \sum_{\sigma\in S_n} c_\sigma \prod_{i=1}^{n} x_{\sigma(i)} = \sum_{\sigma\in S_n} d_\sigma(\prod_{i=1}^{n} x_{\sigma(i)})$$

for every $(x_1, \ldots, x_n) \in (KE)^n$. Let $p \in \mathcal{P}$ with $\ell(p) = 0$. Assume that there exists an element $x \in I_p$ such that $\zeta(x) \neq 0$. Then, $\sum_{\sigma\in S_n} c_\sigma \neq 0$. Hence, we have

$$\zeta(x) = (\sum_{\sigma\in S_n} d_\sigma)(p) = \alpha_p p$$

for every $p \in \mathcal{P}$ and for every $x \in I_p$, where $\alpha_p \in K^*$. Therefore, by linearity, the set of values of ζ on KE is KE itself. Now, to prove the set of values of ζ on KE is V_j, where $0 < j \leq l$, we assume that, for every $q \in \mathcal{P}$ with $\ell(q) < j$, and for every $y \in I_q$, we have $\zeta(y) = 0$ and there exists $x_0 \in I_{p_0}$ for some $p_0 \in \mathcal{P}$ with $\ell(p_0) = j$ such that $\zeta(x_0) \neq 0$. Then, there exists a subset $S_{x_0} = \{\sigma \in S_n : \prod_{i=1}^{n} x_{\sigma(i),0} \neq 0\}$ of S_n such that $\sum_{\sigma \in S_{x_0}} c_\sigma \neq 0$. Hence, we have

$$\zeta(x) = (\sum_{\sigma \in S_{x_0}} d_\sigma)(p) = \alpha_p p$$

for every $p \in \mathcal{P}$ with $\ell(p) \geq j$ and for every $x \in I_p$ with the components of x has a similar decomposition of sub-paths of p as x_0 of p_0, where $\alpha_p \in K^*$. Therefore, the set of values of ζ on KE is V_j. Otherwise, if $\zeta(y) = 0$ for every $q \in \mathcal{P}$ and every $y \in I_q$, then the set of values of ζ on KE is $\{0\}$. ☐

We end this section with the following examples. We assume in these examples that KE has some paths of length greater than or equal to 2 and $K = \mathbb{C}$ or $K = \mathbb{R}$.

Example 3.2. Consider the multi-linear polynomial $\zeta(x_1, x_2, x_3) = (x_1 \circ x_2) \circ x_3$ over K. Then, the set of values of ζ on KE is KE itself. This is due the fact that all coefficients are positive. Therefore, for every $p \in \mathcal{P}$, we have $\zeta(p, 1, 1) = \alpha_p p$, as desired.

In the following example, we use the notation of the proof of Theorem 3.1.

Example 3.3. Consider the multi-linear polynomial

$$\zeta(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2x_4x_3 - x_2x_1x_3x_4 + x_2x_1x_4x_3$$

over K. Then, the set of values of ζ on KE is the space spanned by all paths of a length greater than or equal to 2. This can be checked by choosing a path $p_0 = e_1 \cdots e_l$ in $\mathcal{P}$ with $\ell(p) \geq 2$ and $x_0 = (t(e_1), e_1, t(e_1), e_2 \cdots e_l)$. Hence, $\zeta(x_0) = -p_0$. Therefore, by similar decomposition of all paths with a length greater than or equal to 2 as the decomposition done for p_0 into sub-paths in x_0, we obtain the desired result.

Recall the following definition of Lie polynomials of order 3.

Definition ([2, Definition 4])**.** A non-zero multi-linear Lie polynomial ζ of degree 3 is a polynomial over K that can be written in the form

$$\zeta(x_1, x_2, x_3) = c_1[[x_1, x_2], x_3] + c_2[[x_1, x_3], x_2],$$

where c_1 and c_2 are not both 0 and $c_i \in K$.

Example 3.4. Let ζ be the Lie polynomial of the order 3 defined as:

$$\begin{aligned}\zeta(x_1, x_2, x_3) &= [[x_1, x_2], x_3] + [[x_1, x_3], x_2] \\ &= x_1x_2x_3 + x_1x_3x_2 - 2x_2x_1x_3 + x_2x_3x_1 - 2x_3x_1x_2 + x_3x_2x_1.\end{aligned}$$

Then, the set of values of ζ on KE is the space spanned by all paths with a length greater than or equal to 1. Indeed, for every $p \in \mathcal{P}$ with $\ell(p) = 0$, and every $x \in I_p$, $\zeta(x) = 0$, where I_p is defined as in (3.1). Now, for an edge p_0 in $\mathcal{P}$, we have $x_0 = (p_0, t(p_0), t(p_0)) \in I_{p_0}$ and $\zeta(x_0) = 2p_0 \neq 0$. Hence, for every path p with $\ell(p) > 0$, we have

$$\zeta(p, t(p), t(p)) = 2p.$$

By linearity, we deduce that the set of values of ζ on KE is the space spanned by paths with length at least one.

For the definition of Lie polynomials of order 4, we have the following definition.

Definition ([2, Definition 5])**.** A non-zero multi-linear Lie polynomial ζ of degree 4 is a polynomial over K that can be written in the form

$$\begin{aligned}\zeta(x_1, x_2, x_3, x_4) = {} & c_1[[[x_1, x_2], x_3], x_4] + c_2[[[x_1, x_2], x_4], x_3] \\ & + c_3[[[x_1, x_3], x_2], x_4] + c_4[[[x_1, x_3], x_4], x_2] \\ & + c_5[[[x_1, x_4], x_2], x_3] + c_6[[[x_1, x_4], x_3], x_2],\end{aligned}$$

where c_i are not all 0 and $c_i \in K$.

Example 3.5. Let ζ be the Lie polynomial of the order 4 defined as:

$$\begin{aligned}\zeta(x_1, x_2, x_3, x_4) &= [[[x_1, x_2], x_4], x_3] + [[[x_1, x_3], x_4], x_2] - 2[[[x_1, x_4], x_2], x_3] \\ &= x_1x_2x_4x_3 + x_1x_3x_4x_2 - 2x_1x_4x_2x_3 - x_2x_1x_3x_4 + x_2x_1x_4x_3 \\ &\quad + x_2x_3x_1x_4 - x_2x_4x_1x_3 - x_2x_4x_3x_1 - x_3x_1x_2x_4 + x_3x_1x_4x_2 \\ &\quad - x_3x_2x_1x_4 + 2x_3x_2x_4x_1 - x_3x_4x_1x_2 - x_3x_4x_2x_1 + x_4x_1x_2x_3 \\ &\quad - x_4x_1x_3x_2 + x_4x_2x_1x_3 + x_4x_3x_1x_2.\end{aligned}$$

By the same reasoning as in the previous example, we choose an edge p_0 in $\mathcal{P}$, we have $x_0 = (s(p_0), p, t(p_0), t(p_0)) \in I_{p_0}$ and $\zeta(x_0) = p_0 \neq 0$. Hence, we conclude that the set of values of ζ on KE is the space spanned by paths with length at least one.

Since 2×2-upper triangular matrix algebra $T_2(K)$ is isomorphic to path algebra associated with the line quiver $E_2 : v_1 \xrightarrow{e} v_2$, we have the following result:

Corollary 3.6 ([16, Theorem 1.1])**.** *Let K be a field with characteristic zero. Let $\zeta(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, with $c_\sigma \in K$. Then, the image of ζ on KE_2 is KE_2, Ke or $\{0\}$.*

By similar reasoning, when K is a field with characteristic zero, the main result [8, Theorem 3] is generalized from strictly upper triangular matrix algebras to upper triangular matrix algebras $T_m(K) \cong KE_m$, where $m \geq 2$ and E_m is the line quiver $v_1 \xrightarrow{e_1} v_2 \cdots v_{m-1} \xrightarrow{e_{m-1}} v_m$.

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