J. Korean Math. Soc. **59** (2022), No. 6, pp. 1229–1254
https://doi.org/10.4134/JKMS.j220202
pISSN: 0304-9914 / eISSN: 2234-3008

TWO NEW RECURRENT LEVELS AND CHAOTIC DYNAMICS OF $\mathbb{Z}_+^d$-ACTIONS

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Abstract. In this paper, we introduce the concepts of (quasi-)weakly almost periodic point and minimal center of attraction for $\mathbb{Z}_+^d$-actions, explore the connections of levels of the topological structure the orbits of (quasi-)weakly almost periodic points and discuss the relations between (quasi-)weakly almost periodic point and minimal center of attraction. Especially, we investigate the chaotic dynamics near or inside the minimal center of attraction of a point in the cases of S-generic setting and non S-generic setting, respectively. Actually, we show that weakly almost periodic points and quasi-weakly almost periodic points have distinct topological structures of the orbits and we prove that if the minimal center of attraction of a point is non S-generic, then there exist certain Li-Yorke chaotic properties inside the involved minimal center of attraction and sensitivity near the involved minimal center of attraction; if the minimal center of attraction of a point is S-generic, then there exist stronger Li-Yorke chaotic (Auslander-Yorke chaotic) dynamics and sensitivity ($\aleph_0$-sensitivity) in the involved minimal center of attraction.

1. Introduction

The central problem of the study of dynamical systems is the asymptotic behaviors or topological structures of the orbits. As is known that the most important dynamics are concentrated on a full measure subset from the view of ergodic theory. Nevertheless, only such orbits of points with certain recurrence are of importance indeed. In [15], Zhou introduced the notions of *weakly almost periodic point* and *measure center*, and he proved that the measure center is just the closure of the set of weakly almost periodic points. The notion of *quasi-weakly periodic point* was introduced in [17] and it was proved that weakly almost periodic points and quasi-weakly almost periodic points have completely distinct ergodic properties, i.e., a point is quasi-weakly almost periodic if and

Received April 27, 2022; Accepted July 11, 2022.
2020 *Mathematics Subject Classification.* Primary 58B34, 58J42, 81T75.
Key words and phrases. Weakly almost periodic point, quasi-weakly almost periodic point, minimal center of attraction, chaotic dynamics.
This work was financially supported by the National Natural Science Foundation of China (No. 12061043, 11661054).

only if it belongs to its *minimal center of attraction*, and the support of each invariant measure generated by the orbit of a given weakly almost periodic point is its minimal center of attraction. Since then, the study of weakly almost periodic points and quasi-weakly almost periodic points has been an extremely interesting topic of topological dynamical systems and ergodic theory. For example, in [10], the authors introduced the concepts of weakly almost periodic point, quasi-weakly periodic point and measure center for continuous semi-flows and they proved that quasi-weakly almost periodic points possess especially rich orbit-structure and they presented a necessary and sufficient condition for a point to belong to its own minimal center of attraction. In [3,11], the authors introduced the concepts of weakly almost periodic point and minimal center of attraction for *amenable group actions* and they obtained some similar results of [15, 17]. In the paper, we introduce the notions of weakly almost periodic point, quasi-weakly almost periodic point, measure center and minimal center of attraction of $\mathbb{Z}_+^d$-actions and we investigate the chaotic properties near or inside the minimal center of attraction of a point in the state space for the cases of *S-generic* setting and non S-generic setting, respectively. In this paper, we say that $(X, \mathcal{T})$ is a $\mathbb{Z}_+^d$-action which means that X is a compact metric space with a metric ρ and $\mathcal{T} = \{T^h : X \to X\}_{h\in\mathbb{Z}_+^d}$ is a family of continuous transformations satisfying: $T^{h+k} = T^h \circ T^k$ for all $h, k \in \mathbb{Z}_+^d$, where $\mathbb{Z}_+^d$ is a *countable commutative additive topological semigroup*, d is a positive integer and $\mathbb{Z}_+$ is the set of non-negative integers.

The paper is organized as follows. In Section 2, we recall some necessary notions. In Section 3, we introduce the notions of weakly almost periodic point, quasi-weakly almost periodic point, measure center and minimal center of attraction of $\mathbb{Z}_+^d$-actions and we show that weakly almost periodic points and quasi-weakly almost periodic points have distinct topological structures of the orbits. Finally, we study the chaotic dynamics exhibited near or inside the minimal center of attraction of a point under the cases of S-generic setting and non S-generic setting, respectively.

2. Preliminaries

2.1. Basic concepts of countable additive topological semigroup $\mathbb{Z}_+^d$

Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action. For $k \in \mathbb{Z}_+^d$ and $\Lambda \subset \mathbb{Z}_+^d$, set

$$\Lambda + k = k + \Lambda = \{h + k : h \in \Lambda\}.$$

For $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, let

$$\Lambda_n = \{h = (h_1, \ldots, h_d) \in \mathbb{Z}_+^d : h_i < n \text{ for each } 1 \le i \le d\}$$

and

$$\lambda_n = |\Lambda_n|,$$

where $|A|$ denotes the cardinality of the set A.

For more details of $\mathbb{Z}_+^d$-actions, we refer the readers to see [6, 14].

For $A \subset \mathbb{Z}_+^d$, the *upper density* and *lower density* of A are defined, respectively, by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \Lambda_n|}{\lambda_n}$$

and

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \Lambda_n|}{\lambda_n}.$$

If $\overline{d}(A) = \underline{d}(A)$, then we call this value the *density* of A and denote it by $d(A)$.

For $A, B \subset \mathbb{Z}_+^d$ and $h \in \mathbb{Z}_+^d$, write

$$h^{-1}A = \{l \in \mathbb{Z}_+^d : l + h \in A\} \text{ and } B^{-1}A = \bigcup_{b\in B} b^{-1}A.$$

Definition 2.1 ([9])**.** A subset S of $\mathbb{Z}_+^d$ is called

(1) *thick* if for every $F \in Fin(\mathbb{Z}_+^d)$, there exists $h \in \mathbb{Z}_+^d$ such that $F + h \subseteq S$, where $Fin(\mathbb{Z}_+^d)$ denotes the collection of all finite subsets of $\mathbb{Z}_+^d$;

(2) *syndetic* if there exists $H \in Fin(\mathbb{Z}_+^d)$ such that

$$\mathbb{Z}_+^d = \bigcup_{t\in H} t^{-1}S, \quad \text{i.e.,} \quad \mathbb{Z}_+^d = H^{-1}S;$$

(3) *piecewise syndetic* if there exists $H \in Fin(\mathbb{Z}_+^d)$ such that for every finite subset A of $\mathbb{Z}_+^d$, there is some $h \in \mathbb{Z}_+^d$ satisfying

$$A + h \subseteq H^{-1}S.$$

Remark 2.1. Obviously, $S \subset \mathbb{Z}_+^d$ is syndetic if and only if S intersects every thick subset of $\mathbb{Z}_+^d$.

2.2. Basic concepts of $\mathbb{Z}_+^d$-actions

Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action and $x \in X$. The *orbit* of x under the action of $\mathbb{Z}_+^d$ is denoted by $\mathcal{T}x = \{T^h x : h \in \mathbb{Z}_+^d\}$. A subset Λ of X is called $\mathcal{T}$*-invariant* if $\mathcal{T}\Lambda \subset \Lambda$, i.e., $T^h x \in \Lambda$ for each $x \in \Lambda$ and each $h \in \mathbb{Z}_+^d$. Let $U, V \subset X$, define the *hitting time set* of U and V by

$$N(U, V) = \{h \in \mathbb{Z}_+^d : U \cap T^{-h}V \neq \emptyset\}$$

and the *recurrence time set* of x entering U by

$$N(x, U) = \{h \in \mathbb{Z}_+^d : T^h x \in U\}.$$

A point $x \in X$ is called a *recurrent point* of $(X, \mathcal{T})$ if for every open neighborhood U of x, $N(x, U)$ is infinite; an *almost periodic point* of $(X, \mathcal{T})$ if for every neighborhood U of x, $N(x, U)$ is syndetic. Denote by $R(\mathcal{T})$ and $A(\mathcal{T})$ the sets of recurrent points and almost periodic points of $(X, \mathcal{T})$, respectively. A point $y \in X$ is called an ω*-limit point of* x if $N(x, U)$ is infinite for every neighborhood U of y. The collection of all ω-limit points of x is called the ω*-limit set of* x and we denote it by $\omega_{\mathcal{T}}(x)$. For $x \in X$, the closure of the orbit $\mathcal{T}x$ of $x \in X$ in X, denoted by $\overline{\mathcal{T}x}$, is the union of $\mathcal{T}x$ and $\omega_{\mathcal{T}}(x)$, i.e., $\overline{\mathcal{T}x} = \mathcal{T}x \cup \omega_{\mathcal{T}}(x)$. A point $x \in X$ is called a *transitive point* of $(X, \mathcal{T})$ if $\overline{\mathcal{T}x} = X$.

$(X, \mathcal{T})$ is called

(i) *point transitive* if $(X, \mathcal{T})$ contains at least one transitive point;

(ii) *transitive* if $N(U, V) \neq \emptyset$ for each pair of nonempty open subsets U and V of X;

(iii) *minimal* if $\overline{\mathcal{T}x} = X$ for every $x \in X$; equivalently, there is no proper nonempty closed $\mathcal{T}$-invariant subset of X.

A point $x \in X$ is called minimal if the subsystem $(\overline{\mathcal{T}x}, \mathcal{T})$ of $(X, \mathcal{T})$ is minimal.

In the following, we give several lemmas which are necessary for the proofs of the main results of the paper.

Lemma 2.2 ([6, Theorem 1.15])**.** *Every point of a minimal $\mathbb{Z}_+^d$-action is almost periodic.*

Lemma 2.3 ([6, Theorem 1.17])**.** *Let $x \in X$. Then $\overline{\mathcal{T}x}$ is minimal if x is an almost periodic point of $(X, \mathcal{T})$.*

Lemma 2.4. *For all $x \in X$ and $U \subset X$, it holds that*

$$\underline{d}(N(x, U)) + \overline{d}(N(x, X - U)) = 1$$

and

$$\underline{d}(N(x, U)) + \underline{d}(N(x, X - U)) \leq 1.$$

Proof. Since the proof is simple, we omit it. □

2.3. Ergodic theory of $\mathbb{Z}_+^d$-actions

For convenience, we always assume that $(X, \mathcal{T})$ is a $\mathbb{Z}_+^d$-action.

Denote by $\mathcal{B}(X)$ the Borel σ-algebra of X and by $\mathcal{M}(X)$ the collection of all probability measures on X. A probability measure $\mu \in \mathcal{M}(X)$ is said to be *$\mathcal{T}$-invariant* if $T^h\mu = \mu$ for all $h \in \mathbb{Z}_+^d$, where $T^h\mu(A) := \mu(T^{-h}A)$ for every $A \in \mathcal{B}(X)$. A $\mathcal{T}$-invariant measure $\mu \in \mathcal{M}(X)$ is said to be *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ whenever $A \in \mathcal{B}(X)$ with $T^{-h}A = A$ for every $h \in \mathbb{Z}_+^d$. Denote by $\mathcal{M}(X, \mathcal{T})$ and $\mathcal{M}^e(X, \mathcal{T})$ the sets of all $\mathcal{T}$-invariant measures and ergodic measures of $\mathcal{M}(X)$. It is well known that $\mathcal{M}(X)$ is a convex, compact and metrizable space endowed with the *weak*-topology* and $\mathcal{M}(X, \mathcal{T})$ is a compact convex subset of $\mathcal{M}(X)$ (see [16]). For $\mu \in \mathcal{M}(X, \mathcal{T})$, denote by *supp*$(\mu)$ the *support* of μ, i.e., the smallest closed set $S \subset X$ with $\mu(S) = 1$. A point $x \in X$ is said to be a *support point* of $\mu \in \mathcal{M}(X, \mathcal{T})$ if $\mu(B(x, \epsilon)) > 0$ for each $\epsilon > 0$, where $B(x, \epsilon)$ denotes the ϵ-neighborhood of $x \in X$. It is clear that

$$supp(\mu) = \{x \in X : \mu(B(x, \epsilon)) > 0, \forall \epsilon > 0\}.$$

A point $x \in X$ is said to be a support point of $(X, \mathcal{T})$ if for each $\epsilon > 0$ there exists $\mu \in \mathcal{M}(X, \mathcal{T})$ such that $\mu(B(x, \epsilon)) > 0$. Denote by *supp*$(X, \mathcal{T})$ the set of all support points of $(X, \mathcal{T})$.

$x \in X$ determines an element δ_x of $\mathcal{M}(X)$ as follows: for each $A \in \mathcal{B}(X)$,

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\frac{1}{\lambda_n}\sum_{h\in\Lambda_n}\delta_{T^h x} \in \mathcal{M}(X)$ for each $x \in X$ and $n \in \mathbb{N}$. Let M_x denote the set of all limit points of $\{\frac{1}{\lambda_n}\sum_{h\in\Lambda_n}\delta_{T^h x}\}_{n=1}^{\infty}$. Write $M_{X_0} = \bigcup_{x\in X_0} M_x$ for a nonempty subset X_0 of X. From Lemmas 4.1.2 and 4.1.3 in [7], it follows that $\emptyset \neq M_x \subset \mathcal{M}(X, \mathcal{T})$ for each $x \in X$.

We state the following lemmas which play a considerable role in the proofs of our results.

Lemma 2.5 ([5, Theorem B.11])**.** *Let E be an open subset of X and F be a closed subset of X, μ_i, $\mu \in \mathcal{M}(X)$, $i \geq 1$, and $\mu_i \to \mu$ under the weak*-topology as $i \to \infty$. Then*

$$\liminf_{i\to\infty} \mu_i(E) \geq \mu(E) \quad \text{and} \quad \limsup_{i\to\infty} \mu_i(F) \leq \mu(F).$$

Birkhoff's Ergodic Theorem, Ergodic Decomposition Theorem and Mean Ergodic Theorem are important tools for the study of ergodic theory. In [7], the author gave Birkhoff's Ergodic Theorem and Ergodic Decomposition Theorem for $\mathbb{Z}_+^d$-actions as follows.

Lemma 2.6 (Birkhoff's Ergodic Theorem, [7, Theorem 2.1.5])**.** *Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action and $\mu \in \mathcal{M}(X, \mathcal{T})$. Then for every $f \in L^1(X, \mathcal{B}(X), \mu)$, there is a $\mathcal{T}$-invariant function $\tilde{f} \in L^1(X, \mathcal{B}(X), \mu)$ (that is $\tilde{f} \circ T^h = \tilde{f}$ a.e for each $h \in \mathbb{Z}_+^d$) such that*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} f(T^h x) = \tilde{f}(x)$$

for μ-a.e $x \in X$. In particular, if μ is ergodic, then $\tilde{f} = \int f d\mu$.

Lemma 2.7 (Ergodic Decomposition Theorem, [7, Theorem 2.3.3])**.** *Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action and $(X, \mathcal{B}(X), \mu, \mathcal{T})$ be a measure preserving dynamical system with underlying probability space $(X, \mathcal{B}(X), \mu)$, where μ is a Borel measure on X, and $\mathcal{C} = \{B \in \mathcal{B}(X) : T^{-h}B = B$ for every $h \in \mathbb{Z}_+^d\}$. Then there is a conditional probability distribution $(\mu_x | x \in X)$ for $\mu(\cdot|\mathcal{C})(x)$ such that*

(1) *$\mu_{T^h x} = \mu_x$ for all $x \in X$ and $h \in \mathbb{Z}_+^d$;*
(2) *μ_x is ergodic.*

However, we haven't seen Mean Ergodic Theorem for $\mathbb{Z}_+^d$-actions and we prove it as below since it is necessary for the proofs of our main results.

Lemma 2.8 (Mean Ergodic Theorem)**.** *Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action and $\mu \in \mathcal{M}(X, \mathcal{T})$. Then for every $f \in L^2(X, \mathcal{B}(X), \mu)$, there is a $\mathcal{T}$-invariant function $\tilde{f} \in L^2(X, \mathcal{B}(X), \mu)$ such that*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} f \circ T^h = \tilde{f} \quad \text{in} \quad L^2(X, \mathcal{B}(X), \mu).$$

In particular, if μ is ergodic, then $\tilde{f} = \int f d\mu$.

Proof. Let $\mu \in \mathcal{M}(X, \mathcal{T})$. If g is bounded and measurable, then

$$g \in L^2(X, \mathcal{B}(X), \mu)$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} g(T^h x) = \tilde{g}(x)$$

for μ-a.e $x \in X$ by Birkhoff's Ergodic Theorem. Hence, $\tilde{g} \in L^2(X, \mathcal{B}(X), \mu)$. By Bounded Convergence Theorem, for each $x \in X$,

$$\left\| \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} g(T^h x) - \tilde{g}(x) \right\|_2 \to 0.$$

Let $f \in L^2(X, \mathcal{B}(X), \mu)$ and $A_n(f)(x) = \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} f(T^h x)$ for every $x \in X$. Next we show that $\{A_n(f)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(X, \mathcal{B}(X), \mu)$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$ and $k \in \mathbb{N}$, then

$$\left\| \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} g(T^h x) - \frac{1}{\lambda_{n+k}} \sum_{h\in\Lambda_{n+k}} g(T^h x) \right\|_2 < \epsilon/2$$

for each $x \in X$. Take $g \in L^{\infty}(X, \mathcal{B}(X), \mu)$ such that $\|f - g\|_2 < \epsilon/4$. Then for $n > N$ and $k \in \mathbb{N}$,

$$\begin{aligned} \|A_n(f) - A_{n+k}(f)\|_2 \leq &\|A_n(f) - A_n(g)\|_2 + \|A_n(g) - A_{n+k}(g)\|_2 \\ &+ \|A_{n+k}(g) - A_{n+k}(f)\|_2 < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon, \end{aligned}$$

which implies that $\{A_n(f)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(X, \mathcal{B}(X), \mu)$ and so

$$\|A_n(f) - \tilde{f}\|_2 \to 0$$

for some $\tilde{f} \in L^2(X, \mathcal{B}(X), \mu)$.

Let $\{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{Z}_+^d$. As $f = f^+ - f^-$, it suffices to prove this result for $0 \leq f \in L^2(X, \mathcal{B}(X), \mu)$. For $i = 1, \ldots, d$, we have, for each $x \in X$,

$$\begin{aligned} &A_n(f) \circ T^{e_i}(x) \\ &= A_n(f)(x) - \frac{1}{\lambda_n} \sum_{h\in\Lambda_n, h_i=0} f \circ T^h(x) + \frac{1}{\lambda_n} \sum_{h\in(\Lambda_n+e_i)-\Lambda_n} f \circ T^h(x) \\ &\geq A_n(f)(x) - \frac{1}{n} \left(\frac{1}{n^{d-1}} \sum_{h\in\Lambda_n, h_i=0} f \circ T^h \right)(x). \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} A_n(f) \circ T^{e_i}(x) \geq \lim_{n\to\infty} A_n(f)(x)$$

for μ-a.e $x \in X$. At the same time, $\sum_{h\in\Lambda_n} f(T^{h+e_i}x) \le \sum_{h\in\Lambda_{n+1}} f(T^h x)$ and hence

$$\lim_{n\to\infty} A_n(f)\circ T^{e_i} \le \lim_{n\to\infty} \frac{(n+1)^d}{n^d}\Lambda_{n+1}(f) = \lim_{n\to\infty} A_n(f).$$

So we have $\tilde{f}\circ T^{e_i} = \tilde{f}$ a.e for each $i \in \{1,\dots,d\}$. Then, $\tilde{f}\circ T^h = \tilde{f}$ a.e for each $h \in \mathbb{Z}_+^d$ by $h = \sum_{i=1}^d h_i e_i$ $(h_i \in \mathbb{Z}_+,\ i = 1,2,\dots,d)$. □

3. Weakly almost periodic points and quasi-weakly almost periodic points and minimal center of attractions of $\mathbb{Z}_+^d$-actions

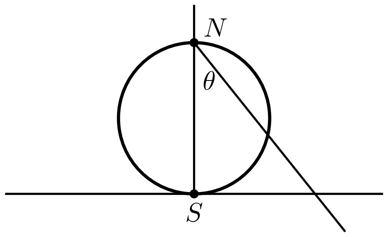
The notions of weakly almost periodic point and measure center were firstly introduced by Zhou [15] for discrete dynamical systems, and it was proved that the measure center of a discrete dynamical system is actually the closure of the set of weakly almost periodic points. The concepts of quasi-weakly periodic point and minimal center of attraction were introduced in [17] and it was proved that weakly almost periodic points and quasi-weakly almost periodic points have completely distinct ergodic properties. In this section, we mainly introduce these notions for $\mathbb{Z}_+^d$-actions and explore the relation between a (quasi-)weakly almost periodic point and its minimal center of attraction. In particular, let $(X,\mathcal{T})$ be a $\mathbb{Z}_+^d$-action, we obtain that a point $x \in X$ is quasi-weakly almost periodic if and only if it belongs to its minimal center of attraction, and the support of each invariant measure generated by the orbit of a given weakly almost periodic point $y \in X$ is the minimal center of attraction of y. The results given in this section are the analogues of the main results of [15] and [17]. Refer to [2] and [3] for the similar notions and results for amenable group actions.

3.1. Minimal center of attractions and measure centers of $\mathbb{Z}_+^d$-actions

Let $(X,\mathcal{T})$ be a $\mathbb{Z}_+^d$-action and for convenience, we denote by $\mathcal{N}_x$ the collection of all open neighborhoods of x in X. In the following, we introduce the concept of minimal center of attraction of a subset of X and we obtain some basic properties of minimal center of attractions.

Definition 3.1. Suppose that X_0 is a nonempty subset of X. A subset E of X is called *a center of attraction of* X_0 if E is closed and $d(N(x, B(E,\epsilon))) = 1$ for each $x \in X_0$ and $\epsilon > 0$, where $B(E,\epsilon)$ denotes the ϵ-neighborhood of E. If E is a center of attraction of X_0 and there is no proper subset of E satisfying the above conditions, then we say that E is *the minimal center of attraction of* X_0, denoted by C_{X_0}. When X_0 is a singleton, say $X_0 = \{x\}$, we denote the minimal center of attraction of $\{x\}$ by C_x.

Example 3.1. Consider the unit circle K as below:



We define $T^h : K \to K$ by $T^h(N) = N$, $T^h(S) = S$ and if $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), T^h(\theta) = \tan^{-1}(\tan(\theta)/2^{h_1+h_2})$ for every $h = (h_1, h_2) \in \mathbb{Z}_+^2$. Let $\mathcal{T} = \{T^h\}_{h\in\mathbb{Z}_+^2}$. Then $(K, \mathcal{T})$ is a $\mathbb{Z}_+^2$-action. It is clear that the singleton $\{S\}$ is the minimal center of attraction of every point of $K - \{N\}$.

Lemma 3.2. *For every $x \in X$, $C_x = \left\{y \in X : \overline{d}(N(x, U)) > 0,\ \forall U \in \mathcal{N}_y\right\}$.*

Proof. Let $x \in X$. Write

$$H = \left\{y \in X : \overline{d}(N(x, U)) > 0,\ \forall U \in \mathcal{N}_y\right\}.$$

Firstly, we prove that $C_x \subset H$, i.e., for every $y \in C_x$ and $U \in \mathcal{N}_y$, we have $\overline{d}(N(x, U)) > 0$. Suppose to the contrary that there exist $y \in C_x$ and $\epsilon_0 > 0$ such that

$$d(N(x, B(y, \epsilon_0))) = 0,$$

which implies that $C_x - B(y, \epsilon_0)$ is also a center of attraction of x. This is contradictory to the minimality of C_x. Hence, $C_x \subset H$.

Next, we show that $H \subset C_x$. Suppose that there exists $y \in H - C_x$. Take $U \in \mathcal{N}_y$ and an open neighborhood V of C_x with $U \cap V = \emptyset$. Since $\overline{d}(N(x, U)) > 0$, by Lemma 2.4, we have $\underline{d}(N(x, V)) < 1$ which is contradictory to $d(N(x, V)) = 1$. Therefore, $H \subset C_x$. This ends the proof. □

Corollary 3.3. *Let $(X, \mathcal{T})$ be a $\mathbb{Z}_+^d$-action. For each $x \in X$, C_x is $\mathcal{T}$-invariant.*

Proof. Let $y \in C_x$, $h \in \mathbb{Z}_+^d$ and $U \in \mathcal{N}_{T^h y}$. Then $T^{-h}U \in \mathcal{N}_y$. By Lemma 3.2, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{|\Lambda_n \cap N(x, U)|}{\lambda_n} &\geq \limsup_{n\to\infty} \frac{|(\Lambda_n + h) \cap N(x, U)|}{\lambda_n} \\ &= \limsup_{n\to\infty} \frac{|\Lambda_n \cap N(x, T^{-h}U)|}{\lambda_n} > 0.\end{aligned}$$

Therefore, $T^h y \in C_x$, namely, C_x is $\mathcal{T}$-invariant. □

Proposition 3.4. *If $C_x = X$ for some $x \in X$, then $\{y \in X : C_y = X\}$ is residual in X.*

Proof. If $x \in X$ with $C_x = X$, then for each nonempty open subset U of X, $N(x, U)$ has positive upper density by Lemma 3.2. Therefore, from Lemma 3.5, it follows that $\overline{\mathcal{T}x} = X$.

Choose a countable basis $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of X and for each $i \in \mathbb{N}$, let $r_i := \overline{d}(N(x, U_i)) > 0$ and let

$$A_i = \bigcap_{m=1}^{\infty} \bigcup_{n>m} \left\{ y \in X : \frac{|\Lambda_n \cap N(y, U_i)|}{\lambda_n} > \frac{r_i}{2} \right\}.$$

Then A_i is a G_δ-subset of X for each $i \in \mathbb{N}$. Since $\mathcal{T}x \subset A_i$ for each $i \in \mathbb{N}$, $\bigcap_{i=1}^{\infty} A_i$ is a dense G_δ-subset of X. Since $C_y = X$ for every $y \in \bigcap_{i=1}^{\infty} A_i$, $\{x \in X : C_x = X\}$ is residual in X. □

Lemma 3.5. *$C_x \subset \overline{\mathcal{T}x}$ for each $x \in X$. In particular, if $x \in C_x$, then $C_x = \overline{\mathcal{T}x}$.*

Proof. Let $x \in X$, it is clear that for each $\epsilon > 0$, $d(N(x, B(\overline{\mathcal{T}x}, \epsilon))) = 1$ which shows that $C_x \subset \overline{\mathcal{T}x}$. If $x \in C_x$, then $\overline{\mathcal{T}x} \subset C_x$ by the definition of C_x, and so $C_x = \overline{\mathcal{T}x}$. □

Proposition 3.6. *Let X_0 be a nonempty subset of X. Then $C_{X_0} = \overline{\bigcup_{x\in X_0} C_x}$. In particular, $C_X = \overline{\bigcup_{x\in X} C_x}$.*

Proof. Let $X_0 \subset X$ be nonempty. Firstly, we show that $C_{X_0} \subset \overline{\bigcup_{x\in X_0} C_x}$. Let U be an open neighborhood of $\overline{\bigcup_{x\in X_0} C_x}$. Then U is also an open neighborhood of C_x for each $x \in X_0$ which indicates that $d(N(x, U)) = 1$ for every $x \in X_0$. Hence, $C_{X_0} \subset \overline{\bigcup_{x\in X_0} C_x}$ by the minimality of C_{X_0}.

Next, we prove that $\overline{\bigcup_{x\in X_0} C_x} \subset C_{X_0}$. Otherwise, there exists $y \in \overline{\bigcup_{x\in X_0} C_x} - C_{X_0}$. Choose an open neighborhood U of y and an open neighborhood V of C_{X_0} with $U \cap V = \emptyset$. Then $C_x \cap U \neq \emptyset$ for some $x \in X_0$. Since $\overline{d}(N(x, U)) > 0$ by Lemma 3.2, $\underline{d}(N(x, V)) < 1$ by Lemma 2.4. This is a contradiction, and so $\overline{\bigcup_{x\in X_0} C_x} \subset C_{X_0}$. □

In the following, we introduce the concept of measure center for $\mathbb{Z}_+^d$-actions and give some basic properties of measure center.

Definition 3.2. Suppose that X_0 is a nonempty subset of X. A subset E of X is called *the measure center of* X_0 if E is closed, $\mathcal{T}$-invariant, $\mu(E) = 1$ for each $\mu \in M_{X_0}$ and there is no proper subset of E with these properties. Denote by $MC(X_0)$ the measure center of X_0. When $X_0 = X$, we call $MC(X)$ the measure center of $(X, \mathcal{T})$.

Lemma 3.7. *Let $X_0 \subset X$ be nonempty. Then $MC(X_0) = \overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)}$. In particular, $MC(X) = \overline{\bigcup_{\mu\in M_X} supp(\mu)}$.*

Proof. Let $X_0 \subset X$ be nonempty. As $\mu(\overline{\bigcup_{\nu\in M_{X_0}} supp(\nu)}) = 1$ for each $\mu \in M_{X_0}$ and $\overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)}$ is closed and $\mathcal{T}$-invariant, then

$$MC(X_0) \subset \overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)}$$

by the minimality of $MC(X_0)$.

Since $\mu(MC(X_0)) = 1$ for each $\mu \in M_{X_0}$ and $MC(X_0)$ is closed and $\mathcal{T}$-invariant, $supp(\mu) \subset MC(X_0)$, $\overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)} \subset MC(X_0)$. Therefore, $\overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)} = MC(X_0)$. □

Lemma 3.8. $MC(X) = supp(X, \mathcal{T})$.

Proof. By Lemma 3.7, it suffices to prove that $supp(X,\mathcal{T}) = \overline{\bigcup_{\mu\in M_X} supp(\mu)}$. Obviously, $\overline{\bigcup_{\mu\in M_X} supp(\mu)} \subset supp(X,\mathcal{T})$. It is left to prove that $supp(X,\mathcal{T}) \subset \overline{\bigcup_{\mu\in M_X} supp(\mu)}$. Suppose that there exists $y \in supp(X,\mathcal{T}) - \overline{\bigcup_{\mu\in M_X} supp(\mu)}$. Then there is $\epsilon > 0$ such that $\mu(B(y,\epsilon)) = 0$ for all $\mu \in M_X$ which is contrary to $y \in supp(X,\mathcal{T})$. This ends the proof. □

Proposition 3.9. *Suppose that $X_0 \subset X$ is nonempty. Then*

$$C_{X_0} = \overline{\bigcup_{\mu\in M_{X_0}} supp(\mu)} = MC(X_0).$$

Particularly,

$$C_X = \overline{\bigcup_{\mu\in M_X} supp(\mu)} = MC(X) \text{ and } C_x = \overline{\bigcup_{\mu\in M_x} supp(\mu)} = MC(\{x\})$$

for all $x \in X$.

Proof. Let $X_0 \subset X$ be nonempty. By Lemma 3.7, it suffices to prove $C_{X_0} = MC(X_0)$.

Firstly, we claim that $MC(X_0)$ is a center of attraction of X_0. Suppose to the contrary that there exist $\epsilon_0 > 0$, $x \in X_0$ and a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap N(x, B(MC(X_0),\epsilon_0))|}{\lambda_{n_k}} < 1.$$

Without loss of generality, we assume that

$$\nu_k := \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x} \to \nu \in M_x \quad \text{as } k \to \infty.$$

Then from Lemmas 2.4 and 2.5,

$$\begin{aligned}\nu(X - B(MC(X_0),\epsilon_0)) &\geq \limsup_{k\to\infty} \nu_k(X - B(MC(X_0),\epsilon_0)) \\ &= 1 - \liminf_{k\to\infty} \nu_k(B(MC(X_0),\epsilon_0)) > 0,\end{aligned}$$

which is contrary to $\nu(MC(X_0)) = 1$. Thus, $C_{X_0} \subset MC(X_0)$.

Next, we prove that $MC(X_0) \subset C_{X_0}$. Assume to the contrary that there exists $y \in MC(X_0) - C_{X_0}$. Then we can choose an open neighborhood U of y and an open neighborhood V of C_{X_0} such that $U \cap V = \emptyset$. By Lemma 3.7, there exists $\mu \in M_{X_0}$ such that $\mu(U) > 0$. Without loss of generality, we assume that $\frac{1}{\lambda_n} \sum\limits_{h \in \Lambda_n} \delta_{T^h x} \to \mu$ for some $x \in X_0$. Then by Lemmas 2.5 and 2.4, we have

$$\begin{aligned}\mu(V) &\leq \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \delta_{T^h x}(V) \\ &\leq 1 - \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \delta_{T^h x}(X - V) \\ &\leq 1 - \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \delta_{T^h x}(U) \\ &= 1 - \mu(U) < 1,\end{aligned}$$

which is a contradiction since C_{X_0} is the minimal center of attraction of X_0. Thus, $C_{X_0} = MC(X_0)$. □

Proposition 3.10. *Let $x \in X$. If $x \in C_x$, then $\omega_{\mathcal{T}}(x) = \overline{\bigcup_{\mu \in M_x} supp(\mu)}$.*

Proof. Let $x \in X$. Then it is clear that $C_x \subset \omega_{\mathcal{T}}(x)$. As $x \in C_x$ and C_x is closed and $\mathcal{T}$-invariant, then $\omega_{\mathcal{T}}(x) \subset C_x$. From Proposition 3.9, it follows that $\omega_{\mathcal{T}}(x) = \overline{\bigcup_{\mu \in M_x} supp(\mu)}$. □

3.2. Quasi-weakly almost periodic points and weakly almost periodic points of $\mathbb{Z}_+^d$-actions

In this subsection, we introduce the concepts of quasi-weakly almost periodic point and weakly almost periodic point of $\mathbb{Z}_+^d$-actions and we give some basic properties of quasi-weakly almost periodic points and weakly almost periodic points. Moreover, we present some equivalent conditions for a point in the state space to be a (quasi-)weakly almost periodic point for $\mathbb{Z}_+^d$-actions.

For the sake of description, we always assume that $(X, \mathcal{T})$ is a $\mathbb{Z}_+^d$-action.

Definition 3.3. $x \in X$ is said to be a *weakly almost periodic point* of $(X, \mathcal{T})$ if for each open neighborhood U of x,

$$\underline{d}(N(x, U)) = \liminf_{n\to\infty} \frac{|\{h \in \Lambda_n : T^h x \in U\}|}{\lambda_n} > 0;$$

$x \in X$ is said to be a *quasi-weakly almost periodic point* of $(X, \mathcal{T})$ if for each open neighborhood U of x,

$$\overline{d}(N(x, U)) = \limsup_{n\to\infty} \frac{|\{h \in \Lambda_n : T^h x \in U\}|}{\lambda_n} > 0.$$

The sets of all weakly almost periodic points and quasi-weakly almost periodic points of $(X, \mathcal{T})$ are denoted by $W(\mathcal{T})$ and $QW(\mathcal{T})$, respectively.

Proposition 3.11. *$W(\mathcal{T})$ and $QW(\mathcal{T})$ are $\mathcal{T}$-invariant.*

Proof. Assume that $x \in W(\mathcal{T})$. Then it is enough to show that $T^l x \in W(\mathcal{T})$ for each $l \in \mathbb{Z}_+^d$. Let $l \in \mathbb{Z}_+^d$ and U be an open neighborhood of $T^l x$, pick an open neighborhood V of x such that $T^l V \subset U$ by the continuity of T^l. Since $x \in W(\mathcal{T})$, $\liminf_{n\to\infty} \frac{|\{h\in\Lambda_n : T^h x \in V\}|}{\lambda_n} > 0$. Therefore,

$$\liminf_{n\to\infty} \frac{|\{h \in \Lambda_n : T^{h+l}x \in U\}|}{\lambda_n} \geq \liminf_{n\to\infty} \frac{|\{h \in \Lambda_n : T^h x \in V\}|}{\lambda_n} > 0.$$

Then $T^l x \in W(\mathcal{T})$ by the arbitrariness of U which implies $W(\mathcal{T})$ is $\mathcal{T}$-invariant.

Similarly, we can prove that $QW(\mathcal{T})$ is $\mathcal{T}$-invariant. □

Proposition 3.12. *$W(\mathcal{T}) \subset QW(\mathcal{T}) \subset supp(X, \mathcal{T})$.*

Proof. It is obvious that $W(\mathcal{T}) \subset QW(\mathcal{T})$ and so it suffices to prove $QW(\mathcal{T}) \subset supp(X, \mathcal{T})$.

Let $x \in QW(\mathcal{T})$. Then for every $\epsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{|\Lambda_n \cap N(x, B(x, \epsilon/2))|}{\lambda_n} > 0.$$

Without loss of generality, we assume that, under the weak*-topology,

$$\frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \delta_{T^h x} \to \mu \in M_x \quad \text{as } n \to \infty.$$

Then by Lemma 2.5,

$$\mu(B(x,\epsilon)) \geq \mu(\overline{B(x,\epsilon/2)}) \geq \limsup_{n\to\infty} \frac{|\Lambda_n \cap N(x, B(x,\epsilon/2))|}{\lambda_n} > 0.$$

And so $x \in supp(X, \mathcal{T})$, i.e., $QW(\mathcal{T}) \subset supp(X, \mathcal{T})$. □

Proposition 3.13. *Let $x \in X$ and C be a nonempty closed subset of X. If $\overline{d}(N(x, C)) > 0$, then there exists $\nu \in M_x$ such that $\nu(C) > 0$.*

Proof. Let $x \in X$ and $C \subset X$ be a nonempty closed set such that $\overline{d}(N(x, C)) > 0$. Then there exists a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap N(x, C)|}{\lambda_{n_k}} > 0.$$

Without loss of generality, assume that $\lim_{k\to\infty} \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x} = \nu$ under the weak*-topology. Clearly, $\nu \in M_x$, and by Lemma 2.5,

$$\nu(C) \geq \limsup_{k\to\infty} \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x}(C) > 0.$$

□

Corollary 3.14. *Let $x \in QW(\mathcal{T})$ and $U \in \mathcal{N}_x$. Then there is $\nu \in M_x$ such that $\nu(\overline{U}) > 0$.*

Proposition 3.15. *Let $x \in R(\mathcal{T})$. If $C_x = \omega_{\mathcal{T}}(x)$, then $x \in QW(\mathcal{T})$.*

Proof. Suppose $x \in R(\mathcal{T})$ and $C_x = \omega_{\mathcal{T}}(x)$. Then $x \in \omega_{\mathcal{T}}(x) = C_x$. If $x \notin QW(\mathcal{T})$, by the definition of quasi-weakly almost periodic point, there exists $\epsilon_0 > 0$ such that $N(x, B(x, \epsilon_0))$ has upper density zero. From Lemma 3.2, $x \notin C_x$ which is a contradiction. Therefore, $x \in QW(\mathcal{T})$. □

Lemma 3.16. *Let $x \in R(\mathcal{T})$. Then $x \in W(\mathcal{T})$ if and only if $\omega_{\mathcal{T}}(x) = supp(\mu)$ for each $\mu \in M_x$.*

Proof. Suppose $x \in W(\mathcal{T})$ and $\mu \in M_x$. Then there exists a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\mu_k = \frac{1}{\lambda_{n_k}} \sum_{h \in \Lambda_{n_k}} \delta_{T^h x} \to \mu \in M_x$$

under the weak*-topology as $k \to \infty$. Let $\epsilon > 0$, choose $\delta > 0$ such that $\delta < \epsilon$. Then we have

$$\mu(B(x,\epsilon)) \geq \mu(\overline{B(x,\delta)}) \geq \limsup_{k\to\infty} \mu_k(\overline{B(x,\delta)}) > 0.$$

Let $h \in \mathbb{Z}_+^d$ and $U \in \mathcal{N}_{T^h x}$. Then $T^{-h}U$ is an open neighborhood of x and $\mu(U) = \mu(T^{-h}U) > 0$. Since $\mathcal{T}x$ is dense in $\omega_{\mathcal{T}}(x)$, for each $y \in \omega_{\mathcal{T}}(x)$ and each open neighborhood V of y, there exists $l \in \mathbb{Z}_+^d$ such that V is an open neighborhood of $T^l x$ which implies that $\mu(V) > 0$ by the above discussions. Hence, every point in $\omega_{\mathcal{T}}(x)$ is a support point of μ, i.e., $\omega_{\mathcal{T}}(x) \subset supp(\mu)$.

Next, we show that $supp(\mu) \subset \omega_{\mathcal{T}}(x)$ for each $\mu \in M_x$. Suppose $\mu \in M_x$ and $y \in supp(\mu)$. Then for each $\epsilon > 0$, we have $\mu(B(y,\epsilon)) > 0$. Without loss of generality, assume that when $k \to \infty$,

$$\frac{1}{\lambda_{n_k}} \sum_{h \in \Lambda_{n_k}} \delta_{T^h x} \to \mu$$

under the weak*-topology. By Lemma 2.5,

$$0 < \mu(B(y,\epsilon)) \leq \liminf_{k\to\infty} \frac{1}{\lambda_{n_k}} \sum_{h \in \Lambda_{n_k}} \delta_{T^h x}(B(y,\epsilon)).$$

It shows that there exists $h \in \mathbb{Z}_+^d$ such that $T^h x \in B(y,\epsilon)$, i.e., $y \in \omega_{\mathcal{T}}(x)$. Therefore, $\omega_{\mathcal{T}}(x) = supp(\mu)$ for every $\mu \in M_x$.

Conversely, it is clear that $x \in \omega_{\mathcal{T}}(x)$ since $x \in R(\mathcal{T})$. If $x \notin W(\mathcal{T})$, then there exist $\epsilon_0 > 0$ and a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{|\{h \in \Lambda_{n_k} : T^h x \in B(x,\epsilon_0)\}|}{\lambda_{n_k}} = 0.$$

Without loss of generality, we assume that when $k \to \infty$,

$$\frac{1}{\lambda_{n_k}} \sum_{h \in \Lambda_{n_k}} \delta_{T^h x} \to \mu \in M_x$$

under the weak*-topology. By Lemma 2.5,

$$\mu(B(x,\epsilon_0)) \le \liminf_{k\to\infty} \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x}(B(x,\epsilon_0)) = 0,$$

which implies that x is not a support point of μ. Thus, $\omega_{\mathcal{T}}(x) \neq supp(\mu)$ by $x \in \omega_{\mathcal{T}}(x)$. □

Theorem 3.17. *Let $x \in R(\mathcal{T})$. Then the following statements are equivalent*:

(1) $x \in QW(\mathcal{T})$;
(2) $x \in C_x$;
(3) $\omega_{\mathcal{T}}(x) = \overline{\bigcup_{\mu\in M_x} supp(\mu)}$;
(4) $C_x = \omega_{\mathcal{T}}(x)$.

Proof. By Lemma 3.2, Definition 3.3 and Propositions 3.9, 3.10 and 3.15, the proof is straightforward. □

Theorem 3.18. *$QW(\mathcal{T})$ is Borel measurable and has full measure for each $\mu \in \mathcal{M}(X,\mathcal{T})$.*

Proof. Let $k,n,m \in \mathbb{N}$ and $t > 0$ and $A_k(t,m,n)$ be the set of all points $x \in X$ with the property that there exists some open neighborhood U of x with $diam(U) < \frac{1}{k}$ such that

$$\frac{|\Lambda_n \cap N(x,U)|}{\lambda_n} > t - \frac{1}{m},$$

where $diam(B)$ denotes the diameter of B. For every $k \in \mathbb{N}$, let

$$A_k = \bigcup_{i=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_k\left(\frac{1}{i}, n, m\right).$$

Then $QW(\mathcal{T}) = \bigcap_{k=1}^{\infty} A_k$ is Borel measurable since every $A_k(\frac{1}{i}, n, m)$ is open.

Next, we show that $\mu(QW(\mathcal{T})) = 1$ for each $\mu \in \mathcal{M}(X,\mathcal{T})$. It suffices to prove that the result holds for each ergodic measure by Lemma 2.7. Let $\mu \in \mathcal{M}^e(X,\mathcal{T})$, we show that $\mu(A_k) = 1$ for each $k \in \mathbb{N}$. If otherwise, there exists $k_0 \in \mathbb{N}$ such that $\mu(A_{k_0}) < 1$. We can choose some measurable set $B \subset X - A_{k_0}$ with $\mu(B) > 0$ and $diam(B) < \frac{1}{3k_0}$. Let

$$f(x) = \limsup_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \chi_{T^{-h}B}(x)$$

for each $x \in X$, where χ_E denotes the characteristic function of $E \subset X$. Then $f(x)$ is Borel measurable and $0 \le f(x) \le 1$ for every $x \in X$. By Mean Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \chi_{T^{-h}B} = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \chi_B \circ T^h = \mu(B)$$

under the L^2-norm $\|\cdot\|_2$ of $L^2(X, \mathcal{B}(X), \mu)$. By Fatou's Lemma, we have

$$\int_B f(x) d\mu(x) \geq \limsup_{n\to\infty} \int_B \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \chi_{T^{-h}B}(x) d\mu(x) = \mu(B)^2 > 0,$$

which implies that there is $x \in B$ such that $f(x) > 0$. Let

$$U = B\left(x, \frac{2}{3k_0}\right) = \left\{y \in X : \rho(x, y) < \frac{2}{3k_0}\right\}.$$

Then $B \subset U$ which indicates that $\bar{d}(N(x, U)) \geq f(x) > 0$. Therefore, $x \in A_{k_0}$ which is contrary to $x \in B \subset X - A_{k_0}$. This contradiction implies $\mu(A_k) = 1$ for every $k \in \mathbb{N}$ and so $\mu(\bigcap_{k=1}^{\infty} A_k) = 1$. Thus, $\mu(QW(\mathcal{T})) = 1$ for each $\mu \in \mathcal{M}(X, \mathcal{T})$. □

For the sake of description of the next result, we introduce the notion of generic point for a given probability measure on X.

Definition 3.4. Let $\mu \in \mathcal{M}(X)$. $x \in X$ is said to be a *generic point* for μ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} f(T^h x) = \int_X f d\mu$$

for each $f \in \mathcal{C}(X, \mathbb{R})$, where $\mathcal{C}(X, \mathbb{R})$ denotes the collection of all continuous real-valued functions on X. Denote by $\mathcal{T}_\mu$ the set of all generic points for $\mu \in \mathcal{M}(X)$.

The following lemma is a consequence of Birkhoff's Ergodic Theorem.

Lemma 3.19. *For every $\mu \in \mathcal{M}^e(X, \mathcal{T})$, almost every point in X is generic for μ.*

Corollary 3.20. $\mathcal{M}^e(X, \mathcal{T}) \subset M_X$.

Theorem 3.21. *$W(\mathcal{T})$ is Borel measurable and has full measure for each $\mu \in \mathcal{M}(X, \mathcal{T})$.*

Proof. Let

$$\psi\left(x, \frac{1}{k}\right) = \liminf_{n\to\infty} \frac{|\Lambda_n \cap N(x, B(x, \frac{1}{k}))|}{\lambda_n}$$

be a map from X to $[0, 1]$ for every $k \in \mathbb{N}$. Then $\psi(x, \frac{1}{k})$ is Borel measurable every $k \in \mathbb{N}$. Write

$$W = \bigcap_{k=1}^{\infty} \left\{x \in X : \psi\left(x, \frac{1}{k}\right) > 0\right\},$$

we obtain that $W(\mathcal{T}) = W$ which shows that $W(\mathcal{T})$ is Borel measurable.

Next we show that $\mu(W(\mathcal{T})) = 1$ for each $\mu \in \mathcal{M}(X, \mathcal{T})$. We need only to prove the result for each $\mu \in \mathcal{M}^e(X, \mathcal{T})$ by Lemma 2.7.

Let $\mu \in \mathcal{M}^e(X, \mathcal{T})$, we claim that $supp(\mu) \cap \mathcal{T}_\mu \subset W(\mathcal{T})$. Let $x \in supp(\mu) \cap \mathcal{T}_\mu$ and $U \in \mathcal{N}_x$. Let $V \in \mathcal{N}_x$ with $x \in V \subset \overline{V} \subset U$. Then $\mu(V) > 0$. By

Urysohn's Lemma, there exists a continuous function $\phi : X \to [0,1]$ such that $\phi(y) = 1$ for each $y \in \overline{V}$ and $\phi(y) = 0$ for each $y \in X - U$. As

$$\begin{aligned}\liminf_{n\to\infty} \frac{|\Lambda_n \cap N(x,U)|}{\lambda_n} &\geq \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \phi(T^h x) \\ &= \int_X \phi d\mu = \mu(\overline{V}) \geq \mu(V) > 0,\end{aligned}$$

$x \in W(\mathcal{T})$ by the arbitrariness of U. Then $supp(\mu) \cap \mathcal{T}_\mu \subset W(\mathcal{T})$. Since $\mu(supp(\mu) \cap \mathcal{T}_\mu) = 1$ by Lemma 3.19 and the definition of $supp(\mu)$, $\mu(W(\mathcal{T})) = 1$. This ends the proof. □

Proposition 3.22. $MC(X) = \overline{W(\mathcal{T})} = \overline{QW(\mathcal{T})}$.

Proof. By Theorem 3.21, $\mu(W(\mathcal{T})) = 1$ for each $\mu \in M_X$ which shows that $MC(X) \subset \overline{W(\mathcal{T})}$. It follows from Lemma 3.8 and Proposition 3.12 that

$$\overline{W(\mathcal{T})} \subset supp(X, \mathcal{T}) = MC(X).$$

Hence, $MC(X) = \overline{W(\mathcal{T})}$. By Proposition 3.12 again, we obtain that $MC(X) = \overline{QW(\mathcal{T})} = \overline{W(\mathcal{T})}$. □

Theorem 3.23. *$QW(\mathcal{T})$ is residual in X if there exists $\mu \in \mathcal{M}^e(X, \mathcal{T})$ such that $supp(\mu) = X$.*

Proof. Let $\mu \in \mathcal{M}^e(X, \mathcal{T})$ with $supp(\mu) = X$ and $\mathcal{U} = \{U_i\}_{i=1}^\infty$ be a countable basis of X. For each $i \in \mathbb{N}$, we can take a nonempty open subset V_i of X satisfying $\overline{V_i} \subset U_i$. Let $r_i = \mu(V_i)$. Then $r_i > 0$ for each $i \in \mathbb{N}$. By Urysohn's Lemma, for each $i \in \mathbb{N}$, there exists a continuous function $\phi_i : X \to [0,1]$ such that $\phi_i(y) = 1$ for each $y \in \overline{V_i}$ and $\phi_i(y) = 0$ for each $y \in X - U_i$. Let $x \in \mathcal{T}_\mu$, then for each $i \in \mathbb{N}$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{|\Lambda_n \cap N(x,U_i)|}{\lambda_n} &\geq \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \phi_i(T^h x) \\ &= \int_X \phi_i d\mu = \mu(\overline{V_i}) \geq \mu(V_i) > 0.\end{aligned}$$

Hence, the upper density of $N(x, U_i)$ is larger than r_i for each $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ and

$$A_i = \bigcap_{m=1}^{\infty} \bigcup_{n>m} \left\{ y \in X : \frac{|\Lambda_n \cap N(y,U_i)|}{\lambda_n} > r_i/2 \right\}.$$

Then every generic point for μ belongs to A_i and A_i is a G_δ-subset of X for each $i \in \mathbb{N}$. By Lemma 3.19, $\mathcal{T}_\mu$ is dense in X. Then, $\bigcap_{i=1}^\infty A_i \subset QW(\mathcal{T})$ and $\bigcap_{i=1}^\infty A_i$ is a dense G_δ-subset of X. Therefore, $QW(\mathcal{T})$ is residual. □

Theorem 3.24. *$W(\mathcal{T})$ contains a dense $F_{\sigma\delta}$-subset of X if there exists $\mu \in \mathcal{M}^e(X, \mathcal{T})$ with $supp(\mu) = X$.*

Proof. Let $\mu \in \mathcal{M}^e(X,\mathcal{T})$ and $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a countable basis of X. For each $i \in \mathbb{N}$, we can take a nonempty open subset V_i of X satisfying $\overline{V_i} \subset U_i$. Let $r_i = \mu(V_i) > 0$ for each $i \in \mathbb{N}$. By Urysohn's Lemma, for each $i \in \mathbb{N}$, there exists a continuous function $\phi_i : X \to [0,1]$ such that $\phi_i(y) = 1$ for each $y \in \overline{V_i}$ and $\phi_i(y) = 0$ for each $y \in X - U_i$. Let $x \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{|\Lambda_n \cap N(x,U_i)|}{\lambda_n} &\geq \liminf_{n\to\infty} \frac{1}{\lambda_n} \sum_{h\in\Lambda_n} \phi_i(T^h x)\\ &= \int_X \phi_i d\mu = \mu(\overline{V_i}) \geq \mu(V_i) > 0.\end{aligned}$$

Hence for each $i \in \mathbb{N}$, the lower density of $N(x,U_i)$ is larger than r_i. For each $i \in \mathbb{N}$, let

$$A_i = \bigcup_{m=1}^{\infty} \bigcap_{n>m} \left\{ y \in X : \frac{|\Lambda_n \cap N(y,U_i)|}{\lambda_n} \geq \frac{r_i}{2} \right\}.$$

Then every generic point for μ belongs to A_i and A_i is a F_σ-subset of X for each $i \in \mathbb{N}$. By Lemma 3.19, $\mathcal{T}_\mu$ is dense in X. Therefore, $\bigcap_{i=1}^{\infty} A_i \subset W(\mathcal{T})$ and $\bigcap_{i=1}^{\infty} A_i$ is a dense $F_{\sigma\delta}$-subset of X, which implies that $W(\mathcal{T})$ contains a dense $F_{\sigma\delta}$-subset of X. □

Inspired by Lemma 3.2, for every $x \in X$, we write

$$I_x = \left\{ y \in X : \liminf_{n\to\infty} \frac{|\Lambda_n \cap N(x,U)|}{\lambda_n} > 0, \forall U \in \mathcal{N}_y \right\}.$$

Proposition 3.25. *If there exists $x \in X$ such that $I_x = X$, then the set $\{x \in X : I_x = X\}$ contains a dense $F_{\sigma\delta}$-subset of X.*

Proof. Let $x \in X$ with $I_x = X$. Then for each nonempty open subset U of X, $N(x,U)$ has positive lower density. By the definition of I_x and Lemma 3.5, we obtain that $\overline{\mathcal{T}x} = X$.

Choose a countable topological base $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of X. For each $i \in \mathbb{N}$, let r_i be the lower density of $N(x,U_i)$ and let

$$A_i = \bigcup_{m=1}^{\infty} \bigcap_{n>m} \left\{ y \in X : \frac{|\Lambda_n \cap N(y,U_i)|}{\lambda_n} \geq r_i/2 \right\}.$$

Then A_i is a F_σ-subset of X for each $i \in \mathbb{N}$. Since $\mathcal{T}x \subset A_i$ for each $i \in \mathbb{N}$, $\bigcap_{i=1}^{\infty} A_i$ is a dense $F_{\sigma\delta}$-subset of X. Since $I_y = X$ for every $y \in \bigcap_{i=1}^{\infty} A_i$, the set $\{x \in X : I_x = X\}$ contains a dense $F_{\sigma\delta}$-subset of X. □

Proposition 3.26. *Let $x \in X$. Then $I_x = \bigcap_{\mu \in M_x} supp(\mu)$.*

Proof. Let $x \in X$. On one hand, we show that $I_x \subset \bigcap_{\mu\in M_x} supp(\mu)$. Let $y \in I_x$ and $\mu \in M_x$. We assume that the subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ satisfies

$$\mu_k := \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x} \to \mu$$

under the weak*-topology as $k \to \infty$. Let U be an open neighborhood of y, we can choose an open neighborhood V of y satisfying $y \in V \subset \overline{V} \subset U$. By Lemma 2.5,

$$\mu(U) \geq \mu(\overline{V}) \geq \limsup_{k\to\infty} \mu_k(\overline{V}) \geq \liminf_{k\to\infty} \mu_k(V) > 0.$$

Thus $y \in \bigcap_{\mu\in M_x} supp(\mu)$ by the arbitrariness of U which implies that $I_x \subset \bigcap_{\mu\in M_x} supp(\mu)$.

On the other hand, we show that $\bigcap_{\mu\in M_x} supp(\mu) \subset I_x$. Assume to the contrary that there exists $y \in \bigcap_{\mu\in M_x} supp(\mu)$ but $y \notin I_x$. Then we can choose an open neighborhood U of y and a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap N(x,U)|}{\lambda_{n_k}} = 0.$$

Without loss of generality,

$$\nu_k := \frac{1}{\lambda_{n_k}} \sum_{h\in\Lambda_{n_k}} \delta_{T^h x} \to \nu \in M_x$$

under the weak*-topology as $k \to \infty$. Then it follows from Lemma 2.5 that

$$\nu(U) \leq \liminf_{k\to\infty} \nu_k(U) = 0.$$

It is a contradiction as $y \in \bigcap_{\nu\in M_x} supp(\nu)$. Hence, $\bigcap_{\nu\in M_x} supp(\nu) \subset I_x$. This ends the proof. □

From the above results of weakly almost periodic point and the definition of minimal center of attraction, we can obtain the following theorem.

Theorem 3.27. *Let $x \in R(\mathcal{T})$. Then the following statements are equivalent.*
(1) $x \in W(\mathcal{T})$;
(2) $x \in I_x$;
(3) $x \in C_x = supp(\mu)$, $\forall \mu \in M_x$;
(4) $supp(\mu) = \omega_{\mathcal{T}}(x)$, $\forall \mu \in M_x$.

4. Chaotic dynamics of minimal center of attractions of $\mathbb{Z}_+^d$-actions

Let $(X,\mathcal{T})$ be a $\mathbb{Z}_+^d$-action. Recall that a pair $(x,y) \in X \times X$ is

(1) *asymptotic* if for every $n \in \mathbb{N}$, there exists a finite subset Λ of $\mathbb{Z}_+^d$ such that $\rho(T^h x, T^h y) < \frac{1}{n}$ for all $h \in \mathbb{Z}_+^d - \Lambda$.

(2) *proximal* if there exists a sequence $\{h_n\}_{n=1}^{\infty}$ of $\mathbb{Z}_+^d$ such that

$$\lim_{n\to\infty} \rho(T^{h_n} x, T^{h_n} y) = 0.$$

(3) *Li-Yorke chaotic* if (x,y) is proximal but not asymptotic.

For any $U \subset X$ and $\delta > 0$, write

$$S_{\mathcal{T}}(U,\delta) = \{h \in \mathbb{Z}_+^d : \text{there exist } x,y \in U \text{ such that } \rho(T^h x, T^h y) > \delta\}.$$

A $\mathbb{Z}_+^d$-action $(X,\mathcal{T})$ is called

(1) *strongly ergodic* if for any pair of nonempty open subsets U, V of X, $N(U, V)$ is syndetic;

(2) *sensitive* if there is $\delta > 0$ such that for any nonempty open subset U of X, $S_{\mathcal{T}}(U, \delta) \neq \emptyset$.

Next, we present some chaotic properties of a $\mathbb{Z}_+^d$-action with proper (quasi-) weakly almost periodic points. To do that, we need the following lemma.

Lemma 4.1 ([6, Theorem 8.7])**.** *For each $x \in X$, there exists $y \in A(\mathcal{T})$ such that (x, y) is proximal.*

Proposition 4.2. *If there exists a proper weakly almost periodic point $x \in X$, i.e., $x \in W(\mathcal{T}) - A(\mathcal{T})$, then $(X, \mathcal{T})$ contains at least countable Li-Yorke pairs.*

Proof. Let $x \in W(\mathcal{T}) - A(\mathcal{T})$. Then it follows from Lemma 4.1 that there exists $y \in A(\mathcal{T})$ such that (x, y) is proximal. Denote $A = \overline{\mathcal{T}y}$. Then there exists $\epsilon > 0$ such that $\rho(B(x, \epsilon), A) > 0$, where $\rho(C, D) = \inf\{\rho(c, d) : c \in C, d \in D\}$ for $C,\ D \subset X$. Set

$$S = \{h \in \mathbb{Z}_+^d : T^h x \in B(x, \epsilon)\}.$$

Then S has positive lower density by $x \in W(\mathcal{T})$. Since for each $h \in S$, $T^h y \in A$ and $T^h x \in B(x, \epsilon)$, then $\rho(T^h x, T^h y) > \rho(B(x, \epsilon), A) > 0$ for each $h \in S$, which implies $\sup_{h \in \mathbb{Z}_+^d} \rho(T^h x, T^h y) > 0$. So (x, y) is a Li-Yorke pair. It is clear that $(T^h x, T^h y)$ is also a Li-Yorke pair for every $h \in \mathbb{Z}_+^d$. Since $\mathbb{Z}_+^d$ is infinitely countable, $(X, \mathcal{T})$ contains countable Li-Yorke pairs. □

Remark 4.3. Similarly, we can prove that if there exists a proper quasi-weakly almost periodic point $x \in X$, i.e., $x \in QW(\mathcal{T}) - W(\mathcal{T})$, then $(X, \mathcal{T})$ contains at least countable Li-Yorke pairs.

4.1. Chaotic dynamics near or inside minimal center of attractions of $\mathbb{Z}_+^d$-actions

In this section, we mainly investigate the chaotic dynamics near the minimal center of attraction of a point in the setting that the involved minimal center of attraction is non S-generic and the chaotic dynamics inside the minimal center of attraction of a point in the setting that the involved minimal center of attraction is S-generic.

For convenience, in the rest of this section, we always assume that $(X, \mathcal{T})$ is a $\mathbb{Z}_+^d$-action.

Definition 4.1. A $\mathcal{T}$-invariant subset Λ of X is called *S-generic* if there exists some point $x \in \Lambda$ such that $\Lambda = C_x$.

Definition 4.2 ([6])**.** A subset $S \subset \mathbb{Z}_+^d$ is called a *central set* if there exist a $\mathbb{Z}_+^d$-action $(Y, \mathcal{T})$, a point $x \in Y$ and an almost periodic point $y \in Y$ of $(Y, \mathcal{T})$ which is proximal to x and $U \in \mathcal{N}_y$ such that $S = \{h \in \mathbb{Z}_+^d : T^h x \in U\}$.

Lemma 4.4. *If $(x, y) \in X \times X$ is asymptotic, then $C_x = C_y$.*

Proof. Suppose that $(x, y) \in X \times X$ is asymptotic. Let $z \in C_x$ and $U \in \mathcal{N}_z$. Then we can choose $\epsilon > 0$ such that $B(z, \epsilon) \subset U$. As (x, y) is asymptotic, there exists a finite subset Λ of $\mathbb{Z}_+^d$ such that $\rho(T^h x, T^h y) < \epsilon/2$ for all $h \in \mathbb{Z}_+^d - \Lambda$. Then, $N(x, B(z, \epsilon/2)) - \Lambda \subset N(y, U)$. By Lemma 3.2, $\overline{d}(N(x, B(z, \epsilon/2))) = \alpha > 0$. Therefore, there exists a subsequence $\{\Lambda_{n_k}\}_{k=1}^\infty$ of $\{\Lambda_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap N(x, B(z, \epsilon/2))|}{\lambda_{n_k}} = \overline{d}(N(x, B(z, \epsilon/2))),$$

which implies that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap (N(x, B(z, \epsilon/2)) - \Lambda)|}{\lambda_{n_k}} = \alpha.$$

Thus,

$$\overline{d}(N(y, U)) \geq \overline{d}(N(x, B(z, \epsilon/2)) - \Lambda) > 0.$$

It follows that $z \in C_y$ from the arbitrariness of U and so $C_x \subseteq C_y$. Similarly, we can prove that $C_y \subseteq C_x$. □

The following two lemmas are necessary for the proofs of the following theorems.

Lemma 4.5 ([12, Proposition 2.11]). *If Λ is a nonempty $\mathcal{T}$-invariant closed subset of $\overline{\mathcal{T}x}$, then for any open neighborhood U of Λ, $N(x, U)$ is thick in $\mathbb{Z}_+^d$.*

Lemma 4.6 ([12, Lemma 3.8]). *Suppose that $(X, \mathcal{T})$ is transitive and x is a transitive point of $(X, \mathcal{T})$. Then $N(U, V) = (N(x, V))^{-1} N(x, U)$ for every pair of nonempty open subsets U, V of X.*

4.1.1. *Non S-generic cases.* In this subsection, we prove that if the minimal center of attraction of a point is non S-generic, then there exist certain Li-Yorke chaotic properties in the involved minimal center of attraction and sensitivity near the involved minimal center of attraction.

Definition 4.3. If $(X, \mathcal{T})$ is point transitive and sensitive, then $(X, \mathcal{T})$ is called *Auslander-Yorke chaotic.*

Theorem 4.7. *Let $x \in X$. If C_x is not S-generic, then one can find some point $y \in \Lambda$ such that (x, y) is Li-Yorke chaotic for each closed $\mathcal{T}$-invariant subset Λ of C_x.*

Proof. Let $x \in X$ such that C_x is not S-generic. Then $x \notin C_x$ by Lemma 3.5. Let $\Lambda \subset C_x$ be a nonempty $\mathcal{T}$-invariant closed set. From Zorn's Lemma, there exists a minimal set $\Lambda_1 \subset \Lambda$. From Lemma 3.2, it follows that x is proximal to Λ_1, that is, there exists a sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{Z}_+^d$ such that $\lim_{n\to\infty} \rho(T^{t_n} x, \Lambda_1) = 0$. Then from Proposition 8.6 of [6], there exists some point $y \in \Lambda_1$ such that (x, y) is proximal. Obviously, (x, y) is not asymptotic. Otherwise, $C_x = C_y$ by Lemma 4.4 which shows that C_x is S-generic. This is a contradiction. Thus (x, y) is a Li-Yorke pair. □

Corollary 4.8. *Let $x \in X$. If C_x is not S-generic, then there exists $y \in C_x$ such that (x, y) is Li-Yorke chaotic and $N(x, B(y, \epsilon))$ is a central set in $\mathbb{Z}_+^d$. In addition, $N(x, B(y, \epsilon))$ has positive upper density.*

Proof. Let $\Lambda \subset C_x$ be a minimal set. Then there is a point $y \in \Lambda$ such that the pair (x, y) is Li-Yorke chaotic by Theorem 4.7. It follows that the set $N(x, B(y, \epsilon))$ is a central set in $\mathbb{Z}_+^d$ for each $\epsilon > 0$ by Definition 4.2. From Lemma 3.2, $N(x, B(y, \epsilon))$ has positive upper density. □

Corollary 4.9. *Let $x \in X$. If there exists a fixed point or a periodic orbit in C_x, then there exists a Li-Yorke chaotic pair near C_x.*

Proof. Let $x \in X$. If C_x is not S-generic, then from Theorem 4.7, there always exists a Li-Yorke chaotic pair near C_x. If C_x is S-generic, then $\mathcal{T}$ restricted to C_x is topologically transitive and has a fixed point or a periodic orbit. From Theorem 1.7 of [13], it follows that $\mathcal{T}$ restricted to C_x is chaotic in the sense of Li-Yorke. □

Motivated by [1, 4, 8], we can obtain the following result on sensitivity near the minimal center of attraction of $x \in X$ if C_x is not S-generic.

Theorem 4.10. *Let $x \in X$. If C_x is not S-generic and the almost periodic points of $(X, \mathcal{T})$ are dense in C_x, then $(X, \mathcal{T})$ is sensitive near C_x in the following sense: one can find an $\epsilon > 0$ such that for any points $x_1 \in X$, $x_2 \in C_x$ and any $U \in \mathcal{N}_{x_2}$, there exist $y \in U$ and $h \in \mathbb{Z}_+^d$ with $\rho(T^h x_1, T^h y) \geq \epsilon$.*

Proof. Let $x \in X$. If C_x is not S-generic, then C_x is not minimal. Therefore, there exist two distinct minimal points $z_1, z_2 \in X$ such that $\overline{\mathcal{T} z_1} \cap \overline{\mathcal{T} z_2} = \emptyset$. Write $\rho(\overline{\mathcal{T} z_1}, \overline{\mathcal{T} z_2}) = 3\delta > 0$. Then for all $a \in C_x$, there exists a corresponding orbit $\mathcal{T} b$ in C_x such that $\rho(a, \overline{\mathcal{T} b}) \geq \delta$. Next, we show that $(X, \mathcal{T})$ is sensitive with a sensitivity constant $\epsilon = \delta/4$.

Let $x_2 \in C_x$ and $U \in \mathcal{N}_{x_2}$. Since $\overline{A(\mathcal{T})} \supseteq C_x$, there exists $p \in A(\mathcal{T})$ such that $p \in U \cap B(x_2, \epsilon/2) \cap C_x$. Thus, there exists another point $z \in C_x$ such that

$$\rho(\overline{\mathcal{T} z}, x_2) \geq 4\epsilon.$$

Since $p \in A(\mathcal{T})$, there exists a finite subset F of $\mathbb{Z}_+^d$ such that

$$\mathbb{Z}_+^d = F^{-1} N(p, B(p, \epsilon/2)).$$

By Lemma 3.2, there exists $h \in \mathbb{Z}_+^d$ such that $T^h x \in U$. By Lemma 4.5, there exists $h_1 \in \mathbb{Z}_+^d$ such that

$$F + h + h_1 \subset N(x, B(\overline{\mathcal{T} z}, \epsilon)).$$

As there exists some nonempty subset $F' \subset F$ such that $(F' + h + h_1)h^{-1} \subset N(p, B(p, \epsilon/2))$. Take $y = T^h x$. Then $y \in U$ and $\rho(T^l p, T^l y) \geq 2\epsilon$ for every $l \in (F' + h + h_1)h^{-1}$. Since both x_2 and U are arbitrary, the result is proved. □

4.1.2. *S-generic cases.* In this subsection, we prove that if the minimal center of attraction of a point is S-generic, then there exist stronger Li-Yorke chaotic dynamics and sensitivity in the involved minimal center of attraction.

Theorem 4.11. *Let $x \in X$. If C_x is S-generic and not minimal, then there exists a dense subset S of C_x such that for each $y \in S$ and each minimal subset $\Lambda \subset C_x$, there is $z \in \Lambda$ satisfying: (z, y) is a Li-Yorke chaotic pair and there exist two sequences $\{h_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ in $\mathbb{Z}_+^d$ such that*

$$\lim_{n\to\infty} \rho(T^{h_n}y, z) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \rho(T^{l_n}y, z) \geq \frac{1}{2} diam(C_x).$$

Proof. Let $x \in X$ and suppose that C_x is S-generic. Then there exists $y \in C_x$ such that $C_y = C_x$. Denote $S = \{y \in C_x : C_y = C_x\}$. Since C_x is not minimal, $\overline{\mathcal{T}y}$ is not minimal for each $y \in S$ by Lemma 3.5. Let Λ be a minimal subset of C_x. Then each $y \in S$ is proximal to Λ by the proof of Theorem 4.7. Therefore, for every $y \in S$, there exists some point $z \in \Lambda$ such that (y, z) is proximal and y is almost periodic by Proposition 8.6 of [6]. Clearly, the pair (y, z) is Li-Yorke chaotic for $(X, \mathcal{T})$ and there exist two sequences $\{h_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ of $\mathbb{Z}_+^d$ such that

$$\lim_{n\to\infty} \rho(T^{h_n}y, z) = 0 \text{ and } \lim_{n\to\infty} \rho(T^{l_n}y, z) \geq \frac{1}{2} diam(C_x).$$

In addition, since $\mathbb{Z}_+^d$ is commutative, S is dense in C_x. □

Lemma 4.12. *Let $x \in X$. If $C_x = X$ and X has no isolated points, then $(X, \mathcal{T})$ is strongly ergodic.*

Proof. Let U, V be any nonempty open subsets of X. Since X has no isolated points, $(X, \mathcal{T})$ is transitive by Lemma 3.5. Then we can take $h \in \mathbb{Z}_+^d$ such that $U_1 = U \cap T^{-h}V \neq \emptyset$. As $N(U, V) \supset h + N(U_1, U_1)$, it suffices to show that $N(U_1, U_1)$ is syndetic. Let P be any thick subset of $\mathbb{Z}_+^d$ with $P \neq \mathbb{Z}_+^d$. Next, we prove that $N(U_1, U_1) \cap P \neq \emptyset$. By Lemma 3.2, there exists a subsequence $\{\Lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{|\Lambda_{n_k} \cap N(x, U_1)|}{\lambda_{n_k}} > 0.$$

By Proposition 3.9, there exists $\mu \in M(X, \mathcal{T})$ such that $\mu(U_1) > 0$. Choose $p_1 \in P$ with $p_1 \neq e = (0, 0, \ldots, 0) \in \mathbb{Z}_+^d$. Since P is thick, there exists $p_2 \in \mathbb{Z}_+^d$ with $p_2 \neq e$ such that $p_2,\ p_1 + p_2 \in P$. Again from the thickness of P, there exists $p_3 \in \mathbb{Z}_+^d$ with $p_3 \neq e$ such that $p_3,\ p_1 + p_3,\ p_2 + p_3,\ p_1 + p_2 + p_3 \in P$. Inductively, we obtain a sequence $\{p_n\}_{n=1}^{\infty}$ of $\mathbb{Z}_+^d$ with $p_n \neq e$ for each $n \in \mathbb{N}$ such that

$$p_1,\ p_2,\ p_1 + p_2,\ p_3,\ p_1 + p_3,\ p_2 + p_3,\ p_1 + p_2 + p_3, \ldots \in P.$$

Set $h_n = p_1 + p_2 + \cdots + p_n$ for all $n \in \mathbb{N}$. If $\{h_n\}_{n=1}^{\infty}$ has only finitely many distinct elements of $\mathbb{Z}_+^d$, then there exist $m, n \in \mathbb{N}$ with $m < n$ such that

$h_m = h_n$. Therefore, $T^{h_m}U_1 = T^{h_n}U_1$ which implies that $p_{m+1} + \cdots + p_n \in N(U_1, U_1)$. Thus, $P \cap N(U_1, U_1) \neq \emptyset$.

If $\{h_n\}_{n=1}^{\infty}$ has infinitely many distinct elements of $\mathbb{Z}_+^d$, then there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ with $h_{n_i} \neq h_{n_j}$ for all $i \neq j$. Since μ is $\mathcal{T}$-invariant, $T^{-h_{n_{k'}}}U_1 \cap T^{-h_{n_{k''}}}U_1 \neq \emptyset$ for some $k' > k''$ which implies that $p_{n_{k''}+1} + \cdots + p_{n_{k'}} \in N(U_1, U_1)$. Thus, $N(U_1, U_1) \cap P \neq \emptyset$. □

The following theorem asserts that the minimal center of attraction of a point exhibits Auslander-Yorke chaotic behaviors if it is S-generic but not minimal.

Theorem 4.13. *Let $x \in X$. If $C_x \subset X$ is S-generic and non-minimal, then $(C_x, \mathcal{T})$ is point transitive and sensitive in the following sense: there is $\delta > 0$ such that for each $y \in X$ there exists a dense subset $B_{\delta,y} \subset X$ such that for each $y_1 \in B_{\delta,y}$ there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset \mathbb{Z}_+^d$ satisfying*

$$\lim_{n\to\infty} \rho(T^{h_n}y, T^{h_n}y_1) \geq \delta.$$

In particular, $(C_x, \mathcal{T})$ is Auslander-Yorke chaotic.

Proof. Suppose $C_x \subset X$ is S-generic and not minimal. Without loss of generality, we assume that $C_x = X$. Then $(X, \mathcal{T})$ is point transitive by Lemma 3.5. Next, we prove that $(X, \mathcal{T})$ is sensitive.

Let M be a minimal subset of $(X, \mathcal{T})$. Take $m \in X$ with $\rho(m, M) > 0$. Let $3r = \rho(m, M)$ and $U \subset X$ be any nonempty open subset of X. Since x is a transitive point of $(X, \mathcal{T})$, there is $h \in \mathbb{Z}_+^d$ such that $x_1 = T^h x \in U$. By Lemmas 4.5 and 4.12, $N(x, B(M, r))$ is thick and $N(U, B(m, r))$ is syndetic and it is not hard to prove that $N(x_1, B(M, r))$ is also thick. Therefore, there exist $h_1 \in N(x_1, B(M, r)) \cap N(U, B(m, r))$ and a nonempty open subset U_1 of X such that

$$\overline{U_1} \subset U \text{ and } \overline{U_1} \subset U \cap T^{-h_1}(B(m, r)).$$

Similarly, there exist $h_2 \neq h_1$ such that $h_2 \in N(x_1, B(M, r)) \cap N(U_1, B(m, r))$ and a nonempty open subset U_2 of X satisfying

$$\overline{U_2} \subset U_1 \text{ and } \overline{U_2} \subset U_1 \cap T^{-h_2}(B(m, r)).$$

Repeating this construction, there are a sequence of nonempty open sets $\{U_n\}_{n=1}^{\infty} \subset X$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset \mathbb{Z}_+^d$ satisfying

$$U \supset \overline{U_1} \supset U_1 \supset \overline{U_2} \supset \cdots, \ \overline{U_{n+1}} \subset U_n \cap T^{-h_{n+1}}(B(m, r)),$$

and $h_i \neq h_j$ for each $i \neq j$. Therefore, $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$. Then for each $y' \in \bigcap_{n=1}^{\infty} U_n$,

$$y' \in U \text{ and } \rho(T^{h_n}x_1, T^{h_n}y') \geq r$$

for all $n \in \mathbb{N}$. Let $\delta = r/2$. Then for each $y \in X$ and each nonempty open set U, there exist $y_1 \in U$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset \mathbb{Z}_+^d$ such that

$$\lim_{n\to\infty} \rho(T^{h_n}y, T^{h_n}y_1) \geq \delta.$$

Let

$$B_y = \{y_1 \in X : \lim_{n\to\infty} \rho(T^{h_n}y, T^{h_n}y_1) \geq \delta \text{ for some sequence } \{h_n\}_{n=1}^{\infty} \subset \mathbb{Z}_+^d\}$$

for $y \in X$. Then B_y is dense in X for each $y \in X$ since the open set U is arbitrary. In particular, $(X, \mathcal{T})$ is Auslander-Yorke chaotic. □

For $n \in \mathbb{N}$, write $X^{(n)} = \overbrace{X \times X \times \cdots \times X}^{n}$ be the n-fold self-product of X. A tuple $(x_1, \ldots, x_n) \in X^{(n)}$ can determine a subset of X, denoted by $L(x_1, \ldots, x_n)$, as follows: $x \in L(x_1, \ldots, x_n)$ if and only if for each $1 \leq i \leq n$, each $U_i \in \mathcal{N}_{x_i}$ and each $U \in \mathcal{N}_x$, there are $h \in \mathbb{Z}_+^d$ and $x_i' \in U$ such that $T^h(x_i') \in U_i$.

The following theorem shows that if the minimal center of attraction C_x of $x \in X$ is S-generic and non-minimal, then C_x exhibits more complicated sensitivity than the non S-generic case given in Theorem 4.10.

Theorem 4.14. *Let $x \in X$. If C_x is S-generic and non-minimal and $A(\mathcal{T})$ is dense in C_x, then $(X, \mathcal{T})$ has $\aleph_0$-sensitivity near C_x in the following sense: one can find an infinitely countable subset K of C_x such that for any k distinct points $x_1, \ldots, x_k \in K$ with $k \geq 2$, it holds $C_x \subset L(x_1, \ldots, x_k)$.*

Proof. Assume that C_x is S-generic. Then there exists $y \in C_x$ such that $C_x = C_y$ which shows that y is a transitive point of $(C_x, \mathcal{T})$. We prove the result by six steps as follows. Let $z \in C_x$ and $U \in \mathcal{N}_z$.

Step 1. We show that $N(y, U)$ is piecewise syndetic in $\mathbb{Z}_+^d$. By Theorem 3.4 of [12], $N(y, U)$ is piecewise syndetic in $\mathbb{Z}_+^d$ since $(C_x, \mathcal{T})$ is an M-system, i.e., $(C_x, \mathcal{T})$ is transitive and the set of minimal points of $(C_x, \mathcal{T})$ is dense in C_x, and y is a transitive point of $(C_x, \mathcal{T})$.

Step 2. We claim that there are infinitely many distinct minimal subsets of C_x. Assume to the contrary that there exist finitely many distinct minimal subsets $A_1, \ldots, A_n$ of C_x. Then $\bigcup_{i=1}^n A_i = C_x$ since the almost periodic points of $(X, \mathcal{T})$ are dense in C_x. It follows that C_x is minimal which is a contradiction. Therefore, there are infinitely many distinct minimal subsets of C_x.

Step 3. Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of minimal subsets of C_x with $M_i \neq M_j$ for every $i \neq j$. Let $\delta > 0$. For every $k \in \mathbb{N}$, as M_k is a $\mathcal{T}$-invariant closed set, $N(y, B(M_k, \delta))$ is thick in $\mathbb{Z}_+^d$ by Lemma 4.5. It follows that $N(U, B(M_k, \delta))$ is thick in $\mathbb{Z}_+^d$ by Lemma 4.6.

Step 4. Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of minimal subsets of C_x with $M_i \neq M_j$ for every $i \neq j$. We are going to prove that $N(U, B(M_1, \delta)) \cap \cdots \cap N(U, B(M_k, \delta)) \neq \emptyset$ for any $\delta > 0$ and $k \geq 2$. In fact, since $N(y, U)$ is piecewise syndetic, there exists a finite subset F of $\mathbb{Z}_+^d$ satisfying that for every finite subset L of $\mathbb{Z}_+^d$ there exists h_L such that $L + h_L \subset F^{-1}N(y, U)$. Then by Step 3, for the finite subset F and each $N(y, B(M_i, \delta))$, we can choose $h_i \in \mathbb{Z}_+^d$ such that $F + h_i \subset N(y, B(M_i, \delta))$ for each $1 \leq i \leq k$. Take $L^* = \{h_i\}_{i=1}^k$. Then there exists h_{L^*} such that $L^* + h_{L^*} \subset F^{-1}N(y, U)$ which implies that for each $i = 1, \ldots, k$, there exists $f_i \in F$ such that $f_i + h_i + h_{L^*} \in N(y, U)$. Therefore, for each $1 \le i \le k$, we have

$$\begin{aligned} N(U, B(M_i, \delta)) &= (N(y, B(M_i, \delta)))^{-1} N(y, U) \\ &\supset \bigcup_{h \in F} (h + h_i)^{-1} N(y, U) \\ &\supset \bigcup_{h \in F} \{x \in \mathbb{Z}_+^d : (h + h_i)x = f_i + h_i + h_{L^*}\} \\ &\supset \{x \in \mathbb{Z}_+^d : (f_i + h_i)x = f_i + h_i + h_{L^*}\} \ni h_{L^*}. \end{aligned}$$

Thus, $N(U, B(M_1, \delta)) \cap \cdots \cap N(U, B(M_k, \delta)) \neq \emptyset$.

Step 5. We claim that for each $k \ge 2$, there exist $x_1 \in M_1, \ldots, x_k \in M_k$ such that $C_x \subset L(x_1, \ldots, x_k)$. Fix $k \in \mathbb{N}$ with $k \ge 2$. For each $n \in \mathbb{N}$, by Step 4, there exist $y_{1,n}, \ldots, y_{k,n} \in B(y, 1/n)$ and $h_n \in \mathbb{Z}_+^d$ such that

$$T^{h_n} y_{1,n} \in B(M_1, 1/n), \ldots, T^{h_n} y_{k,n} \in B(M_k, 1/n).$$

Take

$$x_{1,n} = T^{h_n} y_{1,n}, \ldots, x_{k,n} = T^{h_n} y_{k,n}.$$

Without loss of generality, we may assume that

$$x_1 = \lim_{n\to\infty} x_{1,n}, \ldots, x_k = \lim_{n\to\infty} x_{k,n}.$$

Then $x_1 \in M_1, \ldots, x_k \in M_k$ and $y \in L(x_1, \ldots, x_k)$. As $L(x_1, \ldots, x_k)$ is closed and $\mathcal{T}$-invariant, then $C_x \subset L(x_1, \ldots, x_k)$.

Step 6. By Step 5, there exist $x_{k,1}, \ldots, x_{k,k} \in X$ such that $x_{k,1} \in M_1, \ldots$, $x_{k,k} \in M_k$ and $C_x \subset L(x_{k,1}, \ldots, x_{k,k})$ for each $k \ge 2$. Thus

$$\{x_{2,1}, x_{3,1}, \ldots\} \subset M_1, \quad \{x_{2,2}, x_{3,2}, \ldots\} \subset M_2.$$

Without loss of generality, we assume that

$$\lim_{n\to\infty} x_{n,1} = x_1 \in M_1, \lim_{n\to\infty} x_{n,2} = x_2 \in M_2.$$

Continuing this construction, we obtain an infinite countable set $K = \{x_1, x_2, \ldots\}$. It follows that $C_x \subset L(x_1, \ldots, x_k)$ for each $k \ge 2$ by Step 5 again. The proof is ended. □

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