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ON LIMIT BEHAVIOURS FOR FELLER'S UNFAIR-FAIR-GAME AND ITS RELATED MODEL

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Abstract. Feller introduced an unfair-fair-game in his famous book [3]. In this game, at each trial, player will win 2^k yuan with probability $p_k = 1/2^k k(k+1)$, $k \in \mathbb{N}$, and zero yuan with probability $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Because the expected gain is 1, player must pay one yuan as the entrance fee for each trial. Although this game seemed "fair", Feller [2] proved that when the total trial number n is large enough, player will loss n yuan with its probability approximate 1. So it's an "unfair" game. In this paper, we study in depth its convergence in probability, almost sure convergence and convergence in distribution. Furthermore, we try to take $2^k = m$ to reduce the values of random variables and their corresponding probabilities at the same time, thus a new probability model is introduced, which is called as the related model of Feller's unfair-fair-game. We find out that this new model follows a long-tailed distribution. We obtain its weak law of large numbers, strong law of large numbers and central limit theorem. These results show that their probability limit behaviours of these two models are quite different.

1. Introduction

In recent years, many scholars studied the limit behaviours for random variable sequences with their expectations or variances not existed. They applied these conclusions to study some special random variables, such as Pareto-Zipf distribution ([1]), Feller game ([1, 9–11]), St. Petersburgur game ([6, 10]), etc., and obtained a lot of interesting and meaningful outcomes.

Feller introduced an unfair-fair-game in his famous book [3]. In this game, at each trial, player will win 2^k yuan with probability p_k and zero yuan with

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probability p_0, where

$$(1.1)\ \ p_k = P\left\{X_1 = 2^k\right\} = \frac{1}{2^k k(k+1)},\ k \in \mathbb{N},\ p_0 = P\left\{X_1 = 0\right\} = 1 - \sum_{k=1}^{\infty} p_k.$$

Because the expected gain is 1, the player must pay one yuan as the entrance fee for each trial. Although this game seemed "fair", Feller [2] proved that when the total trial number n is large enough, player will loss n yuan with its probability approximate 1. Let X_i represent the player's gain at the ith trial, $S_n = \sum_{i=1}^{n} X_i$ denote the total gain in its n trials, this result can be expressed as the following theorem (see [3]).

Theorem 1.1. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in* (1.1)*. Then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left\{S_n - n < -\frac{(1-\varepsilon)n}{\mathrm{Log}_2 n}\right\} = 1, \tag{1.2}$$

where $\mathrm{Log}_2 n = \max\{1, \log_2 n\}$*, and* $\log_2 n$ *denotes the logarithm to the base* 2.

Theorem 1.1 shows that this game is unfavorable for players. So it's an "unfair" game. We call it as *Feller's unfair-fair-game.* For this model, the expectation $EX_1 = 1$. But, for any $\alpha > 1$, the moment EX_1^{α} doesn't exist. Therefore, many classical limit theorems don't hold, which arouses our interest in this subject. Unfortunately, its further limit behaviours have not be investigated in literatures. In this paper, some probability limit properties for this model are studied. Concretely peaking, in Section 2, Theorem 2.1 studies the convergence in probability and its rate of convergence for $\{S_n\}$. Theorem 2.2 investigates the convergence in probability and almost sure convergence for $\{X_n\}$. Theorem 2.3 obtains the characteristic function of the limit distribution of $\{S_n\}$, which is similar to Theorem 1 of [8].

As we known, the probability limit properties of random sequences are often closely related to their moment conditions. For the model of Feller's unfair-fair-game, the expectation $EX_1 = 1$, and for any $\alpha > 1$, the moment EX_1^{α} does not exist as stated above. Now if we reduce the values of random variables and their corresponding probabilities at the same time, does the new probability model have the same limit properties? Driven by this motivation, we try to take $2^k = m$, equivalently, $k = [\log_2 m]$, where $[a]$ denotes the largest integer no more than the real number a. For any integer $m \geqslant 2$, we define

$$p_m = P\left\{X_1 = m\right\} = \frac{1}{m \log_2 m \left(\log_2 m + 1\right)},\ m \geqslant 2.$$

Based on this idea, we introduce the following probability model.

Let $X_1, X_2, \ldots$ be independent random variables with the same distribution

$$p_k = P\left\{X_1 = k\right\} = \frac{1}{ck(\mathrm{Log}_a k)^2},\ k \geqslant 1,\ c = \sum_{k=1}^{\infty} \frac{1}{k(\mathrm{Log}_a k)^2},\ a > 1, \tag{1.3}$$

where $\mathrm{Log}_a n = \max\{1, \log_a n\}$. We take the denominator of p_k as $ck(\mathrm{Log}_a k)^2$ instead of $ck(\mathrm{Log}_a k)(\mathrm{Log}_a k{+}1)$, for the sake of the convenience of mathematical processing, however, they are not much different. We call the new model of (1.3) as the *related model of Feller's unfair-fair-game*, abbreviated as *related model of FUFG.*

For the related model of FUFG of (1.3), it's easy to get

$$\overline{F}(x) = P\{X_1 > x\} \sim \frac{\ln a}{c\mathrm{Log}_a x} \quad \text{as } x \to +\infty,$$

where $\ln a = \log_e a$. For any $t > 0$, $\overline{F}(x+t) \sim \overline{F}(x)$ as $x \to +\infty$. Now X_1 follows a long-tailed distribution (see [4]), which is widely used in many fields, such as machine learning, artificial intelligence, finance theory, insurance theory and so on. Therefore it is meaningful to study in depth the probability limit properties of this model.

Since $EX_1 = +\infty$, the conditions of classical limit theorem are not satisfied, it's necessary to specially discuss its limit behaviours. In Section 3, Theorem 3.3, Theorem 3.4 and Theorem 3.6 study the convergence in probability and almost sure convergence for its partial sums $S_n = \sum_{i=1}^n X_i$. Theorem 3.7 obtains its central limit theorem. Our findings show that these two models have completely different probability limit behaviours.

Throughout this paper, $X_n = o_P(Y_n)$ denotes $X_n/Y_n \to 0$ in probability; a.s. is the abbreviation of "almost surely"; i.o. is the abbreviation of "infinitely offen"; C represents a positive constant and its value may be different and unimportant on different occasions; $\xrightarrow{d}$ means convergence in distribution; $a_n = o(b_n)$ represents $a_n/b_n \to 0$ as $n \to \infty$; $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$; $a_n = O(b_n)$ stands for $-\infty < \liminf_{n\to\infty} a_n/b_n \leqslant \limsup_{n\to\infty} a_n/b_n < \infty$.

2. Limit behaviours for Feller's unfair-fair-game

Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.1). Define $b_n = n/\mathrm{Log}_2 n$, $m(n) = \sup\{m \in \mathbb{N} : 2^m \leqslant b_n\}$, $a_n = m(n)/(m(n)+1)$. From

$$2^{m(n)} \leqslant n/\mathrm{Log}_2 n < 2^{m(n)+1},$$

we can get

$$m(n) \leqslant \mathrm{Log}_2 n - \mathrm{Log}_2\mathrm{Log}_2 n < m(n) + 1.$$

Thus

$$1 - \frac{\mathrm{Log}_2\mathrm{Log}_2 n + 1}{\mathrm{Log}_2 n} < \frac{m(n)}{\mathrm{Log}_2 n} \leqslant 1 - \frac{\mathrm{Log}_2\mathrm{Log}_2 n}{\mathrm{Log}_2 n},$$

and

$$\lim_{n\to\infty} \frac{m(n)}{\mathrm{Log}_2 n} = 1, \ \lim_{n\to\infty} a_n = 1. \tag{2.1}$$

We first study the convergence in probability and its rate of convergence for the partial sums S_n and obtain the following result.

Theorem 2.1. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in* (1.1), $S_n = \sum_{i=1}^n X_i$. *Then*

$$\frac{S_n - na_n}{b_n} \to 0 \text{ in probability,} \tag{2.2}$$

and

$$\frac{S_n - na_n}{b_n} = o_P\left(\mathrm{Log}_2 n\right). \tag{2.3}$$

Proof. Define $\widetilde{X}_i = X_i I\ (X_i \leqslant b_n)$. For n large enough, $m(n) > \mathrm{Log}_2 n/2$, $2^{m(n)} > b_n/2$, so

$$E\widetilde{X}_1 = \sum_{k=1}^{m(n)} \frac{1}{k(k+1)} = 1 - \frac{1}{m(n)+1} = \frac{m(n)}{m(n)+1} = a_n,$$

$$\begin{aligned} nP\left\{X_1 > b_n\right\} &= n \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k k(k+1)} \\ &\leqslant \frac{n}{m(n)^2} \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k} \\ &= \frac{n}{m(n)^2 2^{m(n)}} \\ &\leqslant \frac{4n}{(\mathrm{Log}_2 n)^2} \cdot \frac{2\mathrm{Log}_2 n}{n} \\ &= \frac{C}{\mathrm{Log}_2 n} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} E\widetilde{X}_1^2 &= \sum_{k=1}^{m(n)} \frac{2^k}{k(k+1)} \\ &= \sum_{k=1}^{[m(n)/2]} \frac{2^k}{k(k+1)} + \sum_{k=[m(n)/2]+1}^{m(n)} \frac{2^k}{k(k+1)} \\ &:= I_1 + I_2. \end{aligned} \tag{2.4}$$

We start to estimate the order of I_1 and I_2, respectively. Firstly,

$$I_1 \leqslant m(n) 2^{m(n)/2} \leqslant 2\mathrm{Log}_2 n \cdot \sqrt{\frac{n}{\mathrm{Log}_2 n}} = 2\sqrt{n \mathrm{Log}_2 n} = o\left(\frac{n}{(\mathrm{Log}_2 n)^2}\right). \tag{2.5}$$

Secondly,

$$I_2 \leqslant \frac{4}{m(n)^2} \sum_{k=[m(n)/2]+1}^{m(n)} 2^k \tag{2.6}$$

$$\begin{aligned}&\leqslant \frac{4}{m(n)^2}\cdot 2^{m(n)+1}\\ &\leqslant \frac{16}{(\mathrm{Log}_2 n)^2}\cdot\frac{2n}{\mathrm{Log}_2 n}\\ &= o\left(\frac{n}{(\mathrm{Log}_2 n)^2}\right).\end{aligned}$$

Combining (2.5), (2.6) and (2.4), we get that

$$\begin{aligned}\frac{1}{b_n^2}\sum_{i=1}^{n}\mathrm{Var}\widetilde{X}_i &\leqslant \frac{1}{b_n^2}\sum_{i=1}^{n}E\widetilde{X}_i^2\\ &= n\left(\frac{\mathrm{Log}_2 n}{n}\right)^2\cdot o\left(\frac{n}{(\mathrm{Log}_2 n)^2}\right)\to 0 \quad \text{as } n\to\infty.\end{aligned}\tag{2.7}$$

Applying Theorem 6.3.3 of [5] with (2.4) and (2.7), it yields (2.2).

Because $EX_1 = 1$, by the Khintchine's weak law of large numbers we get

$$\frac{S_n - n}{n}\to 0 \quad \text{in probability.}$$

Consequently,

$$\begin{aligned}\frac{S_n - na_n}{b_n} &= \frac{S_n - n}{n}\cdot\frac{n}{b_n}+\frac{n(1-a_n)}{b_n}\\ &= \frac{S_n - n}{n}\cdot \mathrm{Log}_2 n + (1-a_n)\mathrm{Log}_2 n\\ &= \left\{\frac{S_n - n}{n} + (1-a_n)\right\}\mathrm{Log}_2 n\\ &= o_P(\mathrm{Log}_2 n).\end{aligned}$$

Thus (2.3) holds. The proof of Theorem 2.1 is completed. □

Using Theorem 2.1, we can prove Theorem 1.1 easily.

Proof of Theorem 1.1. For any $\varepsilon \in (0,1)$, taking $\varepsilon' \in (0,1)$ such that $0 < 1-\varepsilon+\varepsilon' = 1/\alpha < 1$ for this $\alpha > 1$ and n large enough, the first part of (2.1) implies $m(n)+1 \leqslant \alpha \mathrm{Log}_2 n$. Now

$$\begin{aligned}P\left\{S_n - n < -\frac{(1-\varepsilon)n}{\mathrm{Log}_2 n}\right\} &= P\left\{S_n - n < -\frac{n}{\alpha\mathrm{Log}_2 n}+\frac{n\varepsilon'}{\mathrm{Log}_2 n}\right\}\\ &\geqslant P\left\{S_n - n < -\frac{n}{m(n)+1}+\frac{n\varepsilon'}{\mathrm{Log}_2 n}\right\}\\ &= P\left\{S_n - na_n < \varepsilon' b_n\right\}\\ &\geqslant P\left\{\left|\frac{S_n - na_n}{b_n}\right| < \varepsilon'\right\}\to 1 \quad \text{as } n\to\infty.\end{aligned}$$

So (1.2) holds. □

Next we study the convergence in probability and almost sure convergence for $\{X_n\}$.

Theorem 2.2. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in* (1.1). *Then, we have the following limit behaviour*

$$\frac{X_n - a_n}{b_n} \to 0 \quad \textit{in probability,} \tag{2.8}$$

and

$$\liminf_{n\to\infty} \frac{X_n - a_n}{b_n} = 0 \ \textit{a.s.}, \ \limsup_{n\to\infty} \frac{X_n - a_n}{b_n} = +\infty \ \textit{a.s.}. \tag{2.9}$$

Proof. For any $\varepsilon > 0$ and n large enough, $a_n - \varepsilon n/\mathrm{Log}_2 n < 0$ and

$$P\{X_1 < a_n - \varepsilon n/\mathrm{Log}_2 n\} = 0,$$

since $a_n > 0, a_n \uparrow 1$ as $n \to \infty$, $X_1 \geqslant 0$ a.s.. Thus,

$$P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon\right\} = P\{X_1 < a_n - \varepsilon n/\mathrm{Log}_2 n\} = 0. \tag{2.10}$$

On the other hand, for any $\varepsilon > 0$ and n large enough,

$$P\left\{\frac{X_n - a_n}{b_n} > \varepsilon\right\} = P\{X_1 > a_n + \varepsilon n/\mathrm{Log}_2 n\} \to 0 \text{ as } n \to \infty. \tag{2.11}$$

Consequently, (2.10) and (2.11) lead to (2.8).

Based on the same reason of (2.10) we know that

$$\sum_{n=1}^{\infty} P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon\right\} = \sum_{n=1}^{\infty} P\{X_1 < a_n - \varepsilon n/\mathrm{Log}_2 n\} < \infty.$$

Using the first Borel-Cantlli lemma we can also obtain

$$P\left\{\frac{X_n - a_n}{b_n} < -\varepsilon, \text{ i.o.}\right\} = 0,$$

which yields

$$\liminf_{n\to\infty} \frac{X_n - a_n}{b_n} \geqslant 0 \text{ a.s.}. \tag{2.12}$$

For any $\varepsilon > 0$, $\{X_1 = 0\} \subset \{X_1 < a_n + \varepsilon b_n\}$, thus

$$\sum_{n=1}^{\infty} P\left\{\frac{X_n - a_n}{b_n} < \varepsilon\right\} = \sum_{n=1}^{\infty} P\{X_1 < a_n + \varepsilon b_n\} \geqslant \sum_{n=1}^{\infty} P\{X_1 = 0\} = +\infty.$$

By the second Borel-Cantelli lemma we obtain

$$P\left\{\frac{X_n - a_n}{b_n} < \varepsilon \text{ i.o.}\right\} = 1,$$

which leads to

$$\liminf_{n\to\infty} \frac{X_n - a_n}{b_n} \leqslant 0 \text{ a.s.}. \tag{2.13}$$

Combining (2.12) and (2.13), the first part of (2.9) is proved.

For any $M > 0$, put $k(n) = \inf\{k : 2^k \geqslant (M+1)b_n\}$. Similar to (2.1) we can easily get $k(n) \sim \mathrm{Log}_2 n$ as $n \to \infty$. From the knowledge of mathematical analysis,

$$\int_k^{k+1} \frac{1}{2^x x(x+1)}\mathrm{d}x \leqslant \frac{1}{2^k k(k+1)} \leqslant \int_{k-1}^{k} \frac{1}{2^x x(x+1)}\mathrm{d}x,\ k = 1, 2, \ldots.$$

This leads to

$$\int_{k(n)}^{\infty} \frac{1}{2^x x(x+1)}\mathrm{d}x \leqslant \sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)} \leqslant \int_{k(n)-1}^{\infty} \frac{1}{2^x x(x+1)}\mathrm{d}x. \tag{2.14}$$

Using the law of L'Hospital, it's easy to verify that

$$\int_u^{\infty} \frac{1}{2^x x(x+1)}\mathrm{d}x \sim \frac{1}{2^u u(u+1)\ln 2} \quad \text{as } u \to \infty. \tag{2.15}$$

Combining (2.14) and (2.15), we can obtain that

$$\sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)} = O\left(\frac{1}{2^{k(n)} k(n)(k(n)+1)}\right) \quad \text{as } n \to \infty.$$

Consequently,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\frac{X_n - a_n}{b_n} > M\right\} &= \sum_{n=1}^{\infty} P\left\{X_1 > a_n + M b_n\right\} \\
&\geqslant \sum_{n=1}^{\infty} P\left\{X_1 > (M+1)b_n\right\} (\text{since } a_n \to 1) \\
&= \sum_{n=1}^{\infty} \sum_{k=k(n)}^{\infty} \frac{1}{2^k k(k+1)} \\
&\geqslant C \sum_{n=1}^{\infty} \frac{1}{2^{k(n)} k(n)(k(n)+1)} \\
&\geqslant C \sum_{n=1}^{\infty} \frac{1}{n/\mathrm{Log}_2 n \cdot (\mathrm{Log}_2 n)^2} \\
&= C \sum_{n=1}^{\infty} \frac{1}{n \mathrm{Log}_2 n} = +\infty.
\end{aligned}$$

By the second Borel-Cantelli lemma we have

$$P\left\{\frac{X_n - a_n}{b_n} > M,\ \text{i.o.}\right\} = 1,$$

which yields

$$\limsup_{n\to\infty} \frac{X_n - a_n}{b_n} = +\infty \ \text{ a.s..}$$

Thus the second part of (2.9) holds. The proof of Theorem 2.2 is completed. □

Let $B_n^2 = n\left\{\sum_{k=1}^{m(n)} 2^k/k(k+1) - a_n^2\right\}$, $\widetilde{X}_i = X_i I(X_i \leqslant b_n)$, $\widetilde{S}_n = \sum_{k=1}^{n} \widetilde{X}_i$. It's easy to get

$$P\left\{\widetilde{S}_n \neq S_n\right\} \to 0 \text{ as } n \to \infty.$$

Applying the Lindeberg-Lévy-Feller central limit theorem, we can prove that

$$\frac{\widetilde{S}_n - na_n}{B_n^2} \xrightarrow{d} N(0,1)$$

does not hold, where $N(0,1)$ denotes the standard normal random variable. Consequently

$$\frac{S_n - na_n}{B_n^2} \xrightarrow{d} N(0,1)$$

does not hold.

In order to study the limit distribution for the partial sums, S_n, inspired by Anderson [8], the following theorem obtains its characteristic function of the limit distribution of S_n.

Theorem 2.3. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in* (1.1), *$N_n = 2^n$, $M_n = n^2 2^n$. Then*

$$\frac{S_{M_n} - M_n}{N_n} = \frac{S_{M_n}}{N_n} - n^2 \xrightarrow{d} S,$$

where the characteristic function of random variable S is $\exp\{g(t)\}$, *and*

$$g(t) = \sum_{k=-\infty}^{0} \frac{e^{\mathrm{i}t2^k} - 1 - \mathrm{i}t2^k}{2^k} + \sum_{k=1}^{\infty} \frac{e^{\mathrm{i}t2^k} - 1}{2^k}.$$

Proof. We use the method of the proof of Theorem 1 in Anderson [8] to prove this theorem.

The characteristic function of X_1 is

$$f(t) = E\exp\{\mathrm{i}tX_1\} = p_0 + \sum_{k=1}^{\infty} \frac{\exp\{\mathrm{i}t2^k\}}{2^k k(k+1)} = 1 + \sum_{k=1}^{\infty} \frac{\exp\{\mathrm{i}t2^k\} - 1}{2^k k(k+1)}.$$

Hence the characteristic function of $S_{M_n}/N_n - n^2$ is

$$\begin{aligned} f_{N_n}(t) &= E\exp\left\{\mathrm{i}t\left(\frac{S_{M_n}}{N_n} - n^2\right)\right\} \\ &= E\exp\left\{\frac{\mathrm{i}tS_{M_n}}{N_n}\right\}\exp\{-\mathrm{i}tn^2\} \\ &= \left\{f\left(\frac{t}{N_n}\right)\right\}^{M_n}\exp\{-\mathrm{i}tn^2\}. \end{aligned} \tag{2.16}$$

Next, we will decompose $f(t/N_n)-1$ into two parts.

$$\begin{aligned}(2.17)\qquad & f\left(\frac{t}{N_n}\right)-1\\ &=\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k-n}\}-1}{2^k k(k+1)}\\ &=\sum_{k=-n+1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k+n}(k+n)(k+n+1)}\\ &=\frac{1}{N_n}\left\{\sum_{k=-n+1}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}(k+n)(k+n+1)}+\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}(k+n)(k+n+1)}\right\}\\ &=\frac{1}{N_n}\left(I_n^{(1)}+I_n^{(2)}\right).\end{aligned}$$

Since

$$\left|n^2\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}(k+n)(k+n+1)}\right|\leqslant\sum_{k=1}^{\infty}\frac{\left|\exp\{\mathrm{i}t2^{k}\}-1\right|}{2^{k}(1+\frac{k}{n})(1+\frac{k+1}{n})}\leqslant\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}<\infty,$$

applying the Lesbegue's controlled convergent theorem, we have

$$(2.18)\quad I_n^{(2)}=\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}(k+n)(k+n+1)}\sim\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}}\quad\text{as } n\to\infty.$$

On the other hand,

$$\begin{aligned}(2.19)\qquad I_n^{(1)}&=\sum_{k=-n+1}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1}{2^{k}(k+n)(k+n+1)}\\ &=\sum_{k=-n+1}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1-\mathrm{i}t2^k}{2^{k}(k+n)(k+n+1)}+\sum_{k=-n+1}^{0}\frac{\mathrm{i}t}{(k+n)(k+n+1)}\\ &=\frac{1}{n^2}\sum_{k=-n+1}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1-\mathrm{i}t2^k}{2^{k}(1+\frac{k}{n})(1+\frac{k+1}{n})}+\mathrm{i}t\left(1-\frac{1}{n+1}\right).\end{aligned}$$

Since

$$\frac{\exp\{\mathrm{i}t2^{-n+1}\}-1-\mathrm{i}t2^{-n+1}}{2^{-n+1}(1+\frac{-n+1}{n})(1+\frac{-n+2}{n})}=O\left(\frac{t^2n^2}{2^n}\right)\ \text{as } n\to\infty,$$

we have

$$(2.20)\qquad \lim_{n\to\infty}\sum_{k=-n+1}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1-\mathrm{i}t2^k}{2^{k}(1+\frac{k}{n})(1+\frac{k+1}{n})}=\sum_{-\infty}^{0}\frac{\exp\{\mathrm{i}t2^{k}\}-1-\mathrm{i}t2^k}{2^{k}}.$$

Combining (2.17), (2.18), (2.19) and (2.20), we get

$$\begin{aligned} &f\left(\frac{t}{N_n}\right)-1 \\ &= \frac{1}{n^2N_n}\left\{\sum_{-\infty}^{0}\frac{\exp\{\mathrm{i}t2^k\}-1-\mathrm{i}t2^k}{2^k}+\mathrm{i}n^2t+\sum_{k=1}^{\infty}\frac{\exp\{\mathrm{i}t2^k\}-1}{2^k}+o(1)\right\} \\ &= \frac{1}{n^2N_n}\left\{g(t)+\mathrm{i}n^2t+o(1)\right\} \text{ as } n\to\infty. \end{aligned}$$

By (2.16) we can also get

$$\begin{aligned} f_{N_n}(t) &= \left\{1+\frac{g(t)+\mathrm{i}n^2t+o(1)}{n^2N_n}\right\}^{n^2N_n}\cdot\exp\{-\mathrm{i}n^2t\} \\ &= \exp\{g(t)\}+o(1) \text{ as } n\to\infty. \end{aligned} \tag{2.21}$$

Thus Theorem 2.3 follows from (2.16), (2.21) and the continuity theorem of characteristic function. □

3. Limit behaviours for the related model of FUFG

In this section, we will consider the related model of Feller's unfair-fair-game (1.3) and study its probability limit behaviours. Our findings show that the limit properties of model (1.3) and model (1.1) are quite different. Since the new model follows a long-tailed distribution (see [4]), which is widely used in many fields, therefore it is meaningful to study in depth the probability limit behaviours of this new model.

To prove the next theorems, we will apply the concepts of *regularly varying function*, *slowly varying function* and their key properties.

Definition (see [7])**.** Let U, V be positive monotone functions on $[0,\infty)$ to $[0,\infty)$. We say that U is a regularly varying function (at $+\infty$) with exponent $\alpha\in\mathbb{R}$ if $U(x)=x^\alpha V(x)$ where V is a slowly varying function (at $+\infty$), that is

$$\lim_{x\to\infty}\frac{V(tx)}{V(x)}=1$$

for every $t>0$.

Obviously, regularly variation and slowly variation are tail behaviours for functions so that they are independent of their initial values. $\ln x$, $\ln^2 x$ are slowly varing functions on $(0,\infty)$. Every slowly varying function varies regularly with exponent 0.

Lemma 3.1 (Main Karamata Theorem, see [7])**.** *Let H be positive monotone on $[0,\infty)$ and set*

$$U_s(x)=\int_0^x y^sH(y)\mathrm{d}y,\ V_s(x)=\int_x^\infty y^sH(y)\mathrm{d}y.$$

(i) *If H varies regularly with exponent $\alpha \leqslant -s-1$ and $V_s(x) < \infty$, then, as $x \to \infty$,*

$$\frac{x^{s+1}H(x)}{V_s(x)} \to c = -(s+\alpha+1) \geqslant 0.$$

Conversely, if this limit exists and is positive, then V_s and H vary regularly with exponents $-c = s+\alpha+1$ and α, respectively, while if this limit is 0, then V_s is a slowly varying function.

(ii) *If H varies regularly with exponent $\alpha \geqslant -s-1$, then, as $x \to \infty$,*

$$\frac{x^{s+1}H(x)}{U_s(x)} \to c = s+\alpha+1 \geqslant 0.$$

Conversely, if this limit exists and is positive, then U_s and H vary regularly with exponents $c = s+\alpha+1$ and α, respectively, while if this limit is 0, then U_s is a slowly varying function.

Lemma 3.2. *If $s \geqslant 0$, then*

$$\sum_{k=1}^{n} \frac{k^s}{(\mathrm{Log}_a k)^2} = O\left(\frac{n^{s+1}}{(\mathrm{Log}_a n)^2}\right) \quad \text{as } n \to \infty. \tag{3.1}$$

Proof. For every $s > 0$ and x large enough, $x^s/(\mathrm{Log}_a x)^2$ is monotonously increasing. So, there are constants C_1 and C_2 satisfying

$$C_1 \int_1^n \frac{x^s}{(\mathrm{Log}_a x)^2}\mathrm{d}x \leqslant \sum_{k=2}^{n} \frac{k^s}{(\mathrm{Log}_a k)^2} \leqslant C_2 \int_2^{n+1} \frac{x^s}{(\mathrm{Log}_a x)^2}\mathrm{d}x. \tag{3.2}$$

Since $(\mathrm{Log}_a x)^{-2}$ is regularly varying with exponent $\alpha = 0$ (is also slowly varying), by the (ii) of Lemma 3.1, we can obtain

$$\int_1^n \frac{x^s}{(\mathrm{Log}_a x)^2}\mathrm{d}x = O\left(\frac{n^{s+1}}{(\mathrm{Log}_a n)^2}\right) = \int_2^{n+1} \frac{x^s}{(\mathrm{Log}_a x)^2}\mathrm{d}x \text{ as } n \to \infty. \tag{3.3}$$

Combining (3.2) and (3.3), it's easy to see that (3.1) holds.

If $s = 0$, then we only change the direction of inequalities (3.2) to know that (3.1) is also true. □

With the above preparations, now we start to research the limit properties of model (1.3). Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.3), we obtain the following results about its weak law of large numbers, strong law of large numbers and central limit theorem, respectively.

Theorem 3.3. *Let $X_1, X_2, \cdots$ be independent random variables with the same distribution described in (1.3), $b_n = (\mathrm{Log}_a n)^n$. Then*

$$\frac{S_n}{b_n} \to 0 \quad \textit{in probability.} \tag{3.4}$$

Proof. Let $m_n = [b_n]$, it's easy to verify that $\mathrm{Log}_a m_n \sim n\mathrm{Log}_a\mathrm{Log}_a n$ as $n \to \infty$. Suppose that $\widetilde{X}_i = X_i I(X_i \leqslant b_n)$, $\widetilde{S}_n = \sum_{i=1}^n \widetilde{X}_i$. We have

$$\begin{aligned}
P\left\{\widetilde{S}_n \neq S_n\right\} &= P\left\{\bigcup_{i=1}^n \{\widetilde{X}_i \neq X_i\}\right\} \\
&\leqslant nP\{X_1 > b_n\} \\
&\leqslant n \sum_{k=m_n+1}^{\infty} \frac{1}{ck(\mathrm{Log}_a k)^2} \\
&\leqslant \frac{Cn}{\mathrm{Log}_a m_n} \\
&\leqslant \frac{C}{\mathrm{Log}_a\mathrm{Log}_a n} \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.5}$$

By Lemma 3.2 we also have

$$\widetilde{\mu}_1 = E\widetilde{X}_1 = \sum_{k=1}^{m_n} \frac{1}{c(\mathrm{Log}_a k)^2} = O\left(\frac{b_n}{(n\mathrm{Log}_a\mathrm{Log}_a n)^2}\right).$$

Applying the Chebyshev's inequality and Lemma 3.2 we can get

$$\begin{aligned}
P\left\{\left|\frac{\widetilde{S}_n - n\widetilde{\mu}_1}{b_n}\right| > \varepsilon\right\} &= P\left\{\left|\sum_{i=1}^n \left(\widetilde{X}_i - \widetilde{\mu}_1\right)\right| > \varepsilon b_n\right\} \\
&\leqslant \frac{1}{\varepsilon^2 b_n^2} E\left|\sum_{i=1}^n \left(\widetilde{X}_i - \widetilde{\mu}_1\right)\right|^2 \\
&\leqslant \frac{1}{\varepsilon^2 b_n^2} E\left(\sum_{i=1}^n \widetilde{X}_i^2\right) \\
&= \frac{n}{\varepsilon^2 b_n^2} \sum_{k=1}^{m_n} \frac{k}{c(\mathrm{Log}_a k)^2} \\
&\leqslant \frac{Cn}{b_n^2} \cdot \frac{m_n^2}{(\mathrm{Log}_a m_n)^2} \\
&= \frac{C}{n(\mathrm{Log}_a\mathrm{Log}_a n)^2} \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we can also get

$$\frac{S_n - n\widetilde{\mu}_1}{b_n} \to 0 \ \text{ in probability.} \tag{3.7}$$

Therefore (3.4) follows from (3.7) and $n\widetilde{\mu}_1/b_n \to 0$ as $n \to \infty$. The proof of Theorem 3.3 is completed. □

Theorem 3.4. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in* (1.3)*,* $b_n = (\mathrm{Log}_a n)^n$*. Then*

$$\liminf_{n\to\infty} \frac{S_n}{b_n} = 0 \quad a.s., \quad and \quad \limsup_{n\to\infty} \frac{S_n}{b_n} = +\infty \ a.s.. \tag{3.8}$$

Proof. Since $X_i \geqslant 0$ a.s. thus

$$\liminf_{n\to\infty} \frac{S_n}{b_n} \geqslant 0 \text{ a.s.}.$$

From Theorem 3.3 we know that there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{S_{n_k}}{b_{n_k}} = 0 \text{ a.s.}.$$

Therefore the first part of (3.8) holds.

To prove the second part of (3.8) we first prove

$$\limsup_{n\to\infty} \frac{X_n}{b_n} = +\infty \text{ a.s.}. \tag{3.9}$$

In fact, for every $M > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\{X_n > Mb_n\} &= \sum_{n=1}^{\infty} \sum_{k=[Mb_n]+1}^{\infty} \frac{1}{ck\,(\mathrm{Log}_a k)^2} \\ &\geqslant C \sum_{n=1}^{\infty} \frac{1}{\mathrm{Log}_a([Mb_n]+1)} \\ &\geqslant C \sum_{n=1}^{\infty} \frac{1}{n\mathrm{Log}_a n} = +\infty.\end{aligned}$$

By the second Borel-Cantelli lemma we know that

$$P\left\{\frac{X_n}{b_n} > M, \text{ i.o.}\right\} = 1.$$

So (3.9) holds. Since $\lim_{n\to\infty} b_{n-1}/b_n = 0$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n}{b_n} &= \limsup_{n\to\infty} \left(\frac{X_n}{b_n} + \frac{S_{n-1}}{b_{n-1}} \cdot \frac{b_{n-1}}{b_n}\right) \\ &\geqslant \limsup_{n\to\infty} \frac{X_n}{b_n} + \liminf_{n\to\infty} \frac{S_{n-1}}{b_{n-1}} \cdot \lim_{n\to\infty} \frac{b_{n-1}}{b_n} \\ &= +\infty.\end{aligned}$$

The last step above follows from (3.9) and the first part of (3.8) proved just now. The proof of Theorem 3.4 is completed. □

Remark 3.5. Since $X_n \geqslant 0$ a.s., for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{X_n < \varepsilon b_n\} = \sum_{n=1}^{\infty} P\{X_1 < \varepsilon b_n\} = +\infty,$$

by the second Borel-Cantelli lemma we know $P\{X_n < \varepsilon b_n \text{ i.o.}\} = 1$, consequently

$$\liminf_{n\to\infty} \frac{X_n}{b_n} = 0 \text{ a.s..}$$

Theorem 3.6. *Let $X_1, X_2, \ldots$ be independent random variables with the same distribution described in (1.3), $t_n = a^{n(\mathrm{Log}_a n)^\gamma}$, $\gamma > 1$. Then*

$$\lim_{n\to\infty} \frac{S_n}{t_n} = 0 \quad a.s..$$

Proof. Let $\widetilde{X}_i = X_i I\ (X_i \leqslant t_n)$. Since

$$\begin{aligned}\sum_{n=1}^{\infty} P\{X_n > t_n\} &= \sum_{n=1}^{\infty} P\{X_1 > t_n\} \\ &= \sum_{n=1}^{\infty} \sum_{k=[t_n]+1}^{\infty} \frac{1}{k(\mathrm{Log}_a k)^2} \\ &\leqslant C \sum_{n=1}^{\infty} \frac{1}{\mathrm{Log}_a t_n} \\ &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n(\mathrm{Log}_a n)^\gamma} < \infty,\end{aligned}$$

by the Borel-Cantelli lemma we know

$$P\{\widetilde{X}_n \neq X_n \text{ i.o.}\} = 0. \tag{3.10}$$

Let $\widetilde{S}_n = \sum_{i=1}^n \widetilde{X}_i$. By Lemma 3.2, it's easy to get

$$\widetilde{\mu}_1 = E\widetilde{X}_1 = \sum_{k=1}^{[t_n]} \frac{1}{c(\mathrm{Log}_a k)^2} = O\left(\frac{t_n}{(\mathrm{Log}_a t_n)^2}\right) = O\left(\frac{t_n}{n^2 (\mathrm{Log}_a n)^{2\gamma}}\right).$$

For any $\varepsilon > 0$, applying the Chebyshev's inequality and Lemma 3.2 we have

$$\begin{aligned}\sum_{n=1}^{\infty} P\left\{\frac{\widetilde{S}_n - n\widetilde{\mu}_1}{t_n} > \varepsilon\right\} &= \sum_{n=1}^{\infty} P\left\{\widetilde{S}_n > n\widetilde{\mu}_1 + \varepsilon t_n\right\} \\ &\leq \sum_{n=1}^{\infty} P\left\{\widetilde{S}_n > \varepsilon t_n\right\} \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2 t_n^2} E\widetilde{S}_n^2 \\ &= \sum_{n=1}^{\infty} \frac{n}{\varepsilon^2 t_n^2} E\widetilde{X}_1^2 \\ &= \sum_{n=1}^{\infty} \frac{n}{\varepsilon^2 t_n^2} \sum_{k=1}^{[t_n]} \frac{k}{c(\mathrm{Log}_a k)^2}\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{n}{t_n^2}\cdot\frac{t_n^2}{(\mathrm{Log}_a t_n)^2}$$
$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{n(\mathrm{Log}_a n)^{2\gamma}}<\infty.$$

From the Borel-Cantelli lemma it follows that

$$P\left\{\frac{\widetilde{S}_n-n\widetilde{\mu}_1}{t_n}>\varepsilon \text{ i.o.}\right\}=0,$$

thus

$$\lim_{n\to\infty}\frac{\widetilde{S}_n-n\widetilde{\mu}_1}{t_n}=0 \text{ a.s..}$$

Since

$$\frac{n\widetilde{\mu}_1}{t_n}=O\left(\frac{1}{n\left(\mathrm{Log}_a n\right)^{2\gamma}}\right)\to 0,$$

we have

$$\lim_{n\to\infty}\frac{\widetilde{S}_n}{t_n}=0 \text{ a.s..}$$

It follows from (3.10) that there exists a positive integer N such that $\widetilde{X}_i=X_i$ a.s. for all $i>N$. Now, for $n>N$,

$$\widetilde{S}_n=\widetilde{X}_1+\cdots+\widetilde{X}_N+X_{N+1}+\cdots+X_n=\widetilde{S}_N+S_n-S_N \text{ a.s..}$$

Because $\lim_{n\to\infty}\widetilde{S}_N/t_n=0$ a.s. and $\lim_{n\to\infty}S_N/t_n=0$ a.s., so

$$\lim_{n\to\infty}\frac{S_n}{t_n}=\lim_{n\to\infty}\frac{\widetilde{S}_n}{t_n}-\lim_{n\to\infty}\frac{\widetilde{S}_N}{t_n}+\lim_{n\to\infty}\frac{S_N}{t_n}=0 \text{ a.s.}$$

The proof of Theorem 3.6 is completed. □

Theorem 3.7. *Let $X_1,X_2,\dots$ be independent random variables with the same distribution described in* (1.3). $\widetilde{X}_i=X_iI\ (X_i\leqslant n)$, $\widetilde{S}_n=\sum_{i=1}^n\widetilde{X}_i$, $\widetilde{\mu}_1=E\widetilde{X}_1$, $\widetilde{\sigma}_1^2=\mathrm{Var}\widetilde{X}_1$. *Then*

$$\frac{\widetilde{S}_n-n\widetilde{\mu}_1}{\widetilde{\sigma}_1\sqrt{n}}\xrightarrow{d}N(0,1).$$

Proof. Using Lemma 3.2 and the C_r-inequality, we can get

$$\widetilde{\mu}_1=E\widetilde{X}_1=\sum_{k=1}^{n}\frac{1}{c(\mathrm{Log}_a k)^2}=O\left(\frac{n}{(\mathrm{Log}_a n)^2}\right),$$

$$\widetilde{\sigma}_1^2=\mathrm{Var}\widetilde{X}_1=E\widetilde{X}_1^2-\widetilde{\mu}_1^2=\sum_{k=1}^{n}\frac{k}{c(\mathrm{Log}_a k)^2}-\widetilde{\mu}_1^2=O\left(\frac{n^2}{(\mathrm{Log}_a n)^2}\right),$$

$$E|\widetilde{X}_1-\widetilde{\mu}_1|^3\leqslant 4\left(E\widetilde{X}_1^3+\widetilde{\mu}_1^3\right)=4\sum_{k=1}^{n}\frac{k^2}{c(\mathrm{Log}_a k)^2}+4\widetilde{\mu}_1^3=O\left(\frac{n^3}{(\mathrm{Log}_a n)^2}\right).$$

Therefore

$$\widetilde{\rho} = \frac{E|\widetilde{X}_1 - \widetilde{\mu}_1|^3}{\widetilde{\sigma}_1^3} = O\left(\frac{n^3}{(\mathrm{Log}_a n)^2} \cdot \frac{(\mathrm{Log}_a n)^3}{n^3}\right) = O\left(\mathrm{Log}_a n\right).$$

By the Esseen's inequality (see [12]) we know

$$\sup_{-\infty<x<\infty}\left|P\left\{\frac{\widetilde{S}_n - n\widetilde{\mu}_1}{\widetilde{\sigma}_1\sqrt{n}} \leqslant x\right\} - \Phi(x)\right| \leqslant \frac{C\widetilde{\rho}}{\sqrt{n}} \leqslant \frac{C\mathrm{Log}_a n}{\sqrt{n}} \to 0 \text{ as } n \to \infty,$$

where $\Phi(x)$ denotes the distribution function of the standard normal random variable $N(0,1)$. The proof of Theorem 3.7 is completed. □

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