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$(\mathcal{F}, \mathcal{A})$-GORENSTEIN FLAT HOMOLOGICAL DIMENSIONS

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Abstract. In this paper we develop the homological properties of the Gorenstein $(\mathcal{L}, \mathcal{A})$-flat R-modules $\mathcal{GF}_{(\mathcal{F}(R),\mathcal{A})}$ proposed by Gillespie, where the class $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ sometimes corresponds to a duality pair $(\mathcal{L}, \mathcal{A})$. We study the weak global and finitistic dimensions that come with the class $\mathcal{GF}_{(\mathcal{F}(R),\mathcal{A})}$ and show that over a $(\mathcal{L}, \mathcal{A})$-Gorenstein ring, the functor $- \otimes_R -$ is left balanced over $\mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R)$ by the classes $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})} \times \mathcal{GF}_{(\mathcal{F}(R),\mathcal{A})}$. When the duality pair is $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$ we recover the G. Yang's result over a Ding-Chen ring, and we see that is new for $(\mathrm{Lev}(R), \mathrm{AC}(R^{\mathrm{op}}))$ among others.

1. Introduction

Duality pairs were defined and studied by H. Holm and P. Jørgensen [20]. Recall that for a left R-module M, its character module is defined to be the right R-module $M^+ := \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$. A pair of classes $(\mathcal{L}, \mathcal{A}) \subseteq \mathrm{Mod}(R) \times \mathrm{Mod}(R^{\mathrm{op}})$ is a duality pair if it is such that $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$, and $\mathcal{A}$ is closed under direct summands and finite direct sums. J. Gillespie [17] define the notion of AC-Gorenstein ring which is a generalization of a Gorenstein ring compatible with the Gorenstein AC-projective R-modules [6, §8]. In [17] J. Gillespie has described certain model structures that come from this Gorenstein AC-projective R-modules. In other paper J. Gillespie has introduced the notion of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective, $(\mathcal{L}, \mathcal{A})$-injective and $(\mathcal{L}, \mathcal{A})$-flat R-modules [18] (resp. denoted in this paper by $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$, $\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ and $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$) that come from a duality pair $(\mathcal{L}, \mathcal{A})$. Based on these ideas J. Wang and Z. Di [24] define the concept of $(\mathcal{L}, \mathcal{A})$-Gorenstein ring with respect to a *bi-complete* duality pair showing that such notion unify the notions of Gorenstein, Ding-Chen, AC-Gorenstein, and Gorenstein n-coherent rings taking an

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appropriate duality pair. Among others results J. Wang and Z. Di [24, Theorem 4.8] have been proven that over a *bi-complete* duality pair $(\mathcal{L}, \mathcal{A})$ there is an hereditary and complete cotorsion triple that comes from the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective, and Gorenstein $(\mathcal{L}, \mathcal{A})$-injective R-modules. This result recovers the G. Yang's result [26, Theorem 3.6] when R is Ding-Chen and for the duality pair $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$. More recently J. Gillespe and A. Iacob [19, Corollary 5.1] extend the study of model structures given in [6] and [17] of a general way for the classes of R-modules Gorenstein $(\mathcal{L}, \mathcal{A})$-projective, $(\mathcal{L}, \mathcal{A})$-injective and $(\mathcal{L}, \mathcal{A})$-flat introduced in [18]. Thus, in the current literature there is a treatment of the homotopic properties of the classes $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$, $\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ and $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.

We therefore see the importance to develop their homologic properties, which is the aim of the present paper. We develop this properties of the following manner. In Section 2 we present general concepts from relative homological algebra in terms of an abelian category $\mathcal{C}$ which are important in the study of balance, such as relative homological dimensions, left and right approximations, relative cogenerators, cotorsion pairs. Also recall the notion of GP-admissible pair and see how some duality pairs and their properties give us GP-admissible pairs. Section 3 is devoted to develop relative homological dimensions that come from the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. In Section 4 we study the natural finitistic and weak global dimensions relative to the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, we prove in Theorem 4.1 that the left weak global dimension that comes from $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is characterized by the flat dimension of the class $\mathcal{A}$, we also prove in Proposition 4.6 that under certain conditions there is an under limit given by the flat dimension of the class $\mathcal{L}$. We also compare in Remark 4.9 this left weak global dimension with the finitistic dimension obtained from $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. The main results are presented in Section 5. We first discuss the usefulness of the bi-complete duality pairs to define a $(\mathcal{L}, \mathcal{A})$-Gorenstein ring in the sense of Wang [24, §4]. Also we see how the notion of GP-admissible pair [4] enables us to obtain a relative Hom-balance result for the classes $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$, $\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ in Lemma 5.1. Furthermore we prove in Lemma 5.7 that for a $(\mathcal{L}, \mathcal{A})$-Gorenstein ring, the functor $-\otimes_R-$ is left balanced over the whole category $\mathrm{Mod}(R)$ by the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. Finally in Section 6 we give examples of the relative Hom-balance and several examples of $(\mathcal{L}, \mathcal{A})$-Gorenstein rings where the $(-\otimes_R-)$-balance occurs and we see how one of this examples recover the well-known result on a Ding-Chen ring [26, Theorem 3.23]. Finally in Corollary 6.4 and Proposition 6.5 we answer a question proposed by A. Iacob [21], namely; *When it is true that the Ding projective modules and the Gorenstein projective modules coincide?*. We also give conditions in Proposition 6.7 for other Gorenstein classes to coincide with the Gorenstein projective R-modules.

2. Preliminaries

In this section we will declare some of notation in terms of an abelian category $\mathcal{C}$. The general notation presented here will enable us to give short proofs and more clear concepts in the following sections. We denote by $\mathrm{pd}(C)$ the *projective dimension* of $C \in \mathcal{C}$, and by $\mathcal{P}(\mathcal{C})$ the class of all the objects $C \in \mathcal{C}$ with $\mathrm{pd}(C) = 0$. Similarly, $\mathrm{id}(C)$ stands for the *injective dimension* of $C \in \mathcal{C}$, and $\mathcal{I}(\mathcal{C})$ for the class of all the objects $C \in \mathcal{C}$ with $\mathrm{id}(C) = 0$. Monomorphisms and epimorphisms in $\mathcal{C}$ may sometimes be denoted using arrows $\hookrightarrow$ and $\twoheadrightarrow$, respectively.

Let $\mathcal{X}$ be a class of objects in $\mathcal{C}$ and $M \in \mathcal{C}$. We set the following notation:
Orthogonal classes. For each positive integer i, we consider the right orthogonal classes

$$\mathcal{X}^{\perp_i} := \{N \in \mathcal{C} : \mathrm{Ext}^i_{\mathcal{C}}(-,N)|_{\mathcal{X}} = 0\} \text{ and } \mathcal{X}^{\perp} := \cap_{i>0}\mathcal{X}^{\perp_i}.$$

Dually, we have the left orthogonal classes ${}^{\perp_i}\mathcal{X}$ and ${}^{\perp}\mathcal{X}$.

Given a class $\mathcal{Y} \subseteq \mathcal{C}$, we write $\mathcal{X} \perp \mathcal{Y}$ whenever $\mathrm{Ext}^i_{\mathcal{C}}(X,Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $i > 0$.

Relative homological dimensions. The *relative projective dimension of M, with respect to $\mathcal{X}$*, is defined as

$$\mathrm{pd}_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} : \mathrm{Ext}^j_{\mathcal{C}}(M,-)|_{\mathcal{X}} = 0 \text{ for all } j > n\}.$$

We set $\min\emptyset := \infty$. Dually, we denote by $\mathrm{id}_{\mathcal{X}}(M)$ the *relative injective dimension of M with respect to $\mathcal{X}$*. Furthermore, we set

$$\mathrm{pd}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\mathrm{pd}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\} \text{ and } \mathrm{id}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\mathrm{id}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\}.$$

If $\mathcal{X} = \mathcal{C}$, we just write $\mathrm{pd}(\mathcal{Y})$ and $\mathrm{id}(\mathcal{Y})$.

Resolution and coresolution dimension. The *$\mathcal{X}$-coresolution dimension of $M \in \mathcal{C}$* denoted $\mathrm{coresdim}_{\mathcal{X}}(M)$, is the smallest non-negative integer n such that there is an exact sequence

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_n \to 0$$

with $X_i \in \mathcal{X}$ for all $i \in \{0,\dots,n\}$. If such n does not exist, we set

$$\mathrm{coresdim}_{\mathcal{X}}(M) := \infty.$$

Also, we denote by $\mathcal{X}^{\vee}_n$ the class of objects in $\mathcal{C}$ with $\mathcal{X}$-coresolution dimension at most n. The union $\mathcal{X}^{\vee} := \cup_{n\geq 0}\mathcal{X}^{\vee}_n$ is the class of objects in $\mathcal{C}$ with finite $\mathcal{X}$-coresolution dimension. Dually, we have the *$\mathcal{X}$-resolution dimension* $\mathrm{resdim}_{\mathcal{X}}(M)$ of M, $\mathcal{X}^{\wedge}_n$ the class of objects in $\mathcal{C}$ having $\mathcal{X}$-resolution dimension at most n, and the union $\mathcal{X}^{\wedge} := \cup_{n\geq 0}\mathcal{X}^{\wedge}_n$ is the class of objects in $\mathcal{C}$ with finite $\mathcal{X}$-resolution dimension. We set

$$\mathrm{coresdim}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\mathrm{coresdim}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\},$$

and $\mathrm{resdim}_{\mathcal{X}}(\mathcal{Y})$ is defined dually.

Proper resolutions and balance. Given a class $\mathcal{X} \subseteq \mathcal{C}$ a *left proper $\mathcal{X}$-resolution* of $M \in \mathcal{C}$ is a complex $\mathbf{X}(M) : \cdots \to X^1 \to X^0 \to M \to 0$ such that the complex $\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{X})$ is acyclic for all $X \in \mathcal{X}$. A *right proper $\mathcal{X}$-coresolution* is defined dually. We recall that if $\mathcal{X}, \mathcal{Y}, \mathcal{B}, \mathcal{E}$ are classes of objects in $\mathcal{C}$, then we say that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is *right balanced* on $\mathcal{X} \times \mathcal{Y}$ by $\mathcal{B} \times \mathcal{E}$ if for any objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ there exist complexes

$$\cdots \to B_2 \to B_1 \to B_0 \to X \to 0$$

and

$$0 \to Y \to E^0 \to E^1 \to E^2 \to \cdots$$

such that $B_i \in \mathcal{B}$, $E^i \in \mathcal{E}$ for all $i \geq 0$ and such that $\mathrm{Hom}_{\mathcal{C}}(-, E)$ makes the first complex acyclic whenever $E \in \mathcal{E}$ and such that $\mathrm{Hom}_{\mathcal{C}}(B, -)$ makes the second complex acyclic whenever $B \in \mathcal{B}$. In the case of the category $\mathrm{Mod}(R)$ of left R-modules over an associative ring R the definition is easily modified to give the definition of a left or a right balanced functor $- \otimes_R -$ (see [9, 11, 12] for details).

The class $\mathcal{X}$ is called *precovering* if for each $M \in \mathcal{C}$ there is a homomorphism $f : X \to M$ such that $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, M)$ is surjective for any $Z \in \mathcal{X}$. Dually is defined a *preenveloping* class.

Relative Gorenstein objects. Given a pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects in $\mathcal{C}$, an object $M \in \mathcal{C}$ is $(\mathcal{X}, \mathcal{Y})$*-Gorenstein projective* [4, Definition 3.2] if M is a cycle of an exact complex $\mathbf{X}$ with $X_m \in \mathcal{X}$ for every $m \in \mathbb{Z}$ such that the complex $\mathrm{Hom}_{\mathcal{C}}(\mathbf{X}, Y)$ is an exact complex for all $Y \in \mathcal{Y}$. The class of all $(\mathcal{X}, \mathcal{Y})$-Gorenstein projective objects is denoted by $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$.

The $(\mathcal{X}, \mathcal{Y})$*-Gorenstein projective dimension of* M is defined by

$$\mathrm{Gpd}_{(\mathcal{X},\mathcal{Y})}(M) := \mathrm{resdim}_{\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}}(M),$$

and for any class $\mathcal{Z} \subseteq \mathcal{A}$, $\mathrm{Gpd}_{(\mathcal{X},\mathcal{Y})}(\mathcal{Z}) := \sup\{\mathrm{Gpd}_{(\mathcal{X},\mathcal{Y})}(Z) : Z \in \mathcal{Z}\}$.

Dually, we have the notion of $(\mathcal{X}, \mathcal{Y})$-Gorenstein injective objects and their dimensions.

Let $(\mathcal{X}, \omega) \subseteq \mathcal{C}^2$. The class ω is *$\mathcal{X}$-injective* if $\mathrm{id}_{\mathcal{X}}(\omega) = 0$. It is said that ω is a *relative cogenerator* in $\mathcal{X}$ if $\omega \subseteq \mathcal{X}$ and for any $X \in \mathcal{X}$ there is an exact sequence $0 \to X \to W \to X' \to 0$ with $W \in \omega$ and $X' \in \mathcal{X}$. Dually, we have the notions of *$\mathcal{X}$-projective* and *relative generator* in $\mathcal{X}$. We recall another notion from [4].

Definition 2.1 ([4, Definition 3.1]). A pair $(\mathcal{X}, \mathcal{Y}) \subseteq \mathcal{C}^2$ is *GP-admissible* if for each $C \in \mathcal{C}$, there is an epimorphism $X \to C$ with $X \in \mathcal{C}$, and $(\mathcal{X}, \mathcal{Y})$ satisfies the following conditions:

(a) $\mathcal{X}$ and $\mathcal{Y}$ are closed under finite coproducts in $\mathcal{C}$, and $\mathcal{X}$ is closed under extensions;
(b) $\omega := \mathcal{X} \cap \mathcal{Y}$ is a relative cogenerator in $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{Y}$.

GI-admissible pairs are defined dually (see [4, Definition 3.6]).

A pair $(\mathcal{X}, \mathcal{Y}) \subseteq \mathcal{C}^2$ is a *cotorsion pair* if $\mathcal{X}^{\perp_1} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$. This cotorsion pair is *complete* if for any $C \in \mathcal{C}$, there are exact sequences $0 \to Y \to X \to C \to 0$ and $0 \to C \to Y' \to X' \to 0$, where $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

Moreover, the cotorsion pair is *hereditary* if $\mathcal{X} \perp \mathcal{Y}$. A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \subseteq \mathcal{C}^3$ is called a *cotorsion triple* [8] provided that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are cotorsion pairs; it is *complete* (resp. *hereditary*) provided that both of the two cotorsion pairs are complete (resp. hereditary).

Now we turn our attention to the category $\mathrm{Mod}(R)$ of left R-modules, where R is an associative ring. To refer to an element in $\mathrm{Mod}(R)$ we will say simply *R-module*, while for a right R-module we will say *R^{op}-module* and we denote this last category by $\mathrm{Mod}(R^{\mathrm{op}})$. For short we will write $\mathcal{P}(R)$ and $\mathcal{I}(R)$ for the classes $\mathcal{P}(\mathrm{Mod}(R))$ and $\mathcal{I}(\mathrm{Mod}(R))$, respectively, and $\mathcal{F}(R)$ for the class of flat R-modules.

Recall that for a given R-module M, its *character module* is defined to be the R^{op}-module $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Definition 2.2 ([20, Definition 2.1])**.** A *duality pair* over a ring R is a pair $(\mathcal{L}, \mathcal{A})$, where $\mathcal{L}$ is a class of R-modules and $\mathcal{A}$ is a class of R^{op}-modules, satisfying the following conditions:

(1) $M \in \mathcal{L}$ if and only if $M^+ \in \mathcal{A}$.
(2) $\mathcal{A}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{L}, \mathcal{A})$ is called *perfect* if $\mathcal{L}$ contains the R-module ${}_RR$ and is closed under coproducts and extensions.

By the Lambek's Theorem, the more natural example of a duality pair is when we consider the class $\mathcal{F}(R)$ of all flat R-modules and the class $\mathcal{I}(R^{\mathrm{op}})$ of all injective R^{op}-modules. We are interested when $(\mathcal{P}(R), \mathcal{L})$ give us a GP-admissible pair and thus, it will be possible to obtain a similar result as the one in [24, Lemma 4.6] (by using [4, Proposition 3.16]), among others but with less hypothesis. The following result will be useful to see how to do this.

Proposition 2.3 ([18, Proposition 2.3])**.** *If $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair, then $\mathcal{L}$ contains all projective R-modules and $\mathcal{A}$ contains all injective R^{op}-modules. In fact $\mathcal{L}$ is closed under direct limits and so contains all flat R-modules too.*

The following notion comes originally by Gillespie [18] but it was only stated there for commutative rings. Recently this has been extended to non-commutative rings in [19], but earlier in [24, Definition 3.1(4)].

Definition 2.4 ([19, Definition 2.3])**.** By a *symmetric* duality pair $\{\mathcal{L}, \mathcal{A}\}$ we mean:

(1) $\mathcal{L}$ is a class of R-modules.
(2) $\mathcal{A}$ is a class of R^{op}-modules.
(3) $(\mathcal{L}, \mathcal{A})$ is a duality pair over R and $(\mathcal{A}, \mathcal{L})$ is a duality pair over R^{op}.

An example of a symmetric duality pair is obtained when R is a Noetherian ring by taking $\{\mathcal{F}(R), \mathcal{I}(R^{\mathrm{op}})\}$ (see [24, Example 3.7]).

Another example is obtained by taking $\mathcal{L}$ to be the class of all *level* R-modules and $\mathcal{A}$ to be the class of all *absolute clean* R^{op}-modules [6]. In the last section we will see in detail such classes of R-modules. We denote by $\mathrm{AC}(R)$ the class of all absolutely clean R-modules, and by $\mathrm{Lev}(R)$ the class of all level R-modules. J. Gillespie and A. Iacob call *semi-perfect* to a duality pair $(\mathcal{L}, \mathcal{A})$ if it has all the properties required to be a perfect duality pair *except* that $\mathcal{L}$ may not be closed under extensions [19]. With this in mind we have the following.

Definition 2.5 ([19, Definition 2.5])**.** By a *semi-complete duality pair* $(\mathcal{L}, \mathcal{A})$ we mean that $\{\mathcal{L}, \mathcal{A}\}$ is a symmetric duality pair with $(\mathcal{L}, \mathcal{A})$ being a semi-perfect duality pair. If $(\mathcal{L}, \mathcal{A})$ is indeed perfect, then we call it *complete duality pair*.

By [19, Remark 2.6] we see that when $(\mathcal{L}, \mathcal{A})$ is a semi-complete duality pair then $\mathcal{L}$ contains all projective and flat R-modules and $\mathcal{A}$ contains all injective R^{op}-modules.

3. Modules of $(\mathcal{L}, \mathcal{A})$-Gorenstein type

Recently Gillespie considers in [18] classes of Gorenstein projective, Gorenstein flat R-modules and Gorenstein injective R^{op}-modules, relative to a duality pair $(\mathcal{L}, \mathcal{A})$ as follows. Let M be an R-module and N be an R^{op}-module.

- M is *Gorenstein* $(\mathcal{L}, \mathcal{A})$*-projective* if $M = Z_0(\mathbf{P})$ for some exact complex of projective R-modules $\mathbf{P}$ for which $\mathrm{Hom}_R(\mathbf{P}, L)$ is acyclic for all $L \in \mathcal{L}$.
- M is *Gorenstein* $(\mathcal{L}, \mathcal{A})$*-flat* if $M = Z_0(\mathbf{F})$ for some exact complex of flat R-modules for which $A \otimes_R \mathbf{F}$ is acyclic for all $A \in \mathcal{A}$.
- N is *Gorenstein* $(\mathcal{L}, \mathcal{A})$*-injective* if $N = Z_0(\mathbf{I})$ for some exact complex of injective R^{op}-modules $\mathbf{I}$ for which $\mathrm{Hom}_R(A, \mathbf{I})$ is acyclic for all $A \in \mathcal{A}$.

We denote the previous classes of R-modules by $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$, $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, respectively, and the class of R^{op}-modules by $\mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$. It can be easily seen that when $(\mathcal{L}, \mathcal{A})$ is a perfect or semi-complete duality pair, then $(\mathcal{P}(R), \mathcal{L})$ is a GP-admissible pair in $\mathrm{Mod}(R)$, and $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible pair in $\mathrm{Mod}(R^{\mathrm{op}})$. From this we can obtain a similar result that the one in [24, Lemma 4.6] by applying the theory developed in [4], but for a perfect or semi-complete duality pair instead of a *bi-complete* duality pair (see Definition 4.3).

Since $(\mathrm{Lev}(R), \mathrm{AC}(R^{\mathrm{op}}))$ is a complete duality pair [19, Example 2.7], then $(\mathcal{P}(R), \mathrm{Lev}(R))$ is a GP-admissible pair in $\mathrm{Mod}(R)$ and $(\mathrm{AC}(R^{\mathrm{op}}), \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible in $\mathrm{Mod}(R^{\mathrm{op}})$. The R-module classes $\mathcal{GP}_{(\mathcal{P},\mathrm{Lev}(R))}$ and $\mathcal{GI}_{(\mathrm{AC}(R),\mathcal{I})}$ are called *Gorenstein AC-projective* and *Gorenstein AC-injective* [17], respectively. Note that we can consider the classes $\mathrm{AC}(R)$ and $\mathrm{AC}(R^{\mathrm{op}})$ which are defined in $\mathrm{Mod}(R)$ and in $\mathrm{Mod}(R^{\mathrm{op}})$, respectively, we will see later how this is useful to us. There is another relationship with Gorenstein categories which is set out below.

Remark 3.1. From [18, Theorem 2.5] when $(\mathcal{L}, \mathcal{A})$ is a symmetric duality pair in $\mathrm{Mod}(R)$ and R is a commutative ring, the class $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$ of all Gorenstein

$(\mathcal{L},\mathcal{A})$-projectives is precisely the subcategory $(\mathcal{P}(R),\mathcal{L},\mathcal{A})$-Gorenstein presented in [27, Definition 2.1]. Thus, some results in this paper will be applicable to this Gorenstein categories.

Lemma 3.2. *Consider R a right coherent ring, an exact complex $I^\bullet$ of injective R^{op}-modules and suppose that* $\text{fd}(A) < \infty$ *for all $A \in \mathcal{A} \subseteq \text{Mod}(R^{\text{op}})$. Then the exact complex of flat R-modules $I^{\bullet +}$ is $(\mathcal{A} \otimes_R -)$-acyclic.*

Proof. Let us consider $I^\bullet$ the exact complex of injective R^{op}-modules. Since R is right coherent, the exact complex $I^{\bullet +}$ consists of flat R-modules. We will prove that if N is an R^{op}-module with $\text{fd}(N) = m < \infty$, then $N \otimes_R I^{\bullet +}$ is acyclic. The case $m = 0$ is clear. Let $m > 0$. Then there is an exact sequence $0 \to F_1 \to F \to N \to 0$, with $F \in \mathcal{F}(R^{\text{op}})$ and $\text{fd}(F_1) \leq m-1$, thus we obtain the exact sequence

$$0 \to F_1 \otimes I^{\bullet +} \to F \otimes I^{\bullet +} \to N \otimes I^{\bullet +} \to 0$$

we know that $F \otimes I^{\bullet +}$ is acyclic and $F_1 \otimes I^{\bullet +}$ is acyclic by induction hypothesis. It follows that $N \otimes I^{\bullet +}$ is acyclic. Now let $A \in \mathcal{A}$ with $\text{fd}(A) < \infty$. It is now clear that $A \otimes I^{\bullet +}$ is acyclic. $\square$

Corollary 3.3. *Consider R a right coherent ring and suppose that* $\text{fd}(A) < \infty$ *for all $A \in \mathcal{A} \subseteq \text{Mod}(R^{\text{op}})$. Then for all $G \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\text{op}}))}$ it is satisfied that $G^+ \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.*

The following result will be very useful, as it allows us to use the Auslander-Buchweitz approximation theory.

Proposition 3.4. *Let $\mathcal{A} \subseteq \text{Mod}(R^{\text{op}})$ be a class such that $\mathcal{I}(R^{\text{op}}) \subseteq \mathcal{A}$ with R a right coherent ring and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then the intersection $\mathcal{F}(R) \cap \mathcal{F}(R)^\perp$ is a $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$-injective relative cogenerator for $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.*

Proof. Consider $M \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. We know that there is an exact complex by flat R-modules

$$\cdots \to L_1 \to L_0 \to L^0 \to L^1 \to \cdots$$

which is $(\mathcal{A} \otimes_R -)$-acyclic and with $\text{Ker}(L^0 \to L^1) = M$. Lets take the exact sequence $0 \to M \to L^0 \to N \to 0$. Since $(\mathcal{F}(R), \mathcal{F}(R)^\perp)$ is a complete (and hereditary) cotorsion pair and $\mathcal{F}(R)$ is closed under extensions, for L^0 there is an exact sequence $0 \to L^0 \to K \to L \to 0$ with $K \in \mathcal{F}(R) \cap \mathcal{F}(R)^\perp$ and $L \in \mathcal{F}(R)$. Consider the following p.o. diagram:

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, equal] & L_0 \arrow[r, two heads] \arrow[d, hook] & N \arrow[d, hook] \\
M \arrow[r, hook] & K \arrow[r, two heads] \arrow[d, two heads] & M' \arrow[d, two heads] \\
& L \arrow[r, equal] & L
\end{tikzcd}$$

Since $N \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, $L \in \mathcal{F}(R)$ and $\mathcal{F}(R)$ is closed under extensions, by using [25, Lemma 2.11] we have that $M' \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, thus the exact sequence $0 \to M \to K \to M' \to 0$ is such that $K \in \mathcal{F}(R) \cap \mathcal{F}(R)^{\perp}$ and $M' \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. We still have to prove that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})} \perp (\mathcal{F}(R) \cap \mathcal{F}(R)^{\perp})$, to do this let's choose $X \in \mathcal{F}(R) \cap \mathcal{F}(R)^{\perp}$, we always have the pure exact sequence $\xi : 0 \to X \to X^{++} \to X^{++}/X \to 0$, where $X \in \mathcal{F}(R)$ and $X^{++} \in \mathcal{F}(R)$ (R is right coherent). Since $\mathcal{F}(R)$ is closed by pure quotients we obtain that $X^{++}/X \in \mathcal{F}(R)$, so that the exact sequence ξ splits, and thus for $T \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ we have

$$\begin{aligned}\operatorname{Ext}^i_R(T, X) \oplus \operatorname{Ext}^i_R(T, X^{++}/X) &\cong \operatorname{Ext}^i_R(T, X^{++}) \\ &= \operatorname{Ext}^i_R(T, \operatorname{Hom}_{\mathbb{Z}}(X^{+}, \mathbb{Q}/\mathbb{Z})) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}^R_i(X^{+}, T), \mathbb{Q}/\mathbb{Z}) = 0\end{aligned}$$

the last term is zero, since $\operatorname{Tor}^R_i(\mathcal{A}, \mathcal{GF}_{(\mathcal{F},\mathcal{A})}) = 0$ by [13, Lemma 2.3]. $\square$

From the hypothesis of the previous results we see that it is important to know when the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions, this has been studied recently. From [21, Proposition 7] we know that this occurs when the class $\mathcal{A}$ is *semi-definable* and $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$. Also from [19, Corollary 5.3] when $(\mathcal{L}, \mathcal{A})$ is a semi-complete duality pair then $(\mathcal{GF}_{(\mathcal{F},\mathcal{A})}(R), \mathcal{GF}_{(\mathcal{F},\mathcal{A})}(R)^{\perp_1})$ is a perfect cotorsion pair and thus, $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}(R)$ is closed under extensions.

The $(\mathcal{F}(R), \mathcal{A})$-*Gorenstein flat dimension of* $M \in \operatorname{Mod}(R)$ is defined by

$$\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) := \operatorname{resdim}_{\mathcal{GF}_{(\mathcal{F},\mathcal{A})}}(M),$$

and for any class $\mathcal{Z} \subseteq \mathcal{A}$, $\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(\mathcal{Z}) := \sup\{\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(Z) : Z \in \mathcal{Z}\}$.

Theorem 3.5. *Let* $\mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ *and suppose that* $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ *is closed under extensions. Then for all* $C \in \operatorname{Mod}(R)$ *with* $\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(C) = n < \infty$*, the following statements are true.*

(a) *There are exact sequences in* $\operatorname{Mod}(R)$

$$0 \to K \to X \xrightarrow{\varphi} C \to 0$$

with $\operatorname{resdim}_{\mathcal{F}}(K) = n - 1$ *and* $X \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$*, and*

$$0 \to C \xrightarrow{\varphi'} H \to X' \to 0$$

with $\operatorname{resdim}_{\mathcal{F}}(H) = n$ *and* $X' \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.

(b) *If* R *is a right coherent ring and* $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$*, then*

(i) $\varphi : X \to C$ *is a* $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$*-precover and* $K \in \mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})}$,

(ii) $\varphi' : C \to H$ *is an* $(\mathcal{F}(R) \cap \mathcal{F}(R)^{\perp})^{\wedge}$*-preenvelope and* $X' \in {}^{\perp}((\mathcal{F}(R) \cap \mathcal{F}(R)^{\perp})^{\wedge})$.

Proof. (a) It follows from [4, Theorem 2.8(a)] while the equality $\operatorname{resdim}_{\mathcal{F}}(H) = n$ it holds since $\mathcal{F}(R)$ is a resolving class.

(b) It follows from Proposition 3.4 and [4, Theorem 2.8(b)]. $\square$

The following result makes use of the fact that the flat dimension $\mathrm{fd}(-)$ coincides with $\mathrm{resdim}_{\mathcal{F}}(-)$ [23, Proposition 8.17].

Proposition 3.6. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then for all $M \in \mathcal{A}^+$ the equality $\mathrm{fd}(M) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$ is given.*

Proof. Let us suppose that $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) < \infty$, by Theorem 3.5(a), there is an exact sequence $\eta : 0 \to M \to X \to G \to 0$ such that $\mathrm{fd}(X) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$ and $G \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. Since $M \in \mathcal{A}^+$ then $M = A^+$ for some $A \in \mathcal{A}$, and thus

$$\mathrm{Ext}^i_R(G, M) = \mathrm{Ext}^i_R(G, A^+) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_i(A, G), \mathbb{Q}/\mathbb{Z}).$$

Now $\mathrm{Tor}^R_i(\mathcal{A}, \mathcal{GF}_{(\mathcal{F},\mathcal{A})}) = 0$ by [25, Lemma 2.10], thus the sequence η splits, therefore

$$\mathrm{fd}(M) \leq \mathrm{fd}(X) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M).$$

Finally the inequality $\mathrm{fd}(M) \geq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$ is always true. $\square$

Proposition 3.7. *Consider a class $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$. Then for all $M \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, it holds that $M^+ \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$.*

Proof. Take $F^\bullet$ an exact complex of flat R-modules such that the complex $A \otimes_R F^\bullet$ is acyclic for all $A \in \mathcal{A}$. Since $(-)^+$ is an exact functor and by [12, Theorem 2.1.10] the following isomorphism is natural

$$(A \otimes_R F^\bullet)^+ \cong \mathrm{Hom}_R(A, (F^\bullet)^+),$$

we have that the last term is acyclic if and only if $A \otimes_R F^\bullet$ is acyclic. By Lambek's Theorem [23, Proposition 3.54] the complex $(F^\bullet)^+$ is of injective R^{op}-modules and also is acyclic, thus is $\mathrm{Hom}_R(\mathcal{A}, -)$-acyclic. $\square$

Proposition 3.8. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ be a class such that the pair $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible in $\mathrm{Mod}(R^{\mathrm{op}})$. Then for all $M \in \mathrm{Mod}(R)$ with $\mathrm{pd}(M^+) < \infty$ it is satisfied that $\mathrm{fd}(M) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$.*

Proof. Since $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible, by using the dual of [3, Lemma 3.3] we have in $\mathrm{Mod}(R^{\mathrm{op}})$ that

$$\mathrm{id}(M^+) = \mathrm{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+),$$

and it is true that $\mathrm{fd}(M) = \mathrm{id}(M^+)$ (for every ring R) and it is also true (we prove this below) that $\mathrm{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) \leq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$, thus we obtain the inequality

$$\mathrm{fd}(M) \leq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$$

but the opposite inequality always occurs, since $\mathcal{F}(R) \subseteq \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. It remains to prove that $\mathrm{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) \leq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$ but it follows from Proposition 3.7. $\square$

A natural question is whether all Gorenstein $(\mathcal{L},\mathcal{A})$-projective R-modules are Gorenstein $(\mathcal{L},\mathcal{A})$-flat, for now we will prove this for a symmetric duality pair $\{\mathcal{L},\mathcal{A}\}$. Thus, the inequality $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \leq \mathrm{Gpd}_{(\mathcal{P},\mathcal{L})}(M)$ will be true for all $M \in \mathrm{Mod}(R)$.

Proposition 3.9. *Let $(\mathcal{L},\mathcal{A}) \subseteq \mathrm{Mod}(R) \times \mathrm{Mod}(R^{\mathrm{op}})$ be such that $\mathcal{A}^+ \subseteq \mathcal{L}$. Then the containment $\mathcal{GP}_{(\mathcal{P},\mathcal{L})} \subseteq \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is given.*

Proof. Consider an exact complex $\mathbf{P}$ of projective R-modules which is $\mathrm{Hom}_R(-,\mathcal{L})$-acyclic, we will see below that for all $A \in \mathcal{A}$ it is satisfied that $A \otimes_R \mathbf{P}$ is acyclic. By [12, Theorem 2.1.10] we have the following natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(A \otimes_R \mathbf{P}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(\mathbf{P}, A^+).$$

Since for all $A \in \mathcal{A}$ is satisfied that $A^+ \in \mathcal{L}$, we have that $\mathrm{Hom}_R(\mathbf{P}, A^+)$ is acyclic for all $A \in \mathcal{A}$. Therefore $A \otimes_R \mathbf{P}$ is acyclic for all $A \in \mathcal{A}$. □

Lemma 3.10. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ and suppose that the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then for all $M \in \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}$ the following statements are equivalent.*

(i) $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \leq n$;
(ii) *If $0 \to K_n \to H_{n-1} \to \cdots \to H_0 \to M \to 0$ is an exact sequence with $H_i \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, then $K_n \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.*

Proof. Since the coproduct of flat R-modules is flat and we work in $\mathrm{Mod}(R)$, then the coproduct of exact complexes with flat components is an exact complex of flat components, then as the tensor commutes with the coproducts, we have that the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed by coproducts. It follows from the hypotheses that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is a resolving class, then by the Eilenberg's Swindle we obtain that the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under direct summands. Finally the result follows from the Auslander-Bridger's Lemma [2, Lemma 3.12]. □

We have that the following lemma, a generalization of [5, Lemma 2.5].

Lemma 3.11. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ such that $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$. The following statements are equivalent.*

(1) *$\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions*;
(2) *The class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is preresolving*;
(3) *For all exact sequence of R-modules $0 \to G_1 \to G_0 \to M \to 0$ with $G_1, G_0 \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ if $\mathrm{Tor}_1^R(\mathcal{A}, M) = 0$, then $M \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.*

Proof. We will prove (1) $\Rightarrow$ (3). The remaining implications are similar to [5, Lemma 2.5]. Consider the exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_1, G_0 \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and such that $\mathrm{Tor}_1^R(\mathcal{A}, M) = 0$. Since $G_1 \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, then there is an exact sequence $0 \to G_1 \to F_1 \to H \to 0$ with $F_1 \in \mathcal{F}(R)$ and

$H \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. We have the following p.o. diagram:

$$\begin{tikzcd}
G_1 \arrow[r, hook] \arrow[d, hook] & G_0 \arrow[r, two heads] \arrow[d, hook] & M \arrow[d, equal] \\
F_1 \arrow[r, hook] \arrow[d, two heads] & D \arrow[r, two heads] \arrow[d, two heads] & M \\
H \arrow[r, equal] & H &
\end{tikzcd}$$

From the sequence $0 \to G_0 \to D \to H \to 0$ we have that $G_0, H \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ implies $D \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ (by hypothesis), thus there is an exact sequence $0 \to D \to F \to G \to 0$ with $F \in \mathcal{F}(R)$ and $G \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. Consider the following p.o. diagram:

$$\begin{tikzcd}
F_1 \arrow[r, hook] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, hook] & M \arrow[d, hook] \\
F_1 \arrow[r, hook] & F \arrow[r, two heads] \arrow[d, two heads] & F' \arrow[d, two heads] \\
& G \arrow[r, equal] & G
\end{tikzcd}$$

We will show that $F' \in \mathcal{F}(R)$. Consider the exact sequence $0 \to M \to F' \to G \to 0$. For $A \in \mathcal{A}$ we have the exact sequence

$$0 = \operatorname{Tor}_1^R(A, M) \to \operatorname{Tor}_1^R(A, F') \to \operatorname{Tor}_1^R(A, G) = 0,$$

thus $\operatorname{Tor}_1^R(A, F') = 0$. For another hand, we also have the exact sequence $0 \to F_1 \to F \to F' \to 0$, and from this we obtain the exact sequence β : $0 \to (F')^+ \to F^+ \to (F_1)^+ \to 0$ with $(F_1)^+, F^+ \in \mathcal{I}(R^{\mathrm{op}})$. Now we have the following isomorphism

$$\operatorname{Ext}_R^1((F_1)^+, (F')^+) \cong (\operatorname{Tor}_1^R((F_1)^+, F'))^+ = 0, \text{ since } F_1^+ \in \mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$$

therefore the exact sequence β splits, thus $(F')^+$ is a direct summand of an injective R^{op}-module, and from this $(F')^+$ is an injective R^{op}-module, then by Lambek's Theorem F' is a flat R-module. Finally we have the exact sequence $0 \to M \to F' \to G \to 0$ with $G \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $F' \in \mathcal{F}(R)$, and thus $M \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. $\square$

The following result is a generalization of [5, Theorem 2.8].

Theorem 3.12. *Let $\mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Consider the following statements for $M \in \operatorname{Mod}(R)$.*

(1) $\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \leq n$.

(2) $\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) < \infty$ *and* $\operatorname{Tor}_i^R(A, M) = 0$ *for all* $i > n$ *and all* $A \in \mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$.

(3) $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) < \infty$ *and* $\mathrm{Tor}_i^R(E, M) = 0$ *for all* $i > n$ *and all* $E \in \mathcal{A}^\vee \subseteq \mathrm{Mod}(R^{\mathrm{op}})$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *Furthermore, if* $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$*, then all statements are equivalent.*

Proof. (1) $\Rightarrow$ (2) By induction over n. The case $n = 0$ is satisfied by [13, Lemma 2.5]. Thus, we may assume that $n > 0$. There is therefore an exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(K) = n - 1$. We know that for all $A \in \mathcal{A}$ it is satisfied that $\mathrm{Tor}_i^R(A, G) = 0$ for all $i > 0$ and that $\mathrm{Tor}_i^R(A, K) = 0$ for all $i > n - 1$ (by induction). Then we use the long exact sequence $\mathrm{Tor}_{i+1}^R(A, G) \to \mathrm{Tor}_{i+1}^R(A, M) \to \mathrm{Tor}_i^R(A, K)$ to conclude that $\mathrm{Tor}_{i+1}^R(A, M) = 0$ for all $i > n - 1$.

(2) $\Rightarrow$ (3) It follows by a shifting argument.

(3) $\Rightarrow$ (1) Let us suppose that $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$. Since $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$ is finite, from Lemma 3.10 we can choose, for some $m > n$ an exact sequence

$$0 \to G_m \to \cdots \to G_0 \to M \to 0$$

with $G_0, \dots, G_{m-1} \in \mathcal{F}(R)$ and $G_m \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. Lets take $K_n := \mathrm{Ker}(G_{n-1} \to G_{n-2})$, the goal is to prove that $K_n \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.

We split the sequence $0 \to G_m \to \cdots \to G_n \to K_n \to 0$, in short exact sequences $0 \to H_{i+1} \to G_i \to H_i \to 0$ for $i \in \{n, \dots, m-1\}$, where $H_n = K_n$ and $H_m = G_m$. Thus, let us consider the exact sequence $0 \to H_m \to G_{m-1} \to H_{m-1} \to 0$. We claim that $H_{m-1} \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. From the exact sequence $0 \to H_{m-1} \to \cdots \to G_0 \to M \to 0$, we have that for all $E \in \mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ and all $i > 0$ we have the isomorphism $\mathrm{Tor}_i^R(E, H_{m-1}) \cong \mathrm{Tor}_{(m-1)+i}^R(E, M) = 0$. Therefore, for H_{m-1} there is an exact sequence $0 \to G_m \to G_{m-1} \to H_{m-1} \to 0$ and $\mathrm{Tor}_i^R(E, H_{m-1}) = 0$ for all $i > 0$. Thus by Lemma 3.11 we have that $H_{m-1} \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. Such argument can be repeated to show that $H_{m-2}, \dots, H_n = K_n$ are all in $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. $\square$

4. Finitistic Gorenstein flat and weak Gorenstein global dimensions

Now that we have enough tools, we are ready to study other kinds of dimensions. Given a class $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$, we define the *left weak Gorenstein global dimension* relative to the pair $(\mathcal{F}(R), \mathcal{A})$ as follows:

$$\mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) := \sup\{\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) : M \in \mathrm{Mod}(R)\}.$$

From Theorem 3.12 the following result is easily deduced.

Proposition 4.1. *Let* $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ *such that* $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ *is closed under extensions and* $\mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) < \infty$*. Consider the following assertions.*

(1) $\mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) \le n < \infty$.
(2) $\mathrm{fd}(A) \le n$ *for all* $A \in \mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$.
(3) $\mathrm{fd}(E) \le n$ *for all* $E \in \mathcal{A}^\vee \subseteq \mathrm{Mod}(R^{\mathrm{op}})$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*. Furthermore, if* $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$*, then all assertions are equivalent, i.e., the following equalities are true*:

$$\mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) = \mathrm{fd}(\mathcal{A}) = \mathrm{fd}(\mathcal{A}^{\vee}).$$

To further study of the left weak Gorenstein global dimension relative to the pair $(\mathcal{F}(R),\mathcal{A})$, we need more tool. To do so, let us consider a complementary dimension, which is an adaptation of the *copure flat dimension* defined by E. Enochs and O. M. G. Jenda [10].

Definition 4.2. Consider $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ and $M \in \mathrm{Mod}(R)$. *The $\mathcal{A}$-flat co-dimension of* M, denoted $c\mathrm{fd}_{\mathcal{A}}(M)$ is the largest positive integer n such that $\mathrm{Tor}_n^R(A,M) \neq 0$ for some $A \in \mathcal{A}$, this is

$$c\mathrm{fd}_{\mathcal{A}}(M) := \sup\{n : \mathrm{Tor}_n^R(A,M) \neq 0 \text{ for some } A \in \mathcal{A}\}.$$

Note that if $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions, then by Theorem 3.12 we have that for all $M \in \mathrm{Mod}(R)$ the following inequality holds:

$$c\mathrm{fd}_{\mathcal{A}}(M) \leq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M).$$

At this point we see necessary to consider another kind of duality pair $(\mathcal{L},\mathcal{A})$ with the property that $({}^{\perp_1}\mathcal{A},\mathcal{A})$ will be a cotorsion pair.

Definition 4.3 ([24, Definition 3.2])**.** A duality pair $(\mathcal{L},\mathcal{A})$ in $\mathrm{Mod}(R)$ is called *bi-complete* if $(\mathcal{L},\mathcal{A})$ is a complete duality pair such that the pair $({}^{\perp_1}\mathcal{A},\mathcal{A})$ forms a hereditary cotorsion pair cogenerated by a set.

The theory developed in [24, §3.1] shows that the pairs $(\mathcal{L},\mathcal{L}^{\perp_1})$ and $({}^{\perp_1}\mathcal{A},\mathcal{A})$ are complete and hereditary cotorsion pairs. Thus the coming results with $({}^{\perp_1}\mathcal{A},\mathcal{A})$ as hypothesis can be thought over a bi-complete duality pair. We will give examples of the above mentioned pairs in Example 6.2 below.

Lemma 4.4. *Let* $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ *such that* $({}^{\perp_1}\mathcal{A},\mathcal{A})$ *is a complete cotorsion pair, and suppose that* $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ *is closed under extensions. Then for all* $M \in \mathcal{F}(R)^{\wedge}$*, it holds*

$$c\mathrm{fd}_{\mathcal{A}}(M) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{fd}(M).$$

Proof. By Theorem 3.12 we already know that $c\mathrm{fd}_{\mathcal{A}}(M) \leq \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \leq \mathrm{fd}(M)$, since $\mathcal{F}(R) \subseteq \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$. In particular we have that $c\mathrm{fd}_{\mathcal{A}}(M) \leq \mathrm{fd}(M)$. Suppose now that $n := \mathrm{fd}(M) < \infty$. Then, there is $N \in \mathrm{Mod}(R^{\mathrm{op}})$ such that $\mathrm{Tor}_n^R(N,M) \neq 0$, and for such N there is an exact sequence $\gamma : 0 \to N \to A \to C \to 0$ with $A \in \mathcal{A}$ and $C \in {}^{\perp_1}\mathcal{A}$. Applying $- \otimes_R M$ to the sequence γ we obtain the exact sequence

$$0 = \mathrm{Tor}_{n+1}^R(C,M) \to \mathrm{Tor}_n^R(N,M) \to \mathrm{Tor}_n^R(A,M),$$

where the left-hand side is zero since $\mathrm{fd}(M) = n$, thus $\mathrm{Tor}_n^R(A,M) \neq 0$. Which implies that $c\mathrm{fd}_{\mathcal{A}}(M) \geq n = \mathrm{fd}(M)$. This is $c\mathrm{fd}_{\mathcal{A}}(M) = \mathrm{fd}(M)$. $\square$

We are interested in study the (left and right) weak Gorenstein global dimension relative to a duality pair $(\mathcal{L},\mathcal{A})$. Thus, for a class $\mathcal{L} \subseteq \mathrm{Mod}(R)$ we define the *right weak Gorenstein global dimension* relative to $(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})$ as follows:

$$\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R) := \sup\{\mathrm{Gfd}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(M) : M \in \mathrm{Mod}(R^{\mathrm{op}})\}.$$

From such definition we obtain a dual version of Proposition 4.1 as follows.

Proposition 4.5. *Let $\mathcal{L} \subseteq \mathrm{Mod}(R)$ with $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}$ closed under extensions in $\mathrm{Mod}(R^{\mathrm{op}})$ and $\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R) < \infty$. Consider the following assertions.*

(1) $\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R) \leq n < \infty$.
(2) $\mathrm{fd}(L) \leq n$ *for all* $L \in \mathcal{L} \subseteq \mathrm{Mod}(R)$.
(3) $\mathrm{fd}(E) \leq n$ *for all* $E \in \mathcal{L}^{\vee} \subseteq \mathrm{Mod}(R)$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Furthermore, if $\mathcal{I}(R) \subseteq \mathcal{L}$, then all assertions are equivalent, i.e., the following equalities are true:

$$\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R) = \mathrm{fd}(\mathcal{L}) = \mathrm{fd}(\mathcal{L}^{\vee}).$$

Proposition 4.6. *Consider a pair of classes $(\mathcal{L},\mathcal{A}) \subseteq \mathrm{Mod}(R) \times \mathrm{Mod}(R^{\mathrm{op}})$ that satisfies the following conditions.*

(1) *The pair $({}^{\perp_1}\mathcal{A},\mathcal{A})$ is a complete cotorsion pair.*
(2) *The classes $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}$ and $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ are closed under extensions.*
(3) *The dimension $\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R)$ is finite.*

Then the inequality holds:

$$\mathrm{fd}(\mathcal{L}) \leq \mathrm{l.w.Ggl}_{(\mathcal{F}(R),\mathcal{A})}(R).$$

Proof. Indeed, since $\mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R)$ is finite, by Proposition 4.5 we have that

$$\mathrm{fd}(\mathcal{L}) \leq \mathrm{r.w.Ggl}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{L})}(R) < \infty.$$

From Lemma 4.4 we obtain $\mathrm{Gfd}_{(\mathcal{F}(R),\mathcal{A})}(L) = \mathrm{fd}(L)$ for all $L \in \mathcal{L}$, this is $\mathrm{fd}(\mathcal{L}) \leq \mathrm{l.w.Ggl}_{(\mathcal{F}(R),\mathcal{A})}(R)$. $\square$

In what follows, we recall for a ring R the left finitistic flat dimension $\mathrm{l.FFD}(R)$ and we define with respect to the pair $(\mathcal{F}(R),\mathcal{A})$ the *left finitistic Gorenstein flat dimension*.

$$l.\mathrm{FFD}(R) = \sup\{\mathrm{fd}(M) : M \in \mathrm{Mod}(R) \text{ such that } M \in \mathcal{F}(R)^{\wedge}\},$$

$$l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R) = \sup\{\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) : M \in \mathrm{Mod}(R) \text{ such that } M \in \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}\}.$$

We are interested in compare both finitistic dimensions, for this end we have the following result.

Proposition 4.7. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then the following inequality holds*:

$$l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R) \leq l.\mathrm{FFD}(R).$$

Proof. We can suppose that $n := l.\mathrm{FFD}(R) < \infty$. Consider $M \in \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}$, by Theorem 3.5 we know that there is an exact sequence $0 \to M \to H \to X' \to 0$ with $X' \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $\mathrm{fd}(H) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M)$, but by hypothesis $\mathrm{fd}(H) \leq n$. This is $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \leq n$ for all $M \in \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}$. □

Corollary 4.8. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ such that $({}^{\perp_1}\mathcal{A}, \mathcal{A})$ is a complete cotorsion pair, and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then the following equality holds*:

$$l.\mathrm{FFD}(R) = l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R).$$

Proof. For the inequality $l.\mathrm{FFD}(R) \leq l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R)$ is enough to prove that $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{fd}(M)$ for all $M \in \mathcal{F}(R)^{\wedge}$, which is true by Lemma 4.4.

The other inequality it follows from Proposition 4.7. □

Remark 4.9. Observe that if $\mathrm{l.w.Ggl}_{(\mathcal{F}(R),\mathcal{A})}(R) < \infty$, then

$$\mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) = l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R)$$

and if $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$ and $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions, then by Proposition 4.1 we have that

$$\mathrm{fd}(\mathcal{A}) = \mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) = l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R),$$

and if in addition $({}^{\perp_1}\mathcal{A}, \mathcal{A})$ is a complete cotorsion pair, then by Corollary 4.8 we have the equalities:

$$\mathrm{fd}(\mathcal{A}) = \mathrm{l.w.Ggl}_{(\mathcal{F},\mathcal{A})}(R) = l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R) = l.\mathrm{FFD}(R).$$

Remember that a semi-complete (or perfect) duality pair $(\mathcal{L}, \mathcal{A})$ satisfies that $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$ [19, Remark 2.6] and that the pair $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible. Note also that the containment in the hypothesis of the following result is satisfied when $\mathcal{I}(R^{\mathrm{op}})^{\vee} \subseteq \mathcal{P}(R^{\mathrm{op}})^{\wedge}$.

Proposition 4.10. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ be a class such that the pair $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible in $\mathrm{Mod}(R^{\mathrm{op}})$. If $(\mathcal{F}(R)^{\wedge})^{+} \subseteq \mathcal{P}(R^{\mathrm{op}})^{\wedge}$, then we have the inequality*:

$$l.\mathrm{FFD}(R) \leq l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R).$$

Proof. We can assume that $l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R) := n < \infty$. Lets take $T \in \mathcal{F}(R)^{\wedge}$. Then by hypothesis $T^{+} \in \mathcal{P}(R^{\mathrm{op}})^{\wedge}$. Thus, by Proposition 3.8 we have that $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(T) = \mathrm{fd}(T) < \infty$, but $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(T) \leq n$. This is $\mathrm{fd}(T) \leq n$ for all $T \in \mathcal{F}(R)^{\wedge}$. □

Corollary 4.11. *Let $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$ be a class such that the pair $(\mathcal{A}, \mathcal{I}(R^{\mathrm{op}}))$ is GI-admissible in $\mathrm{Mod}(R^{\mathrm{op}})$ with $(\mathcal{F}(R)^{\wedge})^{+} \subseteq \mathcal{P}(R^{\mathrm{op}})^{\wedge}$ and suppose that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then the following equality holds*:

$$l.\mathrm{FFD}(R) = l.\mathrm{FGFD}_{(\mathcal{F},\mathcal{A})}(R).$$

Proof. It follows from Propositions 4.7 and 4.10. □

Note that similar observation to the previous Remark 4.9 can be made from Corollary 4.11 but with slightly different conditions.

5. Balance situations

Sometimes it is possible to consider a duality pair $(\mathcal{L}, \mathcal{A})$ in $\mathrm{Mod}(R)$ such that $({}_{R^{\mathrm{op}}}\mathcal{L}, \mathcal{A}_{R^{\mathrm{op}}})$ is also a duality pair, but in $\mathrm{Mod}(R^{\mathrm{op}})$. This is the case in [24, §4], when it is established that $(\mathcal{L}, \mathcal{A})$ will be a bi-complete duality pair over $\mathrm{Mod}(R)$ if $(\mathcal{L}, \mathcal{A})$ is a bi-complete duality pair in $\mathrm{Mod}(R)$ and also $({}_{R^{\mathrm{op}}}\mathcal{L}, \mathcal{A}_{R^{\mathrm{op}}})$ is a bi-complete duality pair in $\mathrm{Mod}(R^{\mathrm{op}})$ (see [24, Example 4.3] for examples). In what follows this will be the meaning of a bi-complete duality pair, thus we will use freely the results developed in [24, §4]. As discussed above, for a duality pair $(\mathcal{L}, \mathcal{A})$ in $\mathrm{Mod}(R)$, sometimes it is possible to consider the class $\mathcal{L}$ as one in $\mathrm{Mod}(R^{\mathrm{op}})$, and $\mathcal{A}$ as one in $\mathrm{Mod}(R)$. We use this property within the hypotheses of the following results mentioning it explicitly in each case.

Let us consider the classes of R-modules $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and R^{op}-modules $\mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$ relative to a pair of duality $(\mathcal{L}, \mathcal{A})$. We know that if $(\mathcal{L}, \mathcal{A})$ is a perfect or semi-complete duality pair, then $(\mathcal{P}(R), \mathcal{L})$ is GP-admissible $\mathrm{Mod}(R)$, thus by [4, Theorem 4.2(a)] there exists a left proper $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$-resolution for each $M \in \mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}$. Let's see now that if $\mathcal{A}$ can be considered in $\mathrm{Mod}(R)$ and $\mathcal{L}$ in $\mathrm{Mod}(R^{\mathrm{op}})$, then $- \otimes_R -$ is balanced over $\mathcal{GP}^{\wedge}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{L})} \times \mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}$ by $\mathcal{GP}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{L})} \times \mathcal{GP}_{(\mathcal{P},\mathcal{L})}$. We will need the following result.

Lemma 5.1. *Lets take $\mathcal{L}, \mathcal{A} \subseteq \mathrm{Mod}(R)$ a pair of classes such that $(\mathcal{P}(R), \mathcal{L})$ is a GP-admissible pair. If $M \in \mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}$, then any left proper $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$-resolution of M is $\mathrm{Hom}_R(-, \mathcal{GI}_{(\mathcal{A},\mathcal{I})})$-acyclic.*

Proof. Let $\mathbf{X}(M) \to M$ be a left proper $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$-resolution of M. Since it breaks down in short exact sequences, it is enough to show that $\mathrm{Hom}_R(\eta, Y)$ is exact for all $Y \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R))}$ and for all short exact sequence $\eta : 0 \to K \xrightarrow{i} X \to M \to 0$, where $X \to M$ is a special $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$-precover for $M \in \mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}$ with $K \in \mathcal{P}(R)^{\wedge}$. Therefore, we will prove that $\mathrm{Hom}_R(i, Y) : \mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_R(K, Y)$ is an epimorphism for all $Y \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$. By definition of the class $\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ we know that for $Y \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ there is an exact sequence $\gamma : 0 \to Y' \to V \xrightarrow{\rho} Y \to 0$ with $Y' \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ and $V \in \mathcal{I}(R)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(X, V) & \xrightarrow{(X,\rho)} & \mathrm{Hom}_R(X, Y) \\ \Big\downarrow{\scriptstyle (i,V)} & & \Big\downarrow{\scriptstyle (i,Y)} \\ \mathrm{Hom}_R(K, V) & \xrightarrow{(K,\rho)} & \mathrm{Hom}_R(K, Y) \end{array}$$

We know that $\mathrm{Ext}^1_R(\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}, \mathcal{I}(R)) = 0$, thus from the sequence η we conclude that (i, V) is an epimorphism. It is enough to prove that $\mathrm{Ext}^1_R(\mathcal{P}(R)^{\wedge}, \mathcal{GI}_{(\mathcal{A},\mathcal{I})})$

$=0$, so from the sequence γ we conclude that (K,ρ) is an epimorphism. Therefore (i,Y) is an epimorphism. Indeed take $N\in\mathcal{P}(R)^{\wedge}$, then for Y there are exact sequences $0\to Y_{n+1}\to V_n\to Y_n\to 0$ with $V_n\in\mathcal{I}(R)$ and $Y_n\in\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ for each $n\geq 0$ (with $Y_0:=Y$). By applying $\operatorname{Hom}_R(N,-)$ we have the exact sequences

$$\operatorname{Ext}^i_R(N,V_n)\to\operatorname{Ext}^i_R(N,Y_n)\to\operatorname{Ext}^{i+1}_R(N,Y_{n+1})\to\operatorname{Ext}^{i+1}_R(N,V_n)$$

for all $i>0$, with zero ends since $V_n\in\mathcal{I}(R)$, thus

$$\operatorname{Ext}^i_R(N,Y)\simeq\operatorname{Ext}^{i+n}_R(N,Y_n)\text{ for all }i>0\text{ and for all }n\geq 0,$$

the term on the right-hand side is zero for $i+n>\operatorname{pd}(N)$, therefore $\operatorname{Ext}^i_R(N,Y)$ $=0$ for all $i>0$. ☐

Theorem 5.2. *Let R be a ring, $(\mathcal{L},\mathcal{A})$ a perfect or semi-complete duality pair in* $\operatorname{Mod}(R)$ *such that $\mathcal{A}$ can be considered in* $\operatorname{Mod}(R)$. *Then the functor* $\operatorname{Hom}_R(-,-)$ *is right balanced over $\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}\times\mathcal{GI}^{\vee}_{(\mathcal{A},\mathcal{I})}$ by $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}\times\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$.*

Proof. We know that if $(\mathcal{L},\mathcal{A})$ is perfect or semi-complete duality pair, then the pairs $(\mathcal{P}(R),\mathcal{L})$, $(\mathcal{A},\mathcal{I}(R))$ are GP-admissible and GI-admissible, respectively. The result follows from Lemma 5.1 and its dual. ☐

Theorem 5.3. *Lets take $\mathcal{L},\mathcal{A}\subseteq\operatorname{Mod}(R)$ a pair of classes such that $(\mathcal{P}(R),\mathcal{L})$ is a GP-admissible pair and $\mathbf{D}:\cdots\to G_1\to G_0\to M\to 0$ a left proper $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$-resolution for the R-module $M\in\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{L})}$. Then the complex $G\otimes_R\mathbf{D}$ is acyclic for all R^{op}-module $G\in\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$.*

Proof. We know that the complex $G\otimes_R\mathbf{D}:\cdots\to G\otimes_R G_1\to G\otimes_R G_0\to G\otimes_R M\to 0$ is acyclic if and only if $(G\otimes_R\mathbf{D})^+:0\to(G\otimes_R M)^+\to(G\otimes_R G_0)^+\to(G\otimes_R G_1)^+\to\cdots$ is acyclic. But the latter is naturally isomorphic to $\operatorname{Hom}_R(\mathbf{D},G^+):0\to\operatorname{Hom}_R(M,G^+)\to\operatorname{Hom}_R(G_0,G^+)\to\operatorname{Hom}_R(G_1,G^+)\to\cdots$ from Proposition 3.7 we have that the left R-module $G^+\in\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$, thus the last sequence is acyclic by Lemma 5.1. ☐

Corollary 5.4. *Let R be a ring, $(\mathcal{L},\mathcal{A})$ a perfect or semi-complete duality pair in* $\operatorname{Mod}(R)$ *such that $\mathcal{A}^+\subseteq\mathcal{L}$ and that $\mathcal{A}$ can be considered in* $\operatorname{Mod}(R)$ *and $\mathcal{L}$ in* $\operatorname{Mod}(R^{\mathrm{op}})$. *Then $-\otimes_R-$ is left balanced over $\mathcal{GP}^{\wedge}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{L})}\times\mathcal{GP}^{\wedge}_{(\mathcal{P}(R),\mathcal{L})}$ by $\mathcal{GP}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{L})}\times\mathcal{GP}_{(\mathcal{P}(R),\mathcal{L})}$.*

Proof. It follows from Theorem 5.3 and Proposition 3.9. ☐

We are also interested in considering the balance conditions of the functor $-\otimes_R-$ over $\operatorname{Mod}(R^{\mathrm{op}})\times\operatorname{Mod}(R)$ by $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}\times\mathcal{GF}_{(\mathcal{F}(R),\mathcal{A})}$. To do so, we will use some of the techniques employed in [26], where it is shown that for a Ding-Chen ring R the functor $-\otimes_R-$ is left balanced over $\operatorname{Mod}(R^{\mathrm{op}})\times\operatorname{Mod}(R)$ by $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{I}(R))}\times\mathcal{GF}_{(\mathcal{F}(R),\mathcal{I}(R^{\mathrm{op}}))}$. Since the notion of $(\mathcal{L},\mathcal{A})$-Gorenstein ring [24] is a generalization of AC-Gorenstein and Ding-Chen ring it is natural to ask whether on such rings the above-mentioned balance is given. We recall the notion of $(\mathcal{L},\mathcal{A})$-Gorenstein ring.

Definition 5.5 ([24, Definition 4.2])**.** Let R a ring such that $(\mathcal{L},\mathcal{A})$ is a bi-complete duality pair, and m a nonnegative integer.

(1) R is $(\mathcal{L},\mathcal{A})$-*Gorenstein ring* if $\mathrm{resdim}_{\mathcal{L}}(\mathcal{A}) \leq m$. Equivalently, if $\mathrm{coresdim}_{\mathcal{A}}(\mathcal{L}) \leq m$.

(2) R is called *flat typed* $(\mathcal{L},\mathcal{A})$-Gorenstein ring if $\mathrm{resdim}_{\mathcal{L}}(\mathcal{A}), \mathrm{fd}(\mathcal{A}) \leq m$.

Remark 5.6. Note that if R is a flat typed $(\mathcal{L},\mathcal{A})$-Gorenstein ring with respect to a bi-complete duality pair $(\mathcal{L},\mathcal{A})$ and $\mathrm{l.w.Ggl}_{(\mathcal{F}(R),\mathcal{A})}(R) < \infty$, then from [24, Proposition 4.5] and Proposition 4.1, we have the equalities:

$$\mathrm{l.w.Ggl}_{(\mathcal{F}(R),\mathcal{A})}(R) = \mathrm{fd}(\mathcal{A}) = \mathrm{resdim}_{\mathcal{L}}(\mathcal{A}) = \mathrm{coresdim}_{\mathcal{A}}(\mathcal{L}).$$

Now by [24, Theorem 4.8] we know that in a $(\mathcal{L},\mathcal{A})$-Gorenstein ring R with respect to a bi-complete duality pair $(\mathcal{L},\mathcal{A})$ the triple

$$(\mathcal{GP}_{(\mathcal{P},\mathcal{L})}, \mathcal{W}, \mathcal{GI}_{(\mathcal{A},\mathcal{I})})$$

forms a hereditary and complete cotorsion triple in $\mathrm{Mod}(R)$, this in particular tells us that $\mathcal{GP}^{\perp}_{(\mathcal{P},\mathcal{L})} = {}^{\perp}\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$. Furthermore, since $(\mathcal{L},\mathcal{A})$ is a symmetric duality pair, then Proposition 3.9 implies that $\mathcal{GP}_{(\mathcal{P},\mathcal{L})} \subseteq \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ (see also [24, Lemma 4.6(1)]) which gives the containment

$$\mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GP}^{\perp}_{(\mathcal{P},\mathcal{L})} = {}^{\perp}\mathcal{GI}_{(\mathcal{A},\mathcal{I})}.$$

On the other hand, J. Gillespie and A. Iacob [19, Proposition 4.2] have recently proven that if $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed by extensions and $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$, then

$$(\mathcal{GF}_{(\mathcal{F},\mathcal{A})}, \mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})})$$

form an hereditary and perfect cotorsion pair in $\mathrm{Mod}(R)$. From the latter it can be concluded that for all $M \in \mathrm{Mod}(R)$ there is an exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $K \in \mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})}$, this is, M possesses a $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$-precover. From all this we can establish the following result.

Lemma 5.7. *Let R be an $(\mathcal{L},\mathcal{A})$-Gorenstein ring with respect to a bi-complete duality pair $(\mathcal{L},\mathcal{A})$ and for an R-module M take a left proper $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$-resolution $\mathbf{D} : \cdots \to G_1 \to G_0 \to M \to 0$. Then $G \otimes_R \mathbf{D}$ is acyclic for all R^{op}-module $G \in \mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$.*

Proof. From [19, Corollary 5.3] the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Now, since $(\mathcal{L},\mathcal{A})$ is a perfect duality pair, then $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$ [18, Theorem 2.3] and $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions, from what has been said above, is enough to consider the short exact sequence $0 \to K \to G_0 \to M \to 0$ with $K \in \mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})}$ and $G_0 \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, and prove that for all $G \in \mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$ the sequence $0 \to G \otimes_R K \to G \otimes_R G_0 \to G \otimes_R M \to 0$ is exact. Note that such a sequence is exact if and only if the sequence

$$0 \to (G \otimes_R M)^+ \to (G \otimes_R G_0)^+ \to (G \otimes_R K)^+ \to 0$$

is exact. But the latter is naturally isomorphic to the sequence

$$0 \to \mathrm{Hom}_R(M, G^+) \to \mathrm{Hom}_R(G_0, G^+) \to \mathrm{Hom}_R(K, G^+) \to 0.$$

Since $G \in \mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$ is a right R-module, then by Proposition 3.7 we have that $G^+ \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ is a left R-module. Thus there is an exact sequence $0 \to \tilde{G} \to I \to G^+ \to 0$ with $I \in \mathcal{I}(R)$ and $\tilde{G} \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$. Consider the following exact and commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(G_0,\tilde{G}) \arrow[d, hook] & \mathrm{Hom}_R(K,\tilde{G}) \arrow[d, hook] & \\
\mathrm{Hom}_R(M,I) \arrow[r, hook] & \mathrm{Hom}_R(G_0,I) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K,I) \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(M,I)=0 \\
\mathrm{Hom}_R(M,G^+) \arrow[r, hook] & \mathrm{Hom}_R(G_0,G^+) \arrow[r] & \mathrm{Hom}_R(K,G^+) \arrow[d] & \\
 & & \mathrm{Ext}^1_R(K,\tilde{G}) &
\end{tikzcd}$$

Thus, to prove that $\mathrm{Hom}_R(G_0,G^+) \to \mathrm{Hom}_R(K,G^+)$ is an epimorphism, it is enough to see that $\mathrm{Ext}^1_R(K,\tilde{G}) = 0$. To this end, let us remember that $K \in \mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})}$, $\tilde{G} \in \mathcal{GI}_{(\mathcal{A},\mathcal{I})}$ and note that $\mathcal{GF}^{\perp}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GP}^{\perp}_{(\mathcal{P},\mathcal{L})} = {}^{\perp}\mathcal{GI}_{(\mathcal{A},\mathcal{I})}$. □

We can declare the following results.

Proposition 5.8. *Let R be an $(\mathcal{L},\mathcal{A})$-Gorenstein ring with respect to a bicomplete duality pair $(\mathcal{L},\mathcal{A})$. Then the functor $-\otimes_R -$ is left balanced over* $\mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R)$ *by* $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})} \times \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.

Proof. It follows from Lemma 5.7 applied for $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$. □

Proposition 5.9. *Let R be a coherent ring (on both sides) with $l.\mathrm{FPD}(R) < \infty$, $r.\mathrm{FPD}(R) < \infty$ and $(\mathcal{L},\mathcal{A})$ a semi-complete duality pair in* $\mathrm{Mod}(R)$ *and such that $({}_{R^{\mathrm{op}}}\mathcal{L}, \mathcal{A}_{R^{\mathrm{op}}})$ also semi-complete in* $\mathrm{Mod}(R^{\mathrm{op}})$. *Then the functor $-\otimes_R-$ is left balanced over $\mathcal{GF}^{\wedge}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})} \times \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}$ by $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})} \times \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$.*

Proof. From [19, Corollary 5.3] the classes $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ and $\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}),\mathcal{A})}$ are closed under extensions. Now from Theorem 3.5(b) there exist a left proper $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$-resolution for each $M \in \mathcal{GF}^{\wedge}_{(\mathcal{F},\mathcal{A})}$. Observe that in the proof of Lemma 5.1, it has been showed that $\mathrm{Ext}^1_R(\mathcal{P}(R)^{\wedge}, \mathcal{GI}_{(\mathcal{A},\mathcal{I})}) = 0$, thus the proof in Lemma 5.7 can be adapted by observing that $\mathrm{Ext}^1_R(K,\tilde{G}) = 0$ since by [22, Proposition 6] we have that $K \in \mathcal{F}(R)^{\wedge} \subseteq \mathcal{P}(R)^{\wedge}$. □

6. Applications

We will present here some consequences of the theory developed above. Although several of the preceding results can be rewritten in terms of the following duality pairs, we have chosen only a few of those we believe to be most significant. We begin remember some notions, and next give examples of duality pairs, some of which have been considered in [20, 24].

Let R be an arbitrary ring and $n \in \mathbb{N}$.

(1) Recall that an R-module M is called *FP-injective* [15] if $\mathrm{Ext}_R^1(N, M) = 0$ for all finitely presented R-modules N. Denote by $\mathcal{FP}Inj(R)$ the class of all FP-injective R-modules, and by $\mathcal{F}(R)$ the class of all flat R-modules. The pair $(\mathcal{P}(R), \mathcal{F}(R))$ is GP-admissible, the pair $(\mathcal{FP}Inj(R), \mathcal{I}(R))$ is GI-admissible, and the classes $\mathcal{GP}_{(\mathcal{P}(R),\mathcal{F}(R))}$ and $\mathcal{GI}_{(\mathcal{FP}Inj(R),\mathcal{I}(R))}$ are called *Ding projective modules* and *Ding injective modules*, respectively (also denoted by $\mathcal{DP}(R)$ and $\mathcal{DI}(R)$, respectively).

(2) An R-module F is of *type* FP_∞ [6] if it has a projective resolution $\cdots \to P_1 \to P_0 \to F \to 0$, where each P_i is finitely generated. An R-module A is *absolutely clean* if $\mathrm{Ext}_R^1(F, A) = 0$ for all R-modules F of type FP_∞. Also an R-module L is *level* if $\mathrm{Tor}_1^R(F, L) = 0$ for all right R-module F of type FP_∞. We denote by $\mathrm{AC}(R)$ the class of all absolutely clean R-modules, and by $\mathrm{Lev}(R)$ the class of all level R-modules. The pair $(\mathcal{P}(R), \mathrm{Lev}(R))$ is GP-admissible and the pair $(\mathrm{AC}(R), \mathcal{I}(R))$ is GI-admissible. The classes of R-modules $\mathcal{GP}_{(\mathcal{P}(R),\mathrm{Lev}(R))}$ and $\mathcal{GI}_{(\mathrm{AC}(R),\mathcal{I}(R))}$ are called *Gorenstein AC-projective* and *Gorenstein AC-injective*, respectively.

(3) $M \in \mathrm{Mod}(R)$ is called *finitely n-presented* if M possesses a projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ for which each P_i is finitely generated free. We denote by $\mathcal{FP}_n(R)$ the class of all finite n-presented R-modules. A ring R is said *left n-coherent* if $\mathcal{FP}_n(R) \subseteq \mathcal{FP}_{n+1}(R)$.

(4) An R-module E is called $\mathcal{FP}_n$*-injective* if for all $M \in \mathcal{FP}_n(R)$, $\mathrm{Ext}_R^1(M, E) = 0$. Also an R^{op}-module H is called $\mathcal{FP}_n$*-flat* if for all $M \in \mathcal{FP}_n(R)$, $\mathrm{Tor}_1^R(H, M) = 0$.

(5) Given an R-module M, it is said that M is (resp. strongly) w*-FI-flat* if $\mathrm{Tor}_1^R(E, M) = 0$ (resp. $\mathrm{Tor}_i^R(E, M) = 0$ for all $i > 0$) for any absolutely w-pure R^{op}-module E (for details see [1]). Also is said that M is (resp. strongly) w*-FI-injective* if $\mathrm{Ext}_R^1(E, M) = 0$ (resp. $\mathrm{Ext}_R^i(E, M) = 0$ for all $i > 0$) for any w-absolutely pure R-module E. In a similar way it is also possible to define (strongly) w*-FI-projective* R-modules.

We recall examples of perfect and complete duality pairs.

Example 6.1. Let R be a ring.

(1) Take $(n \geq 2)$, then by [7, Theorems 5.5 and 5.6] the pair

$$(\mathcal{FP}_n\text{-}Flat(R), \mathcal{FP}_n\text{-}Inj(R^{\mathrm{op}}))$$

of $\mathcal{FP}_n$-flat R-modules and $\mathcal{FP}_n$-injective R^{op}-modules form a duality pair in $\mathrm{Mod}(R)$. And from [6, Corollarry 3.7] this is a complete duality pair for any ring R.

(2) Let R be commutative and Noetherian with a semidualizing R-complex C. There is associated two classes called the Auslander class, denoted $\mathcal{A}_0^C$, and the Bass class, denoted $\mathcal{B}_0^C$. Then by [20, Proposition 2.4] the pair $(\mathcal{A}_0^C, \mathcal{B}_0^C)$ is a complete duality pair.

(3) By [1, Remark 1, Proposition 6] we have that the pair of (strongly) w-FI-flat R-modules and (strongly) w-FI-injective R^{op}-modules

$$(w\text{-}FI\text{-}Flat(R), w\text{-}FI\text{-}Inj(R^{\mathrm{op}}))$$

is a perfect duality pair (resp. the pair $(\mathcal{S}w\text{-}FI\text{-}Flat(R), \mathcal{S}w\text{-}FI\text{-}Inj(R^{\mathrm{op}}))$ is a perfect duality pair).

From Theorem 5.2 and the previous list of complete and perfect duality pairs, we have the following classes of R-modules that makes the $\mathrm{Hom}_R(-,-)$ right balanced.

- For any ring R, the functor $\mathrm{Hom}_R(-,-)$ is right balanced over
$$\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{FP}_n\text{-}Flat(R))} \times \mathcal{GI}^{\vee}_{(\mathcal{FP}_n\text{-}Inj(R),\mathcal{I})}$$
by $\mathcal{GP}_{(\mathcal{P},\mathcal{FP}_n\text{-}Flat(R))} \times \mathcal{GI}_{(\mathcal{FP}_n\text{-}Inj(R),\mathcal{I})}$.
- For a commutative and Noetherian ring R with a semidualizing R-complex C, the functor $\mathrm{Hom}_R(-,-)$ is right balanced over $\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{A}_0^C)} \times \mathcal{GI}^{\vee}_{(\mathcal{B}_0^C,\mathcal{I})}$ by $\mathcal{GP}_{(\mathcal{P},\mathcal{A}_0^C)} \times \mathcal{GI}_{(\mathcal{B}_0^C,\mathcal{I})}$.
- For any ring R, the functor $\mathrm{Hom}_R(-,-)$ is right balanced over
$$\mathcal{GP}^{\wedge}_{(\mathcal{P},w\text{-}FI\text{-}Flat(R))} \times \mathcal{GI}^{\vee}_{(w\text{-}FI\text{-}Inj(R),\mathcal{I})}$$
by $\mathcal{GP}_{(\mathcal{P},w\text{-}FI\text{-}Flat(R))} \times \mathcal{GI}_{(w\text{-}FI\text{-}Inj(R),\mathcal{I})}$. Furthermore is right balanced over
$$\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{S}w\text{-}FI\text{-}Flat(R))} \times \mathcal{GI}^{\vee}_{(\mathcal{S}w\text{-}FI\text{-}Inj(R),\mathcal{I})}$$
by $\mathcal{GP}_{(\mathcal{P},\mathcal{S}w\text{-}FI\text{-}Flat(R))} \times \mathcal{GI}_{(\mathcal{S}w\text{-}FI\text{-}Inj(R),\mathcal{I})}$.

Now from Corollary 5.4 and Example 6.1 we have the following classes of R-modules that makes the $-\otimes_R -$ left balanced.

- For any ring R, the functor $-\otimes_R -$ left balanced over
$$\mathcal{GP}^{\wedge}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{FP}_n\text{-}Flat(R^{\mathrm{op}}))} \times \mathcal{GP}^{\wedge}_{(\mathcal{P}(R),\mathcal{FP}_n\text{-}Flat(R))}$$
by $\mathcal{GP}_{(\mathcal{P}(R^{\mathrm{op}}),\mathcal{FP}_n\text{-}Flat(R^{\mathrm{op}}))} \times \mathcal{GP}_{(\mathcal{P}(R),\mathcal{FP}_n\text{-}Flat(R))}$.
- For a commutative and Noetherian ring R with a semidualizing R-complex C the functor $-\otimes_R -$ left balanced over
$$\mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{A}_0^C)} \times \mathcal{GP}^{\wedge}_{(\mathcal{P},\mathcal{A}_0^C)}$$
by $\mathcal{GP}_{(\mathcal{P},\mathcal{A}_0^C)} \times \mathcal{GP}_{(\mathcal{P},\mathcal{A}_0^C)}$.

The following are examples of bi-complete duality pairs.

Example 6.2. Let R be a ring.

(1) The pair $(\mathrm{Lev}(R), \mathrm{AC}(R^{\mathrm{op}}))$ forms a duality pair in $\mathrm{Mod}(R)$, since from [6, Propositions 2.7 and 2.10] $M \in \mathrm{Lev}(R)$ if and only if $M^+ \in \mathrm{AC}(R^{\mathrm{op}})$, and both classes are closed under direct summands. Furthermore $\mathrm{Lev}(R)$ is projective resolving and closed under arbitrary coproducts and from [24, Example 3.6] is a bi-complete duality pair.

(2) From [14, Theorem 2.1] the pair $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$ forms a duality pair in $\mathrm{Mod}(R)$. If R is a right coherent ring, then from [14, Theorem 2.2] $(\mathcal{FP}Inj(R^{\mathrm{op}}), \mathcal{F}(R))$ is a duality pair in $\mathrm{Mod}(R^{\mathrm{op}})$. Furthermore from [24, Example 3.8(1)] $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$ is a bi-complete duality pair when R is right coherent.

(3) Now suppose that R is right n-coherent ($n \geq 2$). Then from [24, Example 3.8(2)] the pair $(\mathcal{FP}_n\text{-}Flat(R), \mathcal{FP}_n\text{-}Inj(R^{\mathrm{op}}))$ of $\mathcal{FP}_n$-flat R-modules and $\mathcal{FP}_n$-injective R^{op}-modules is a bi-complete duality pair when R is right n-coherent.

From Proposition 5.8 and Example 6.2 we have the following balance situations for $- \otimes_R -$, in the whole category $\mathrm{Mod}(R)$.

- For an *AC-Gorenstein ring* R [17, Definition 4.1] the functor $- \otimes_R -$ is left balanced over $\mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R)$ by
$$\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}), \mathrm{AC}(R))} \times \mathcal{GF}_{(\mathcal{F}(R), \mathrm{AC}(R^{\mathrm{op}}))}$$
since from [24, Example 4.3 (1)] the *AC-Gorenstein rings* coincide with the called $(\mathrm{Lev(R)}, \mathrm{AC(R^{op})})$-Gorenstein rings.
- For a right coherent and $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$-Gorenstein ring R the functor $- \otimes_R -$ is left balanced over $\mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R)$ by
$$\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}), \mathcal{FP}Inj(R))} \times \mathcal{GF}_{(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))},$$
note that by [24, Example 4.3 (3)] the $(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))$-Gorenstein rings are exactly the Ding-Chen rings. Thus this example recovers the result given in [26, Theorem 3.23], since by [15, Proposition 3.13]
$$\mathcal{GF}_{(\mathcal{F}(R), \mathcal{FP}Inj(R^{\mathrm{op}}))} = \mathcal{GF}_{(\mathcal{F}(R), \mathcal{I}(R^{\mathrm{op}}))}.$$
- For a Gorenstein n-coherent ring R ($n \geq 2$) the functor $- \otimes_R -$ is left balanced over $\mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R)$ by
$$\mathcal{GF}_{(\mathcal{F}(R^{\mathrm{op}}), \mathcal{FP}_n\text{-}Inj(R))} \times \mathcal{GF}_{(\mathcal{F}(R), \mathcal{FP}_n\text{-}Inj(R^{\mathrm{op}}))},$$
this is true since by [24, Example 4.3(4)] the Gorenstein n-coherent rings are exactly the flat-typed $(\mathcal{FP}_n\text{-}Flat(R), \mathcal{FP}_n\text{-}Inj(R^{\mathrm{op}}))$-Gorenstein rings.

We conclude this work by considering the situation when the Ding-projective R-modules $\mathcal{DP}(R)$ coincide with the Gorenstein projective R-modules $\mathcal{GP}(R)$. By definition and without conditions on the ring R it is known that $\mathcal{DP}(R) \subseteq \mathcal{GP}(R)$. Recently A. Iacob [21] has considered this question as part of the tools developed around the study of class $\mathcal{GF}_{(\mathcal{P},\mathcal{B})}(R)$ where some occasions $\mathcal{B}$ corresponds to a *definable class*. It is known that over a Ding-Chen ring R the class of Ding-projective R-modules coincides with the class of Gorenstein projective R-modules [16, Theorem 1.1], here we will see that the above also occurs when the finitistic projective dimension of R is finite.

Lemma 6.3. *Let R be a ring with $l.\mathrm{FPD}(R) < \infty$ and $M \in \mathrm{Mod}(R)$. If for all $m > 0$, $\mathrm{Ext}^m_R(M, \mathcal{P}(R)) = 0$, then $\mathrm{Ext}^m_R(M, \mathcal{F}(R)^{\wedge}) = 0$ for all $m > 0$.*

Proof. Consider $F \in \mathcal{F}(R)$, then by [22, Proposition 6] we have that $\mathrm{pd}(F) < \infty$, thus $\mathcal{F}(R) \subseteq \mathcal{P}(R)^\wedge$. From this, it follows that $\mathcal{F}(R)^\wedge \subseteq \mathcal{P}(R)^\wedge$. Now for $T \in \mathcal{F}(R)^\wedge$ there is a projective resolution

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to T \to 0,$$

since $\mathrm{Ext}_R^m(M, Q_i) = 0$ for all $m > 0$ and all $Q_i \in \mathcal{P}(R)$, by a dimension shifting we have $\mathrm{Ext}_R^j(M,T) \cong \mathrm{Ext}_R^{j+n}(M, Q_n) = 0$, where the last term is zero since $Q_n \in \mathcal{P}(R)$. Thus, $\mathrm{Ext}_R^j(M,T) = 0$ for all $j > 0$. ☐

Corollary 6.4. *Let R be a ring such that $l.\mathrm{FPD}(R) < \infty$. Then $\mathcal{DP}(R) = \mathcal{GP}(R)$.*

Proof. Indeed, by definition and without conditions on the ring $\mathcal{DP}(R) \subseteq \mathcal{GP}(R)$. Now let's consider an exact complex of projectives $\mathbf{P}$, such that the complex $\mathrm{Hom}_R(\mathbf{P}, \mathcal{P}(R))$ is an acyclic complex. By using [4, Lemma 3.10] we have that all the cycles of the complex $Z_{\mathbf{P}}^i \in {}^\perp\mathcal{P}(R)$, accordingly by Lemma 6.3 we have that $Z_{\mathbf{P}}^i \in {}^\perp\mathcal{F}(R)$, and again using [4, Lemma 3.10], we obtain that the complex $\mathrm{Hom}_R(\mathbf{P}, \mathcal{F}(R))$ is acyclic, thus we obtain the containment $\mathcal{GP}(R) \subseteq \mathcal{DP}(R)$. ☐

The following result shows how to know when equality $\mathcal{GP}(R) = \mathcal{DP}(R)$ occurs, namely, when $\mathrm{id}(\mathcal{F}(R)) < \infty$.

Proposition 6.5. *Let R be a ring such that $\mathcal{F}(R) \subseteq \mathcal{I}(R)^\vee$. Then $\mathcal{GP}(R) = \mathcal{DP}(R)$.*

Proof. Consider an exact complex $\mathbf{P}$ of projective R-modules such that the complex $\mathrm{Hom}_R(\mathbf{P}, \mathcal{P}(R))$ is acyclic. From the proof of [3, Lemma 3.6] we see that for each $F \in \mathcal{F}(R) \subseteq \mathcal{I}(R)^\vee$ we have that $\mathrm{Hom}_R(\mathbf{P}, F)$ is an acyclic complex. Thus we obtain the containment $\mathcal{GP}(R) \subseteq \mathcal{DP}(R)$. ☐

Remark 6.6. We can see from [3, Proposition 3.10] and the previous result, that for a ring R if the global Gorenstein projective dimension $\mathrm{gl.GP}(R)$ is finite, and $\mathcal{F}(R) \subseteq \mathcal{I}(R)^\vee$, then $\mathrm{id}(\mathcal{F}(R)) < \infty$.

Now we sill see that Corollary 6.4 is more general than how it was stated. As we see below, it can be stated in terms of GP-admissible pairs in the following result.

Proposition 6.7. *Consider a GP-admissible pair $(\mathcal{X}, \mathcal{Y})$ in an abelian category $\mathcal{C}$ with $\omega := \mathcal{X} \cap \mathcal{Y}$ such that $\mathcal{Y}^\wedge$, ω and $\mathcal{X} \cap \mathcal{Y}^\wedge$ are closed under direct summands in $\mathcal{C}$. If $(\mathcal{X}, \overline{\mathcal{Y}}) \subseteq \mathcal{C}^2$ is a pair with $\mathcal{Y} \subseteq \overline{\mathcal{Y}} \subseteq \mathcal{Y}^\wedge$, then equality $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} = \mathcal{GP}_{(\mathcal{X},\overline{\mathcal{Y}})}$ is given.*

Proof. From the containment $\mathcal{Y} \subseteq \overline{\mathcal{Y}} \subseteq \mathcal{Y}^\wedge$ we have that

$$\mathcal{GP}_{(\mathcal{X},\mathcal{Y}^\wedge)} \subseteq \mathcal{GP}_{(\mathcal{X},\overline{\mathcal{Y}})} \subseteq \mathcal{GP}_{(\mathcal{X},\mathcal{Y})},$$

while by [4, Theorem 3.34(d)] we have that $\mathcal{GP}_{(\mathcal{X},\mathcal{Y}^\wedge)} = \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$, from which we obtain the desired equality. ☐

It is also naturally interesting to know when the class of the Gorenstein AC-projectives $\mathcal{GP}_{AC}(R)$ coincides with the class $\mathcal{GP}(R)$, we establish this as follows.

Corollary 6.8. *Let us consider the class* $\mathcal{GP}_{\mathrm{AC}}(R)$ *of all the Gorenstein AC-projective R-modules. If any of the conditions* $\mathrm{Lev}(R) \subseteq \mathcal{P}(R)^{\wedge}$ *or* $\mathrm{Lev}(R) \subseteq \mathcal{I}(R)^{\vee}$*, then equality* $\mathcal{GP}(R) = \mathcal{GP}_{\mathrm{AC}}(R)$ *is given.*

Proof. It follows from the Proposition 6.7 or with a similar proof to the one given in Proposition 6.5, respectively. □

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