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MAXIMAL DOMAINS OF SOLUTIONS FOR ANALYTIC QUASILINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

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Abstract. We study the real-analytic continuation of local real-analytic solutions to the Cauchy problems of quasi-linear partial differential equations of first order for a scalar function. By making use of the first integrals of the characteristic vector field and the implicit function theorem we determine the maximal domain of the analytic extension of a local solution as a single-valued function. We present some examples including the scalar conservation laws that admit global first integrals so that our method is applicable.

1. Introduction

An important feature of analytic functions, either in complex or real variables, is that a germ of a function determines the function globally. In complex analysis, special functions like the Riemann zeta function or the gamma function are defined by defining equations in part of the complex plane and then extended by analytic continuation. For those two special functions the maximal domain is the whole complex plane except for the poles. However, for functions like $\log z$ or $\sqrt{z}$ analytic continuation leads to a multi-valued function so that the domain of maximal analytic continuation is a Riemann surface that cannot be embedded in the complex plane. We shall say there is no maximal domain for $\log z$ or for $\sqrt{z}$. A domain $U \subset \mathbb{C}$ is the maximal domain of an analytic function f if f cannot be analytically continued across any of the boundary point of U. It is not difficult to show that any domain in $\mathbb{C}$ is a maximal domain for some complex analytic function. As for an analytic function of several complex variables a domain being maximal is the notion of domain of holomorphy.

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We study in this paper how to determine the maximal domains for real-analytic functions given as solutions of differential equations with analytic data. This problem seems to be difficult in general. However, for quasi-linear partial differential equations that have globally defined first integrals one can decide the maximal domain by applying the implicit function theorem to the first integrals determined by the initial data (Theorem 3.3).

A function $f(x)$ in n real variables $x = (x_1, \ldots, x_n)$ defined on a connected open set $U \subset \mathbb{R}^n$ is said to be analytic if at every point $p = (p_1, \ldots, p_n) \in U$, $f(x)$ is representable as a convergent power series in $(x - p) := (x_1 - p_1, \ldots, x_n - p_n)$, that is, $f(x)$ is locally equal to its Taylor series. We shall denote by $C^\omega(U)$ the set of analytic functions in U. A function $f_1 \in C^\omega(U_1)$, where U_1 is a connected open set with $U \cap U_1 \neq \varnothing$, is called a direct analytic continuation of f if $f(x) = f_1(x)$ for all $x \in U \cap U_1$. If $U_1 \supset U$, the direct analytic continuation f_1 shall be called an *analytic extension* of f. Now consider a sequence $f_k \in C^\omega(U_k)$, $k = 1, \ldots, N$, where f_k is a direct analytic continuation of f_{k-1} such that $U \cap U_N \neq \varnothing$. For any point $x \in U \cap U_N$, $f_N(x)$ need not be the same as $f(x)$ as we see in the following example. Thus in general, a sequence of direct analytic continuations yields a multi-valued function.

Example 1.1. Let $U \subset \mathbb{R}^2$ be an open disk of radius $1/2$ centered at $(1, 0)$ and let $f(x_1, x_2) := \tan^{-1}(x_2/x_1)$. Then $f_N(x)$ and $f(x)$ may differ by a multiple of 2π.

A boundary point p of U is said to be regular if p has a neighborhood to which f continues analytically, that is, there is a neighborhood $N(p)$ of p and a direct analytic continuation $f_1 \in C^\omega(N(p))$ of f. If a boundary point of U is not regular it is said to be singular. If every boundary point of U is singular, then U is the maximal domain of extension and f is maximally extended as a single-valued function.

We observe that for an analytic differential equation an analytic extension of a local analytic solution is also a solution, thus the maximal extension gives the global solution (Theorem 2.5).

A differential equation is said to be quasi-linear if it is linear in the highest order derivatives of the unknown function. Thus a quasi-linear PDE of first order for $u(t, x)$, $x = (x_1, \ldots, x_n)$, can be written as

$$\alpha(t, x, u)u_t \; + \; \sum_{k=1}^{n} a_k(t, x, u)u_{x_k} = b(t, x, u), \tag{1.1}$$

where $u_t := \frac{\partial u}{\partial t}$, $u_{x_k} := \frac{\partial u}{\partial x_k}$. We assume α is nowhere vanishing and find a solution subject to the initial condition

$$u(0, x) = h(x), \quad |x| < \epsilon \tag{1.2}$$

for an arbitrarily small $\epsilon > 0$. We shall call

$$\Gamma := \{(0, x, h(x)) : |x| < \epsilon\} \tag{1.3}$$

the initial set. We assume that all the coefficients α, a_k and b are real-analytic (C^ω) in a connected open set $\Omega \subset \mathbb{R}^{n+2} = \{(t, x, u)\}$ that contains Γ. We also assume that $h(x)$ is C^ω in a small ball $|x| < \epsilon$. Then by the Cauchy-Kowalevski theorem there is a unique C^ω solution $u(t, x)$ on a neighborhood of the origin of $\mathbb{R}^{n+1} = \{(t, x)\}$.

The Cauchy-Kowalevski theorem is an existence theory that is applicable to a wider class of analytic differential equations with analytic initial data. But for quasi-linear PDEs of first order, one can prove the existence of C^ω solution on an open neighborhood of $\{(0, x) : |x| < \epsilon\}$ in $\mathbb{R}^{n+1}$ by the method of characteristics. By the uniqueness of analytic solutions the solution obtained by the method of characteristics and the one by the Cauchy-Kowalevski theorem are equal in their common domain. Then the uniqueness of the analytic extension implies the followings, which are rather surprising.

i) An analytic extension $\hat{u}(t, x)$ of this local solution $u(t, x)$ satisfies (1.1).
ii) Its initial value $\hat{u}(0, x)$ is an analytic extension of $h(x)$ if the set $\{(0, x) \in$ domain of $\hat{u}\}$ is connected.

Suppose that a local solution to (1.1)-(1.2) is given implicitly as

$$F(t, x, u) = 0, \tag{1.4}$$

where F is analytic in Ω, vanishes on Γ, and satisfies the nondegeneracy condition

$$F_u(t, x, u) \neq 0,$$

so that one obtains an explicit solution $u(t, x)$ by the implicit function theorem. Then the zero locus of F is locally the graph of the explicit solution $u(t, x)$. Now let Σ be the set defined by (1.4) and $\sigma \subset \Sigma$ be the subset given by

$$F_u(t, x, u) = 0.$$

If needed we specify the dimensions by superscripts as in σ^n, Σ^{n+1}, and Γ^n, respectively. Let π be the projection $(t, x, u) \xrightarrow{\pi} (t, x)$. We shall show in § 3 that the connected component of $\Sigma \backslash \sigma$ that contains Γ is the graph of the maximally extended solution so that its image under π is the maximal domain of analytic extension of the local solutions to (1.1)-(1.2) assuming that the characteristic vector field of (1.1) admits $n + 1$ first integrals that are defined globally on Ω (see § 2 and § 3 for definitions). In particular, our method is useful for scalar conservation laws, where (1.1) has the form

$$u_t + \sum_{k=1}^{n} a_k(u) u_{x_k} = 0. \tag{1.5}$$

The functions

$$\rho_1 := u,$$
$$\rho_k := x_k - a_k(u)t, \qquad k = 1, \ldots, n$$

are the first integrals of (1.5) that are defined globally.

2. Preliminaries

2.1. Real analyticity

We review some basic facts on real analytic functions.

Theorem 2.1 (Identity theorem). *Suppose that $V \subset \mathbb{R}^n$, for any positive integer n, is a connected open set and $f \in C^\omega(V)$ is identically zero on a small open ball that is contained in V. Then f is identically zero on V.*

Theorem 2.1 implies the following.

Theorem 2.2 (Uniqueness of the analytic extension). *Let U and V be connected open subsets of $\mathbb{R}^n$ with $U \subset V$ and let $f \in C^\omega(U)$. Suppose that $F_1, F_2 \in C^\omega(V)$ are extensions of f. Then $F_1 = F_2$.*

The implicit function theorem states that if $F(x,u)$ is a smooth function in the variables $x = (x_1, \ldots, x_n)$ and u, and if $F_u \neq 0$ at a point (a,b), $a = (a_1, \ldots, a_n)$, then $F(x,u) = 0$ is solvable for u as a function of x, namely, there is a function $f(x)$ with $f(a) = b$ such that $F(x, f(x)) = 0$ for all x in a neighborhood of a. The analytic implicit function theorem states that if F is C^ω, then f is C^ω. More precisely,

Theorem 2.3 (Analytic implicit function theorem). *Let F be C^ω on an open subset of $\mathbb{R}^{n+1} = \{(x,u)\}$, $x = (x_1, \ldots, x_n)$. Suppose that $F(x,u)$ is C^ω in its arguments and that $F_u(a,b) \neq 0$. Then there exist an open neighborhood $U \subset \mathbb{R}^n$ of a and $f \in C^\omega(U)$ with $f(a) = b$ such that*

$$F(x, f(x)) = 0, \quad \forall x \in U.$$

One can prove Theorem 2.3 by the method of majorants for the power series expansion of F:

$$F(x,u) = \sum_{\alpha,k} a_{\alpha,k} x^\alpha u^k,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and k is a non-negative integer. Recall that the Cauchy-Kowalevski theorem is proved by the method of the majorant. Theorem 2.3 can be also proved by an application of the Cauchy-Kowalevski theorem. We refer the readers to [6] for the proofs.

Proofs for the existence and uniqueness of solutions of ODEs are based on the convergence of the iteration of the integral operator associated to the differential equation. An analytic version is the following.

Theorem 2.4 (Existence theorem for C^ω ODEs). *Let $g = (g_1, \ldots, g_n)$ be a system of C^ω functions in $(t,x) \in U \subset \mathbb{R}^{n+1}$, $x = (x_1, \ldots, x_n)$. Then for the initial value problem*

$$\frac{dx}{dt} = g(t,x), \quad x(0) = x_0,$$

for any x_0 *with* $(0, x_0) \in U$, *there exists a unique system of analytic functions* $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$ *that satisfies* $\phi(0) = x_0$ *and*

$$\frac{d\phi}{dt} = g(t, \phi(t)), \quad |t| < \epsilon \ \text{ for some } \epsilon > 0.$$

One can prove Theorem 2.4 by complexifying the variables t and x and using the fact that a uniform limit of complex analytic functions is complex analytic (see [1]).

2.2. Analytic extension of local solutions

Let us consider a system of analytic differential equations in its most general setting

$$\triangle(x, u^{(m)}) = 0, \tag{2.1}$$

where $\triangle = (\triangle_1, \ldots, \triangle_\ell)$ is a system of ℓ partial differential equations of order m for unknown functions $u = (u^1, \ldots, u^q)$ in n independent variables $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$, and $u^{(m)}$ is the partial derivatives of u of order up to m. We assume that Ω is a connected open set. Our basic observation is the following.

Theorem 2.5. *Suppose* (2.1) *is analytic, that is,* $\triangle$ *is* C^ω *in its arguments, and that an analytic function* $u = f(x)$ *defined on a small neighborhood of* $p \in \Omega$ *satisfies* (2.1). *Let* $\hat{f}(x)$ *be an analytic extension of* $f(x)$ *to* Ω. *Then* $\hat{f}(x)$ *satisfies* (2.1).

Proof. On a small neighborhood of p,

$$0 = \triangle(x, f^{(m)}(x)) = \triangle(x, \hat{f}^{(m)}(x)).$$

Hence by the identity theorem $\triangle(x, \hat{f}^{(m)}(x)) = 0$ for all $x \in \Omega$. □

2.3. First integrals and invariant submanifolds

In this subsection we define the notions of first integral and invariant submanifold in smooth (C^∞) category and from the local viewpoint, namely, our functions, vector fields, and submanifolds are in C^∞-category and defined on a small open set of $\mathbb{R}^N$. Thanks to Theorem 2.4, the Cauchy-Kowalevski theorem, and other basic facts on analytic functions, all the statements in this subsection hold true in analytic category as well.

A system of C^∞ real-valued functions $\vec{\rho} := (\rho_1, \ldots, \rho_d)$ that are defined on an open subset $U \subset \mathbb{R}^N$ is said to be *non-degenerate* if

$$d\rho_1 \wedge \cdots \wedge d\rho_d \neq 0.$$

Then their common zero set $\vec{\rho} = 0$ is a smooth submanifold of U of codimension d. Conversely, a submanifold of codimension d is locally the common zero-set of a non-degenerate system of d real-valued functions $\vec{\rho} = (\rho_1, \ldots, \rho_d)$, which we call local defining functions of the submanifold.

Given a C^∞ nowhere vanishing vector field

$$X = \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j}$$

with C^∞ coefficients a_j defined on an open subset of $\mathbb{R}^N = \{(x_1, \ldots, x_N)\}$, a real-valued function ρ is called a *first integral* of X if ρ is invariant under the flow of X, that is, if

$$X\rho = 0.$$

Proposition 2.6. *Suppose that X is a C^∞ nowhere vanishing vector field defined on an open subset of $\mathbb{R}^N$. Then*

i) *there exists locally a non-degenerate set of $N-1$ first integrals*

$$\vec{\rho} = (\rho_1, \ldots, \rho_{N-1}).$$

ii) *A C^∞ function F is invariant under the flow of X if and only if F is a function of ρ_j's, that is, there exists a C^∞ function f in $N-1$ variables so that*

$$F = f \circ \vec{\rho}.$$

See [8] for the proof.

A C^∞ curve $x(t) = (x_1(t), \ldots, x_N(t))$ for t in some interval is an integral curve of X if

$$\frac{dx(t)}{dt} = X(x(t)).$$

A submanifold $M \subset \mathbb{R}^N$ is said to be an invariant submanifold of X if M is invariant under the flow of X, that is, for any integral curve $x(t)$ with $x(0) \in M$ we have $x(t) \in M$ for all t. M is an invariant submanifold of X if and only if X is tangent to M at every point of M. An integral curve is an invariant submanifold of dimension 1. Every level set of a non-degenerate system of first integrals $(\rho_1, \ldots, \rho_d)$ is an invariant submanifold of codimension d. The following is rather obvious:

Proposition 2.7. *Let ρ be a C^∞ function with $d\rho \neq 0$ defined on an open subset of $\mathbb{R}^N$ and M be the zero set of ρ. Then M is an invariant hypersurface of a non-vanishing vector field X if and only if*

$$(X\rho)(x) = 0 \quad \textit{for all } x \textit{ with } \rho(x) = 0.$$

3. Solution by means of invariant submanifolds

In this section we present an analytic (C^ω) version of the method of characteristics for quasi-linear equations of first order. Some relevant results are found in [4] and [5]. Coming back to the Cauchy problem (1.1)-(1.2) consider the vector field

$$X := \alpha \frac{\partial}{\partial t} + \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + b \frac{\partial}{\partial u}, \tag{3.1}$$

which we call the characteristic vector field of (1.1). The method of characteristics is to construct the graph of the solution of (1.1)-(1.2) by finding integral curves of (3.1) starting from each point of the initial set Γ. It is the problem of solving the following system of initial value problems of ordinary differential equations:

$$\begin{aligned} \dot{t} &= \alpha, \quad t(0) = 0, \\ \dot{x_k} &= a_k, \quad x_k(0) = s_k, \quad k = 1, \ldots, n, \\ \dot{u} &= b, \quad u(0) = h(s),\ s := (s_1, \ldots, s_n), \end{aligned} \tag{3.2}$$

where $\dot{x}$ means the derivative of x with respect to the parametrization of the curves. The solution of (3.2) is an n-parameter family parameterized by $s = (s_1, \ldots, s_n)$ of integral curves of the characteristic vector field. Therefore, solving (3.2) is equivalent to finding a hypersurface $\Sigma \subset \mathbb{R}^{n+2}$ such that

i) $\Gamma \subset \Sigma$, where $\Gamma = \{(0, s, h(s)) : |s| < \epsilon\}$ is the initial set.
ii) Σ is invariant under the flow of X.

The graph of a solution to (1.1)-(1.2) is part of Σ. We first prove the following.

Theorem 3.1. *Let $F(t, x, u)$ be an analytic real-valued function on an open set $\Omega \subset \mathbb{R}^{n+2} = \{(t, x, u)\}$ such that $F_u \neq 0$. Then $F(t, x, u) = 0$ is an implicit solution of* (1.1) *if and only if the zero set of F is invariant under the flow of* (3.1).

Proof. Suppose $F(t, x, u) = 0$ is an implicit solution. Let $p = (t_0, x_0, u_0)$ be an arbitrary point of the zero set of F. To show that the zero set of F is invariant under the flow of the characteristic vector X, it is enough by Proposition 2.7 to show that

$$X(p)F = 0.$$

Let $u(t, x)$ be as in Theorem 2.3, namely, $u(t_0, x_0) = u_0$ and

$$F(t, x, u(t, x)) = 0. \tag{3.3}$$

Differentiating (3.3) with respect to t and x_k, respectively, we have

$$\begin{aligned} F_t + F_u u_t &= 0, \\ F_{x_k} + F_u u_{x_k} &= 0,\ k = 1, \ldots, n. \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} X(p)F &= \alpha(p)F_t + \sum_{k=1}^{n} a_k(p)F_{x_k} + b(p)F_u \\ &= F_u(p)\left(-\alpha(p)u_t - \sum_{k=1}^{n} a_k(p)u_{x_k} + b(p)\right) \quad \text{by (3.4)} \end{aligned} \tag{3.5}$$

where the last line is zero because $u(x, t)$ is an explicit solution to (1.1).

Conversely, suppose that the zero set of F is invariant under the flow of the characteristic vector (3.1). It suffices to show that at an arbitrary point

$p = (t_0, x_0, u_0)$ of the zero set of F, the function $u(t,x)$ as in Theorem 2.3 with $u(t_0, t_0) = u_0$ is an explicit solution to (1.1). Again, by differentiating $F(t, x, u(t, x)) = 0$ with respect to t and with respect to x_k, we have (3.4). Then as in (3.5) we have

$$
\begin{aligned}
0 &= X(p)F \\
&= F_u(p)\left(-\alpha(p)u_t - \sum_{k=1}^{n} a_k(p)u_{x_k} + b(p)\right).
\end{aligned} \tag{3.6}
$$

The last line of (3.6) being zero and $F_u \neq 0$ imply that $u(t,x)$ is the explicit solution to (1.1). □

Now we find $F(t,x,u)$ that defines an invariant hypersurface as in Theorem 3.1 by using the first integrals of (3.1) as follows: Let $\vec{\rho} = (\rho_1, \ldots, \rho_{n+1})$ be a nondegenerate system of C^ω first integrals of (3.1) defined on a neighborhood of the initial set Γ as in (1.3).

Consider the mapping

$$\mathbb{R}^{n+2} \ni (t, x_1, \ldots, x_n, u) \overset{\vec{\rho}}{\mapsto} (\rho_1, \ldots, \rho_{n+1}) \in \mathbb{R}^{n+1}.$$

Since $\vec{\rho}$ is nondegenerate, $\vec{\rho}(\Gamma)$ is a C^ω hypersurface in $\mathbb{R}^{n+1}$, thus there exists an analytic function f so that $\vec{\rho}(\Gamma)$ is contained in the zero set of f. Now let

$$F := f \circ \vec{\rho}.$$

Then F itself is a first integral of (3.1) that is non-degenerate and vanishes on Γ. Thus we proved the following.

Proposition 3.2. *Let $\vec{\rho} := (\rho_1, \ldots, \rho_{n+1})$ be a nondegenerate system of C^ω first integrals of the characteristic vector field* (3.1) *defined on a neighborhood of the initial set Γ. Let f be an analytic local defining function of the hypersurface $\vec{\rho}(\Gamma) \subset \mathbb{R}^{n+1}$. Then the zero set of $F(t,x,u) := f \circ \vec{\rho}$ has the following properties*:

i) $F(\Gamma) = 0$,
ii) *the zero set of F is invariant under the flow of* (3.1).

Now let $\pi : (t,x,u) \mapsto (t,x)$ be the projection. From the invariant set $F(t,x,u) = 0$ as in Proposition 3.2 we find the maximal domain of analytic extension of the local explicit solution $u(t,x)$ as follows: Let $\Omega \subset \mathbb{R}^{n+2}$ be the open set where the coefficients α, a_k and b of (1.1) are defined and $\Sigma^{n+1} \subset \Omega$ be the connected component of the zero set of F that contains Γ^n, and $\sigma^n \subset \Sigma^{n+1}$ be the set of points where $F_u(t,x,u) = 0$. Then $\Sigma^{n+1} \backslash \sigma^n \overset{\pi}{\to} \pi(\Sigma^{n+1} \backslash \sigma^n)$ is a local diffeomorphism by the implicit function theorem. Now let Σ_Γ be the connected component of $\Sigma^{n+1} \backslash \sigma^n$ that contains Γ^n. We fix a point $P \in \Gamma^n$ and let $Q \in \Sigma^{n+1}$ be any point. If $Q \in \Sigma_\Gamma$, then there is a curve in Σ_Γ that connects P to Q, which implies that the local solution $u(t,x)$ to (1.1)-(1.2) analytically continues to $\pi(Q)$ by the analytic implicit function theorem. If $Q \in \Sigma^{n+1} \backslash \Sigma_\Gamma$, then any curve in Σ^{n+1} that connects P to Q intersects σ^n, which implies that

$u(t,x)$ does not continue analytically to $\pi(Q)$ because at the intersecting point $|\nabla u|$ blows up by the implicit function theorem. Thus we proved the following.

Theorem 3.3. *Given a quasi-linear PDE of first order* (1.1) *where the coefficients* α, a_j *and* b *are analytic on an open set* $\Omega \subset \mathbb{R}^{n+2} = \{(t,x,u)\}$, $x = (x_1, \dots, x_n)$, *let* X *be the characteristic vector field on* Ω *given by* (3.1), *and let* $\Gamma^n = \{(0,x,h(x)) : |x| < \epsilon\}$ *be given an initial set where* h *is analytic. Suppose that there exists a set of first integrals* $\vec{\rho} = (\rho_1, \dots, \rho_{n+1})$ *of* X *that are defined in* Ω *and nondegenerate on a neighborhood of* Γ^n. *Let* f *be a defining function of* $\vec{\rho}(\Gamma^n)$ *that is analytic in* $\vec{\rho}(\Omega)$ *and let* $F = f \circ \vec{\rho}$. *Let* Σ^{n+1} *be the connected component of the zero set of* F *that contains* Γ^n *and* $\sigma^n := \{(t,x,u) \in \Sigma^{n+1} \mid F_u(t,x,u) = 0\}$ *and* $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ *be the projection* $(t,x,u) \xrightarrow{\pi} (t,x)$. *Then we have*

i) $F(t,x,u) = 0$ *is an implicit solution to* (1.1)-(1.2),

ii) $\Sigma^{n+1} \backslash \sigma^n \xrightarrow{\pi} \pi(\Sigma^{n+1} \backslash \sigma^n)$ *is a local diffeomorphism,*

iii) *Let* Σ_Γ *be the connected component of* $\Sigma^{n+1} \backslash \sigma^n$ *that contains* Γ^n. *Then* Σ_Γ *is the graph of the maximal analytic extension of the local solution* $u(t,x)$ *to* (1.1)-(1.2),

iv) $\pi(\Sigma_\Gamma)$ *is the maximal domain of analytic extension of the local solution to* (1.1)-(1.2).

Example 3.4. Case $n = 0$: This is the case of ordinary differential equations. For a real-valued function $u(t)$ consider

$$u' = u^2 \tag{3.7}$$

subject to the initial condition

$$u(0) = 1. \tag{3.8}$$

It is easy to find the solution $u = \frac{1}{1-t}$, so that the maximal domain is $(-\infty, 1)$. Now we shall obtain the same by our theory and Theorem 3.3. The characteristic vector field is

$$X = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial u}$$

so that

$$\rho := t + \frac{1}{u}$$

is a first integral. On the initial data set $\Gamma^0 = \{(0,1)\}$ ρ has value 1, so that $f(y) = y - 1$ and

$$F := f \circ \rho = \rho - 1 = t + \frac{1}{u} - 1.$$

Now $F = 0$ is an implicit solution to (3.7)-(3.8) and the connected component of Γ^0 in $F = 0$ is the set

$$\Sigma^1 = \{(t,u) \in \mathbb{R}^2 : t + \frac{1}{u} - 1 = 0,\ t < 1\}.$$

We see further that σ^0 is empty, which implies $\Sigma_\Gamma = \Sigma^1$, and that $\pi(\Sigma_\Gamma) = (-\infty, 1)$.

Example 3.5. Case $n = 1$: For a real-valued function $u(t, x)$ of two real variables consider

$$uu_t = -t \tag{3.9}$$

subject to the initial condition

$$u(0, x) = \sqrt{1 - x^3}, \quad |x| < \epsilon.$$

Then the set of initial data is $\Gamma^1 = \{(0, x, \sqrt{1 - x^3}) : |x| < \epsilon\}$ and the characteristic vector field is

$$X := u\frac{\partial}{\partial t} - t\frac{\partial}{\partial u}.$$

Since X has no $\frac{\partial}{\partial x}$ component, $\rho_1(t, x, u) := x$ is obviously a first integral. It is easy to find another first integral $\rho_2(t, x, u) := t^2 + u^2$. This implies that each characteristic curve is a circle $t^2 + u^2 = c_1$ and $x = c_2$ for constants c_1 and c_2, so that the graph of the solution is contained in a surface of revolution obtained by rotating about x-axis. Now let

$$\vec{\rho} = (\rho_1, \rho_2) : \mathbb{R}^3 \longrightarrow \mathbb{R}^2.$$

Since $\vec{\rho}(\Gamma^1) = \{(x, 1 - x^3)\}$,

$$f(\rho_1, \rho_2) := \rho_2 - 1 + (\rho_1)^3 = 0 \quad \text{on } \Gamma^1.$$

Hence

$$F := f(\vec{\rho}) = t^2 + u^2 - 1 + x^3 = 0 \tag{3.10}$$

is the implicit solution of (3.9). Let Σ^2 be the surface defined by (3.10). To find the set σ^1 of the singular points solve (3.10) together with

$$F_u = 2u = 0$$

simultaneously. We have

$$\sigma^1 = \{(t, x, u) : u = 0,\ x^3 + t^2 - 1 = 0\}.$$

Therefore, the maximal domain of analytic extension is

$$1 - t^2 - x^3 > 0,$$

as shown in Figure 1 and the maximally extended single-valued solution is

$$u(t, x) = (1 - t^2 - x^3)^{1/2}.$$

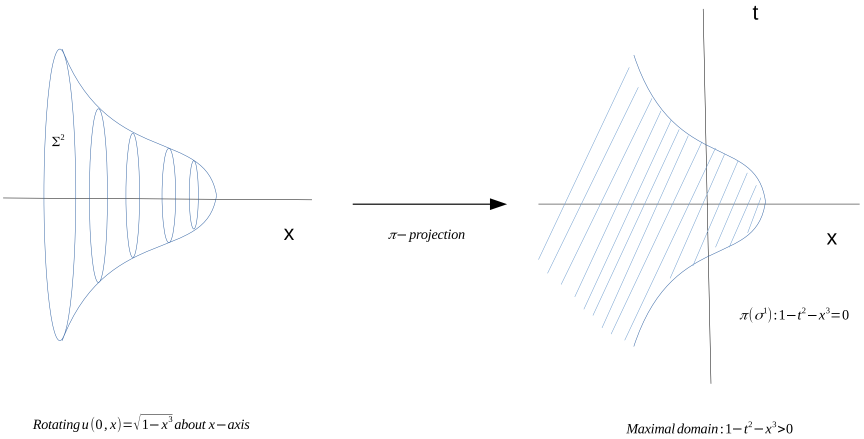

FIGURE 1.

4. Conservation laws

A quasi-linear PDE of first order for $u(t, x)$ of the form

$$u_t + \Phi(u)_x = 0$$

is called a 1-*dimensional conservation law*. This describes the motion of 1-dimensional flow of fluid where u is the conserved density and $\Phi(u)$ is the flux. Let us consider the case $\Phi(u) = u^2/2$ so that the equation becomes

$$u_t + uu_x = 0, \tag{4.1}$$

which is the inviscid Burgers' equation. We assume (4.1) holds for all time $-\infty < t < \infty$. The characteristic vector field of (4.1) is

$$X = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}. \tag{4.2}$$

Observe that X has no $\frac{\partial}{\partial u}$ component and thus $\rho_1(t, x, u) := u$ is a first integral. We see that $\rho_2(t, x, u) := x - ut$ is another first integral and that $\vec{\rho} = (\rho_1, \rho_2)$ is non-degenerate and globally defined. Now we consider the Cauchy problem (4.1) with the initial condition $u(0, x) = h(x) : |x| < \epsilon$. It is well known that if h is strictly decreasing, then the solution $u(t, x)$ has singularities for some positive t (see [2, 3]). In this case we observe that the boundary of the maximal domain is the envelope of the projections of the characteristic lines (see Example 4.1 and Example 4.2).

Example 4.1. Consider (4.1) with the initial condition

$$u(0,x) = -kx, \ |x| < \epsilon,$$

where $k > 0$ is a constant. As in Example 3.5 and in Example 4.2 we have $F(t,x,u) = (1-kt)u + kx$ and σ^1 is the line $t = 1/k, x = 0$ that is parallel to u-axis. The maximal domain of analytic extension of solution is the half plane $\{(t,x) : t < 1/k\}$.

Example 4.2. We consider the same equation (4.1) with a decreasing initial data

$$u(0,x) = \frac{1}{x+1}, \quad |x| < \epsilon.$$

Let $\Gamma^1 = \{(0, x, \frac{1}{x+1}) : |x| < \epsilon\}$ be the initial set. Then

$$\vec{\rho}(\Gamma^1) = \{(1/(x+1), x) : |x| < \epsilon\}$$

is given by

$$f(\vec{\rho}) := \rho_1 - \frac{1}{\rho_2 + 1} = 0.$$

Therefore, the implicit solution to (4.1) is

$$F(t,x,u) := f \circ \vec{\rho} = u - \frac{1}{x - ut + 1} = 0 \tag{4.3}$$

and (4.3) defines Σ^2. Solving (4.3) for u we have

$$u = \begin{cases} \dfrac{x + 1 \pm \sqrt{(x+1)^2 - 4t}}{2t} & \text{if } t \neq 0, \\ \dfrac{1}{x+1} & \text{if } t = 0. \end{cases}$$

To figure out the shape of Σ^2 we recall that the quantities u and $x - ut$ remain constant along the integral curves of (4.2). These curves are given by (3.2) with $n = 1$, $\alpha = 1$, $a = u$, $b = 0$, $h(s) = \frac{1}{s+1}$, namely,

$$\begin{cases} \dot{t} = 1, & t(0) = 0, \\ \dot{x} = u, & x(0) = s, \\ \dot{u} = 0, & u(0) = \frac{1}{s+1}, \end{cases}$$

which give 1-parameter family of lines

$$x = s + ut, \quad u = \frac{1}{s+1}. \tag{4.4}$$

Given each value s, (4.4) is the *characteristic line* for (4.1) through $(0, s, \frac{1}{s+1}) \in \Gamma^1$. Projections to the (t,x)-plane of some characteristic lines are shown in Figure 2. On the other hand, by solving (4.3) and

$$F_u = 1 - \frac{t}{(x - ut + 1)^2} = 0$$

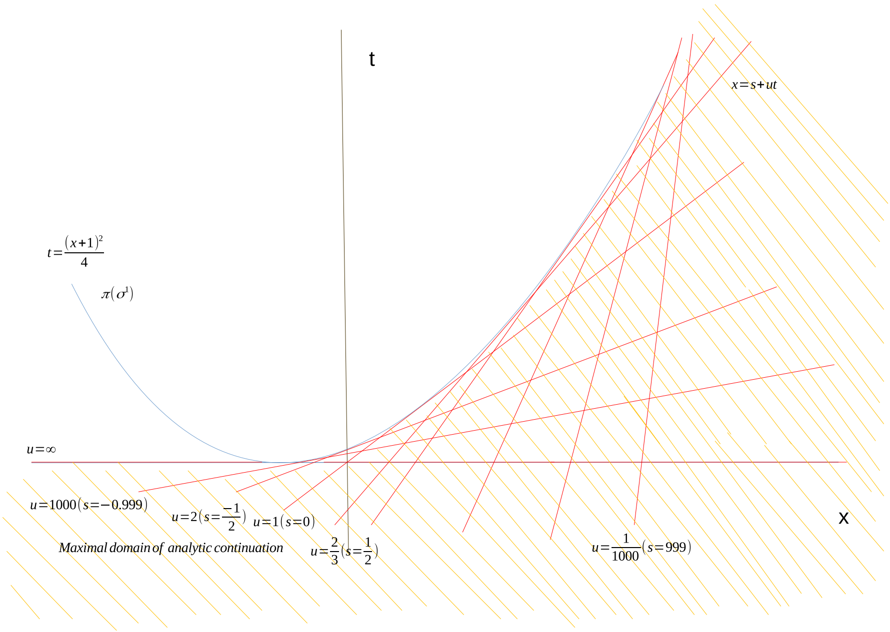

FIGURE 2.

simultaneously, we find that σ^1 is the curve $t = \frac{(x+1)^2}{4}$, $u = \frac{2}{x+1}$, so that $\pi(\sigma^1)$ is the curve

$$t = \frac{(x+1)^2}{4}. \tag{4.5}$$

Easy computation shows that for each s the line (4.4) is tangent to (4.5) at the point

$$t = (s+1)^2, \;\; x = 2(s+1) - 1. \tag{4.6}$$

Observe that $\pi(\sigma^1)$ is part of the envelope of the projections to (t, x)-plane of (4.4) with $s \neq -1$. The maximal domain of analytic extension is the shaded area of Figure 2, which is the union of characteristic projections with positive slope. In order for the analytic continuation to be single-valued we take each characteristic projection up to the point (4.6). By differentiating (4.5) with respect to t we obtain

$$\frac{dx}{dt} = \frac{1}{\sqrt{t}} = \frac{2}{x+1},$$

which gives an alternative way of finding the speed of propagation of singularity for analytic cases. As for the singularities of solutions of 1-D conservation laws we refer the readers to [7].

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