J. Korean Math. Soc. **59** (2022), No. 6, pp. 1067–1082
https://doi.org/10.4134/JKMS.j210607
pISSN: 0304-9914 / eISSN: 2234-3008

A NEW OPTIMAL EIGHTH-ORDER FAMILY OF MULTIPLE ROOT FINDERS

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Abstract. This paper presents a new optimal three-step eighth-order family of iterative methods for finding multiple roots of nonlinear equations. Different from the all existing optimal methods of the eighth-order, the new iterative scheme is constructed using one function and three derivative evaluations per iteration, preserving the efficiency and optimality in the sense of Kung-Traub's conjecture. Theoretical results are verified through several standard numerical test examples. The basins of attraction for several polynomials are also given to illustrate the dynamical behaviour and the obtained results show better stability compared to the recently developed optimal methods.

1. Introduction

Approximating the roots of the nonlinear equation $f(x) = 0$ is one of the most important tasks in numerical mathematics with many applications in engineering and science. There is a great amount of literature that deals with the problem of determining the simple root (say α) of the nonlinear equation, but not so many papers address the case when α is the root of multiplicity $m > 1$ (which means that $f^{(i)}(\alpha) = 0$ for $i = 0, 1, \ldots, m-1$, and $f^{(m)}(\alpha) \neq 0$).

A very basic multiple root finding method is modified Newton's method [15] (also known as Rall's method [14])

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, \ldots. \tag{1}$$

This one-point method is quadratically convergent and therefore optimal in the sense of Kung-Traub's conjecture [9] which states that any multipoint iterative scheme that requires s function/derivative evaluations per iteration can reach at most 2^{s-1} convergence order.

In the last decade, many researchers have developed the multistep methods of the higher convergence order using method (1) as the first step in their

Received October 1, 2021; Revised July 2, 2022; Accepted September 14, 2022.
2020 *Mathematics Subject Classification.* Primary 65H05.
Key words and phrases. Nonlinear equation, multiple root, optimal methods, eighth-order of convergence.

iterative schemes. For example, such Newton-type methods of optimal fourth-order can be found in [10, 11, 25] and some more efficient optimal eighth-order methods are described in [1, 3, 4, 7, 23, 24]. Nevertheless, the majority of those methods could be considered as the generalizations of the optimal two or three-step optimal methods constructed for finding the simple roots. Thus, several well-known fourth and eighth-order multiple root finders have been derived in [13] directly from the previously published methods for simple roots by using a relatively simple technique for generalization. All of the eighth-order methods mentioned above require three function evaluations and one derivative evaluation per iteration.

Very recently, Sharma and Kumar [17] have constructed the optimal eighth-order iterative scheme

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mQ(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - mu_nw_nW(u_n,w_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{2}$$

for $m > 1$, where $u_n = \left(\frac{f'(y_n)}{f'(x_n)}\right)^{\frac{1}{m-1}}$, $v_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $w_n = \frac{v_n}{u_n}$, while $Q(u)$ and $W(u,w)$ are analytic functions in a neighborhood of 0 and $(0,0)$ that satisfy conditions summarized in the theorem [17, page 319]. Despite all existing optimal eighth-order methods, method (2) uses two function and two derivative evaluations per iteration.

In the next section, we present the new three-step family of iterative methods of the optimal eighth-order. Uniqueness of the family lies in the fact that the iterative scheme requires one function and three derivative evaluations per iteration, in contrast to the all other eighth-order methods including Sharma and Kumar's method (2). In the last two sections, the numerical efficiency and the dynamic behaviour of the new family members are compared to the other recently developed optimal methods.

2. A new iterative family

The first two steps of method (2) are actually the optimal fourth-order derived by Liu and Zhou [11]. They established the following conditions function Q must satisfy to provide the optimal fourth-order of convergence of Liu-Zhou method,

$$Q(0) = 0, \quad Q'(0) = 1, \quad Q''(0) = \frac{4m}{m-1}. \tag{3}$$

Sharma and Kumar have improved the Liu-Zhou method by adding the third step that involves the additional function evaluation $f(z_n)$. In contrast to this, the new iterative family consists of the Liu-Zhou method and the third step that requires the additional evaluation of $f'(z_n)$. Hence, the general form of the new iterative scheme is

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - mQ(u_n)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= z_n - mG(u_n, w_n)\frac{f(x_n)}{f'(x_n)},
\end{aligned}
\tag{4}
$$

where $u_n = \big(\frac{f'(y_n)}{f'(x_n)}\big)^{\frac{1}{m-1}}$ and $w_n = \big(\frac{f'(z_n)}{f'(y_n)}\big)^{\frac{1}{m-1}}$, while $Q(u)$ and $G(u,w)$ are analytic in neighborhoods of 0 and $(0,0)$, respectively.

Since the new scheme should provide eighth convergence order, we need proper forms of Taylor's expansions of $f(x_n)$ and $f'(x_n)$ about α, given by

$$
f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m \cdot \big(1 + \sum_{i=1}^{8} c_i e_n^i + O(e_n^9)\big),
\tag{5}
$$

$$
f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1} \cdot \big(1 + \sum_{i=1}^{8} \frac{m+i}{m} c_i e_n^i + O(e_n^9)\big),
\tag{6}
$$

where $e_n = x_n - \alpha$ and $c_i = (m!/(m+i)!) \cdot (f^{(m+i)}(\alpha)/f^{(m)}(\alpha))$ for $i \geq 1$. Let $\widehat{e}_n = y_n - \alpha$ and $\widetilde{e}_n = z_n - \alpha$ be the errors of the first and second step in the n-th iteration. From (5) and (6), the error of the first step $\widehat{e}_n$ in terms of e_n equals

$$
\begin{aligned}
\widehat{e}_n = e_n^2\Big[&\frac{c_1}{m} + \big(2mc_2 - (1+m)c_1^2\big)\frac{e_n}{m^2}\\
&+ \big((1+m)^2c_1^3 - m(4+3m)c_1c_2 + 3m^2c_3\big)\frac{e_n^2}{m^3}\\
&+ \big(-(1+m)^3c_1^4 + 2m(1+m)(3+2m)c_1^2c_2\\
&- 2m^2(2+m)c_2^2 - 2m^2(3+2m)c_1c_3 + 4m^3c_4\big)\frac{e_n^3}{m^4}\\
&+ \big((1+m)^4c_1^5 - m(1+m)^2(8+5m)c_1^3c_2 + m^2(1+m)(9+5m)c_1^2c_3\\
&+ m^2c_1((2+m)(6+5m)c_2^2 - m(8+5m)c_4)\\
&+ m^3(5mc_5 - (12+5m)c_2c_3)\big)\frac{e_n^4}{m^5} + \cdots\Big] + O(e_n^9).
\end{aligned}
\tag{7}
$$

It is clear that

$$
f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}\widehat{e}_n^{m-1} \cdot \big(1 + \sum_{i=1}^{8} \frac{m+i}{m} c_i \widehat{e}_n^i + O(\widehat{e}_n^9)\big).
\tag{8}
$$

Therefore, using (6) and (8), after substituting (7) into (8), we get

$$
\begin{aligned}
u_n = {} & \sqrt[m-1]{f'(y_n)/f'(x_n)} \\
= {} & e_n\Big[\frac{c_1}{m} + \big(2(m-1)c_2 - (1+m)c_1^2\big)\frac{e_n}{m(m-1)} \\
& + \big((-2-m+2m^2+3m^3+2m^4)c_1^3 - 2m^2(-4+m+3m^2)c_1c_2 \\
& + 6(m-1)^2m^2c_3\big)\frac{e_n^2}{2m^3(m-1)^2} \\
& + \big((1+m)^2(6-16m+7m^2-m^3+6m^4)c_1^4 \\
& - 6m(4-m-8m^2-3m^3+4m^4+4m^5)c_1^2c_2 + 12(m-1)^2m^3c_1c_3 \\
& + 12(m-1)^2m^3((2+m)c_2^2 - 2(m-1)c_4)\big)\frac{e_n^3}{6m^4(m-1)^3} \\
& + \cdots\Big] + O(e_n^9).
\end{aligned} \tag{9}
$$

If the function $Q(\cdot)$ satisfies conditions (3), then from (5), (6), (9) and Taylor's expansion of $Q(u_n)$ about 0, the error of the second step $\widetilde{e}_n$ has a form

$$
\begin{aligned}
\widetilde{e}_n = {} & \Big(\big(3(2+m+8m^2+m^3) - (m-1)^2Q'''(0)\big)c_1^3 \\
& + 6m^2(1-m)c_1c_2\Big)\frac{e_n^4}{6m^3(m-1)^2} \\
& + \Big[-48m^3(m-1)^2c_2^2 - 48m^3(m-1)^2c_1c_3 \\
& + 24m(m-1)\big(4+2+24m^2+4m^3-(m-1)^2Q'''(0)\big)c_1^2c_2 \\
& + \big(4(1+m)(12+5m+9m^2-63m^3-7m^4) \\
& + 4(m-1)^2(3m^2+4m-1)Q'''(0) \\
& - (m-1)^3Q^{(4)}(0)\big)c_1^4\Big]\frac{e_n^5}{24m^4(m-1)^3} + \cdots + O(e_n^9).
\end{aligned} \tag{10}
$$

Substituting $\widetilde{e}_n$ instead e_n into (6) to get $f'(z_n)$, using (8) and Taylor's expansion of $(m-1)$st root, we have

$$
\begin{aligned}
w_n = {} & \sqrt[m-1]{f'(z_n)/f'(y_n)} \\
= {} & \Big((6+3m+24m^2+m^3-(m-1)^2Q'''(0))c_1^2 \\
& + 6m^2(1-m)c_2\Big)\frac{e_n^2}{6m^2(m-1)^2} \\
& + \Big[16m(m-1)(3+3m+24m^2+3m^3-(m-1)^2Q'''(0))c_1c_2 \\
& + \Big(8(1+m)(3+4m-6m^2-21m^3-2m^4) \\
& + 8m(2+m)(m-1)^2Q'''(0) - (m-1)^3Q^{(4)}(0)\Big)c_1^3
\end{aligned} \tag{11}
$$

$$- 48m^3(m-1)^2c_3\Big]\frac{e_n^3}{24m^3(m-1)^3} + \cdots + O(e_n^6).$$

Since $\widetilde{e}_n = z_n - \alpha = O(e_n^4)$ and $f(x_n)/f'(x_n) = O(e_n)$, from the third step of (4) it is clear that the function $G(u_n, w_n)$ should be of third-order, which means that, with respect to (9) and (11), in Taylor's expansion of G about $(0, 0)$ we have

$$G(0,0) = 0,\ \frac{\partial G}{\partial u}(0,0) = 0,\ \frac{\partial^2 G}{\partial u^2}(0,0) = 0 \text{ and } \frac{\partial G}{\partial w}(0,0) = 0. \tag{12}$$

Moreover, to achieve the optimal eighth-order, all coefficients of $e_n, e_n^2, \ldots, e_n^7$ should vanish in error equation $e_{n+1} = x_{n+1} - \alpha$. Thus, from the third step of (4), taking into account (5), (6), (10) and Taylors expansion of $G(u_n, w_n)$, after simple computation we get the following conditions which provide optimality of the method:

$$\begin{aligned}
&G(0,0)=\frac{\partial^i G}{\partial u^i}(0,0)=\frac{\partial^j G}{\partial w^j}(0,0)=0 \text{ for } i\in\{1,2,3,4,5,6\} \text{ and } j\in\{1,2,3\}\\
&\frac{\partial^2 G}{\partial u \partial w}(0,0) = 1,\ \frac{\partial^3 G}{\partial u^2 \partial w}(0,0) = \frac{4m}{m-1},\ \frac{\partial^3 G}{\partial u \partial w^2}(0,0) = 2,\\
&\frac{\partial^4 G}{\partial u^2 \partial w^2}(0,0) = \frac{8(2m^2-1)}{m(m-1)},\ \frac{\partial^4 G}{\partial u^3 \partial w}(0,0) = 6\frac{m+1}{m-1} + Q'''(0),\\
&\frac{\partial^5 G}{\partial u^4 \partial w}(0,0) = \frac{-48(m+1)(2m^2+1)}{m(m-1)^2} + 8\frac{m+1}{m}Q'''(0) + Q^{(4)}(0).
\end{aligned} \tag{13}$$

Therefore, if conditions (13) are satisfied, it yields

$$x_{n+1} = z_n - mG(u_n, w_n)\frac{f(x_n)}{f'(x_n)} = x_n - \alpha + O(e_n^8),$$

i.e., $e_{n+1} = x_{n+1} - \alpha = O(e_n^8)$.

The above discussion is summarized in the following theorem.

Theorem 2.1. *Let α be a multiple root of known multiplicity m of a sufficiently differentiable function $f(x)$. If the initial iteration x_0 is close enough to α, and if functions Q and G satisfy conditions* (3) *and* (13) *respectively, then family of methods defined by* (4) *is of optimal eighth convergence order.*

Remark 1. Some parts of the expressions in the previous analysis are intentionally omitted for the sake of simplicity. All the results have been done and verified with the aid of *Mathematica*'s symbolic computation system.

Remark 2. Since family (4) possesses eighth convergence order, and requires one function and three derivative evaluations per iteration, it is clear that the family is optimal in the sense of Kung-Traub's conjecture.

Three special cases of (4) based on the different choices of $Q(u_n)$ and $G(u_n, w_n)$ have been considered for the numerical comparisons. The first new method is defined as

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n - m\Big(u_n + \frac{2m}{m-1}u_n^2\Big)\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= z_n - m\frac{\Big(u_n + \frac{2}{m(m-1)}u_n^2\Big)w_n}{1 - \frac{2(m+1)}{m}u_n + \frac{3(m+1)}{m-1}u_n^2 - w_n}\cdot\frac{f(x_n)}{f'(x_n)} \end{aligned}$$

and denoted by **NM1**. The second one, denoted by **NM2**, has the following form

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n - m\frac{(m-1)u_n}{m-1-2mu_n}\cdot\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= z_n - m\frac{\Big(u_n + \frac{2}{m(m-1)}u_n^2\Big)w_n}{1 - \frac{2(m+1)}{m}u_n - \frac{m^2+3}{(m-1)^2}u_n^2 - w_n}\cdot\frac{f(x_n)}{f'(x_n)}. \end{aligned}$$

The third one is **NM3** given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n - m\frac{(m-1)u_n}{m-1-2mu_n}\cdot\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= z_n - m\frac{\Big(u_n + \frac{2}{m(m-1)}u_n^2\Big)w_n + u_n w_n^2}{1 - \frac{2(m+1)}{m}u_n - \frac{m^2+3}{(m-1)^2}u_n^2 - \frac{2m}{m-1}u_n w_n}\cdot\frac{f(x_n)}{f'(x_n)}. \end{aligned}$$

3. Numerical comparison

In this section, several test examples are employed in order to verify the theoretic results from the previous section and to illustrate the effectiveness of the methods NM1, NM2 and NM3. The results are compared with the very recently developed Newton-type methods of the optimal eighth-order.

Such existing method is the one proposed by Zafar et al. [22], denoted by **ZCJT**, with the following structure

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n - mu_n\frac{1+8u_n+11u_n^2}{1+6u_n}\cdot\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= z_n - mw_n\big(1 + t_n + \frac{1}{2}t_n^2 + u_n(2+4t_n)\big)\frac{f(x_n)}{f'(x_n)}, \end{aligned}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $t_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$.

Behl et al. [2] have developed an efficient family (14) and tested several special cases. We choose two members of this family with the best performance in the original research. This family has the following general form

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mH(v_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - w_n u_n\Big(G(u_n) + \frac{mw_n}{1-4u_n}\Big)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{14}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $w_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, $v_n = \frac{1+\alpha u_n}{1+\beta u_n}$ for some real numbers $\alpha \neq \beta$ and H and G are analytic functions in neighborhoods of 1 and 0. The first chosen special case is denoted by **BAASA1** for and $\alpha = 1/2$, $\beta = -3/2$ and functions

$$H(v_n) = m\frac{\alpha - \beta + 2v_n - 2}{\alpha - \beta},\ G(u_n) = m\big(1 + 2u_n + (1-2\beta)u_n^2 + 2(\beta^2 - 2\beta - 2)u_n^3\big).$$

The second special case **BAASA2** uses $\alpha = 0$, $\beta = -2$ and functions

$$H(v_n) = m\frac{\alpha - \beta + 2v_n - 2}{\alpha - \beta},\ G(u_n) = \frac{m\big(2\beta^2 u_n + \beta(2 - 4u_n^2) - (3u_n + 1)^2\big)}{2\beta^2 u_n + \beta(2 - 4u_n) - 4u_n - 1}.$$

Kumar et al. [8] have constructed the eighth-order family

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_n(1 + 2u_n - u_n^2)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - m(1+u_n)v_n H(v_n)\frac{f(x_n)}{f'(x_n)} - m(u_n + w_n)v_n G(u_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $w_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, while H and G are analytic functions in neighborhood of 0. The special case (**KKSdA**) with the best performance in the original research is the one where

$$H(v_n) = \frac{1}{1 - 4v_n} \quad \text{and} \quad G(u_n) = \frac{1 + 6u_n}{1 + 6u_n + 6u_n^2}.$$

Finally, the variant of Sharma-Kumar's family (2) that we use for the comparison has been established in [17] for

$$\begin{aligned} Q(u_n) &= u_n + \frac{2m}{m-1}u_n^2 + \frac{6m^4 + m^3 - 5m^2 - 3m - 3}{3(m-1)^2(m^2 - m - 1)}u_n^3, \\ W(u_n, w_n) &= 1 + 2u_n + \frac{m-1}{m}w_n + \frac{u_n}{3}\Big(\frac{k_1 w_n}{m^2} + \frac{k_2 u_n}{m^3 - 2m^2 + 1}\Big), \end{aligned}$$

where $k_1 = 6(2m^2 - 2m - 1)$ and $k_2 = 9m^3 - 8m^2 - 5m + 6$. This method is denoted by **SK**.

Table 1 displays the test functions used for the comparison, with appropriate root α and its multiplicity m.

Table 1. Test functions

$f(x)$	α	m
$f_1(x) = \big(x - x^3 \cos\frac{\pi x}{3} + \frac{1}{1+x^2} - 30.1\big)(x-3)^4$	3	5
$f_2(x) = \exp\big(\frac{((x-0.5)^2+3)^2}{x^5+\cos((x-0.5)^2+3)}\big) - 1$	$0.5+\sqrt{3}i$	2
$f_3(x) = x^4 + 11.5x^3 + 47.49x^2 + 83.06325x + 51.23266875$	-2.85	2
$f_4(x) = (\cos x - x)^3$	0.7390851...	3
$f_5(x) = (\arcsin(x^2-1) + e^x - 3)^2$	1.0579494...	2
$f_6(x) = (2x - e^{-x} + \sin x^2 - 3)^5$	3.8173523...	5

Tables 2-7 present the number of iterations (*it*) required to satisfy stopping criterion $|f(x_n)| < 10^{-1000}$. Along with that, errors $|x_n - \alpha|$ and residual errors $|f(x_n)|$ are given for each method after the third iteration. The tables also show the computational order of convergence [21] given by

$$\mathrm{COC} = \frac{\log|(x_n - \alpha)/(x_{n-1} - \alpha)|}{\log|(x_{n-1} - \alpha)/(x_{n-2} - \alpha)|},$$

which has been used to numerically check the convergence order of the proposed methods. The last columns display CPU time computed as the average of 25 performances of each method. If an algorithm fails to find the root within 100 iterations, it is denoted by "–".

All computations have been done using *Mathematica* program package with the aid of *SetPrecision* function with 10000 precision digits. The performances of the computer have been 64-bit Windows 10 Pro operating system and AMD Ryzen 7 1700 eight-core CPU 3.00 GHz processor.

Table 2. Numerical results for $f_1(x)$ and $x_0 = 2.87$

method	it	$\lvert x_3 - \alpha\rvert$	$\lvert f(x_3)\rvert$	COC	CPU
ZCJT	3	$1.4577 \cdot 10^{-818}$	$1.8392 \cdot 10^{-4088}$	8.0000	0.0312
BAASA1	3	$4.5649 \cdot 10^{-803}$	$5.5386 \cdot 10^{-4011}$	8.0000	0.0306
BAASA2	3	$2.3604 \cdot 10^{-803}$	$2.0472 \cdot 10^{-4012}$	8.0000	0.0294
KKSdA	3	$9.0274 \cdot 10^{-800}$	$1.6751 \cdot 10^{-3994}$	8.0000	0.0325
SK	3	$1.4194 \cdot 10^{-796}$	$1.6096 \cdot 10^{-3978}$	8.0000	0.0412
NM1	3	$1.0260 \cdot 10^{-857}$	$3.1769 \cdot 10^{-4284}$	8.0000	0.0469
NM2	3	$1.5370 \cdot 10^{-865}$	$2.3963 \cdot 10^{-4323}$	8.0000	0.0488
NM3	3	$8.9639 \cdot 10^{-782}$	$1.6170 \cdot 10^{-3904}$	8.0000	0.0506

Table 3. Numerical results for $f_2(x)$ and $x_0 = 0.495 + 1.72i$

method	it	$\|x_3 - \alpha\|$	$\|f(x_3)\|$	COC	CPU
ZCJT	5	$7.1869 \cdot 10^{-216}$	$3.0946 \cdot 10^{-431}$	6.0104	0.155
BAASA1	4	$1.7861 \cdot 10^{-229}$	$1.9113 \cdot 10^{-458}$	14.050	0.132
BAASA2	5	$9.3159 \cdot 10^{-123}$	$5.1996 \cdot 10^{-245}$	6.0250	0.157
KKSdA	4	$4.2989 \cdot 10^{-413}$	$1.1072 \cdot 10^{-825}$	4.0000	0.142
SK	4	$2.7323 \cdot 10^{-199}$	$4.4729 \cdot 10^{-398}$	4.0000	0.156
NM1	3	$1.3399 \cdot 10^{-808}$	$1.0757 \cdot 10^{-1616}$	8.0000	0.140
NM2	3	$1.3120 \cdot 10^{-790}$	$1.0313 \cdot 10^{-1580}$	8.0000	0.143
NM3	3	$2.1424 \cdot 10^{-804}$	$2.7499 \cdot 10^{-1608}$	8.0000	0.143

Table 4. Numerical results for $f_3(x)$ and $x_0 = -3.4$

method	it	$\|x_3 - \alpha\|$	$\|f(x_3)\|$	COC	CPU
ZCJT	13	12.452	23981	6.0026	0.0950
BAASA1	–	–	–	–	–
BAASA2	–	–	–	–	–
KKSdA	11	1.4981	0.29865	13.981	0.0844
SK	4	$3.3798 \cdot 10^{-64}$	$2.3988 \cdot 10^{-127}$	8.0000	0.0081
NM1	4	$6.4848 \cdot 10^{-181}$	$8.8311 \cdot 10^{-361}$	8.0000	0.0069
NM2	4	$3.0560 \cdot 10^{-229}$	$1.9612 \cdot 10^{-457}$	8.0000	0.0075
NM3	4	$2.8531 \cdot 10^{-267}$	$1.7095 \cdot 10^{-533}$	8.0000	0.0081

Table 5. Numerical results for $f_4(x)$ and $x_0 = 1$

method	it	$\|x_3 - \alpha\|$	$\|f(x_3)\|$	COC	CPU
ZCJT	3	$1.3542 \cdot 10^{-496}$	$1.1642 \cdot 10^{-1487}$	8.0000	0.0575
BAASA1	3	$3.3886 \cdot 10^{-592}$	$1.8240 \cdot 10^{-1774}$	8.0000	0.0581
BAASA2	7	$1.6438 \cdot 10^{-39}$	$2.0821 \cdot 10^{-116}$	2.0000	0.1540
KKSdA	3	$2.8142 \cdot 10^{-483}$	$1.0448 \cdot 10^{-1447}$	8.0000	0.0594
SK	3	$1.7382 \cdot 10^{-492}$	$2.4620 \cdot 10^{-1475}$	8.0000	0.0431
NM1	3	$3.2879 \cdot 10^{-501}$	$1.6661 \cdot 10^{-1501}$	8.0000	0.0569
NM2	4	$1.9335 \cdot 10^{-193}$	$3.3884 \cdot 10^{-578}$	2.2831	0.0663
NM3	3	$5.5417 \cdot 10^{-527}$	$7.9779 \cdot 10^{-1579}$	8.0000	0.0494

Table 6. Numerical results for $f_5(x)$ and $x_0 = 0.9$

method	it	$\|x_3 - \alpha\|$	$\|f(x_3)\|$	COC	CPU
ZCJT	5	$1.7022 \cdot 10^{-35}$	$7.2776 \cdot 10^{-69}$	14.183	0.097
BAASA1	6	$1.2364 \cdot 10^{-7}$	$3.8394 \cdot 10^{-13}$	8.0000	0.188
BAASA2	8	$9.8845 \cdot 10^{-6}$	$2.4539 \cdot 10^{-9}$	6.0159	0.159
KKSdA	6	$2.7924 \cdot 10^{-5}$	$1.9585 \cdot 10^{-8}$	8.0000	0.119
SK	4	$2.9970 \cdot 10^{-144}$	$2.2559 \cdot 10^{-286}$	3.4290	0.086
NM1	4	$7.2622 \cdot 10^{-341}$	$1.3246 \cdot 10^{-679}$	8.0000	0.093
NM2	4	$1.6300 \cdot 10^{-416}$	$6.6730 \cdot 10^{-831}$	8.0000	0.094
NM3	4	$5.6748 \cdot 10^{-423}$	$8.0881 \cdot 10^{-844}$	8.0000	0.095

Table 7. Numerical results for $f_6(x)$ and $x_0 = 0.75$

method	it	$\|x_3 - \alpha\|$	$\|f(x_3)\|$	COC	CPU
ZCJT	7	$1.1372 \cdot 10^{-24}$	$3.3738 \cdot 10^{-118}$	1.9960	0.301
BAASA1	6	$5.9075 \cdot 10^{-8}$	$1.2765 \cdot 10^{-34}$	8.0000	0.271
BAASA2	6	$5.9072 \cdot 10^{-8}$	$1.2762 \cdot 10^{-34}$	8.0000	0.269
KKSdA	4	$1.2927 \cdot 10^{-28}$	$6.4040 \cdot 10^{-138}$	7.9998	0.188
SK	5	$1.8431 \cdot 10^{-16}$	$3.7735 \cdot 10^{-77}$	14.045	0.243
NM1	4	$7.5331 \cdot 10^{-93}$	$4.3038 \cdot 10^{-459}$	8.0000	0.127
NM2	4	$4.0614 \cdot 10^{-93}$	$1.9604 \cdot 10^{-460}$	8.0000	0.122
NM3	4	$6.9922 \cdot 10^{-27}$	$2.9652 \cdot 10^{-129}$	7.9998	0.121

Obviously, the values of the COC columns confirm eighth-order of convergence of the new proposed methods. According to the numerical results computed for the inner three columns, all new methods are very competitive compared to the existing ones.

4. Dynamical comparison

Another very frequent way of comparing the iterative methods is the analysis of their basins of attraction in the complex plane (see, for example [6,12,16,18]). Through the basins of attraction analysis, researchers are able to visualize the areas of convergence of particular roots of $f(x)$ in the complex plane for the iterative methods under consideration. Here we only give a brief review of some basic concepts related to basins of attraction, while the underlying ideas as well as the description of the dynamical behaviour of the methods in more details can be found in [5, 18–20].

For a function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (where $\hat{\mathbb{C}}$ is the Riemann sphere), $z_0 \in \hat{\mathbb{C}}$ is a fixed point if $F(z_0) = z_0$. Fixed point can be attracting, repelling or neutral, if $|F'(z_0)| < 1$, $|F'(z_0)| > 1$ or $|F'(z_0)| = 1$, respectively. The orbit of any point z is defined as a set $orb(z) = \{z, F(z), F^2(z), \ldots\}$, and if there exist some point $\tilde{z}$ and $k \in \mathbb{N}$ where $F^k(\tilde{z}) = \tilde{z}$ and $F^s(\tilde{z}) \neq \tilde{z}$, $s < k$, then such point $\tilde{z}$ is called periodic with period k. Therefore, the fixed point is periodic with period 1.

If α is an attracting fixed point of F, then its corresponding basin of attraction can be defined as a set $A(\alpha)$ given by

$$A(\alpha) = \{z_0 \in \hat{\mathbb{C}} : F^n(z_0) \to \alpha,\ n \to \infty\},$$

which means that the basin of attraction consists of the starting points whose orbits tend to the attractor α. The set of such points whose orbits converge to any attractor is called the Fatou set, while the Julia set is its complementary set and it establishes the borders between different basins of attraction.

In the ideal cases, if a function has several distinct roots, every initial point should converge to the nearest root applying iterative method, and consequently, the basins boundaries should have smooth form. Nevertheless, for concrete functions and multiple iterative schemes, the dynamical behaviour of the methods are not so predictable, the overlapping of the basins of attraction

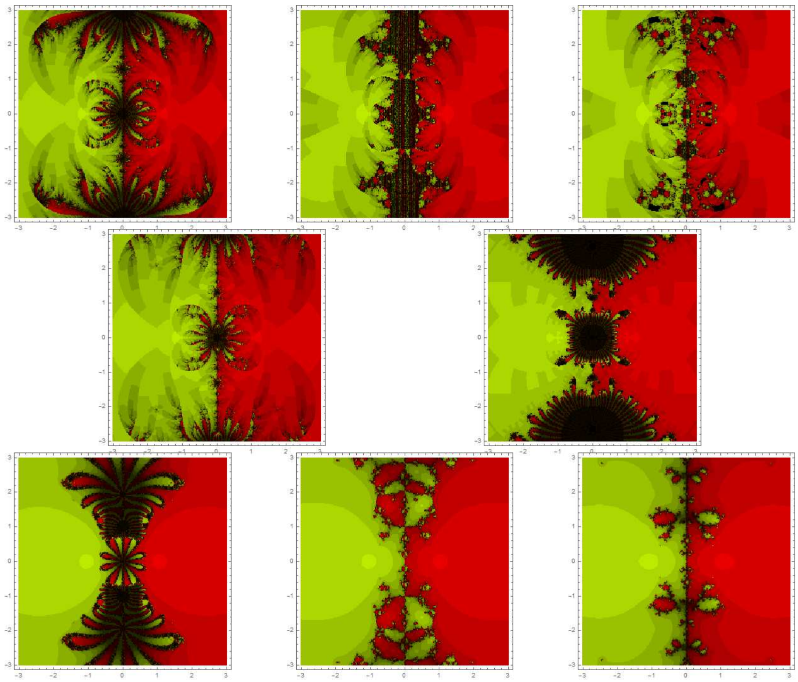

FIGURE 1. Basins of attraction of different methods for polynomial p_1 (the first row: ZCJT(left), BAASA1(middle), BAASA2(right), the second row: KKSdA(left), SK(right), the third row: NM1(left), NM2(middle), NM3(right))

is noticeable as well as the chaotic structure of the basins boundaries. Hence, iterative methods with less fractal 'decorations' along boundaries are considered as more desirable ones.

The following functions with associated multiple roots and their multiplicity are observed:

(1) $p_1(z) = (z^2 - 1)^2$; $\alpha_1 = 1, \alpha_2 = -1, m = 2$,
(2) $p_2(z) = (z^3+4z^2-10)^3$; $\alpha_1 \approx 1.3652, \alpha_{2,3} \approx -2.6826 \pm i \cdot 0.3582, m = 3$,
(3) $p_3(z) = (z^3 - z)^4$; $\alpha_1 = 0, \alpha_{2,3} = \pm 1, m = 4$.

In those examples, we consider the region $[-3, 3] \times [-3, 3]$ of the complex plane, with 256×256 equally distributed initial points. We picture the dynamical planes for every method described in the previous sections where each inital point is colored associated to the root which it converges to. If the method does not converge (here, this means that the distance after at most 100 iterations is still greater than 10^{-5} to any of the roots), than that point is marked black. The intensity of the color suggests the number of iterations (fewer iterations $-$ lower intensity).

Table 8. The number of black points (in %) for p_1, p_2 and p_3

method	$p_1(z)$	$p_2(z)$	$p_3(z)$	total average
ZCJT	0	1.794	1.448	1.081
BAASA1	0.003	0.027	0.018	0.016
BAASA2	1.511	0	0	0.504
KKSdA	0	0.397	0.366	0.254
SK	0.629	10.948	4.834	5.470
NM1	0	0	0	0
NM2	0	0	0.024	0.008
NM3	0	0	0	0

Tables 8, 9 and 10 show the computed values related to the depiction of the basins of attraction given in Figures 1, 2 and 3. Table 8 displays the percentage of the black points (out of 65536 starting points) for each graph. The values displayed in Table 9 represent the average number of iterations per starting point calculated without black starting points, which means that the average

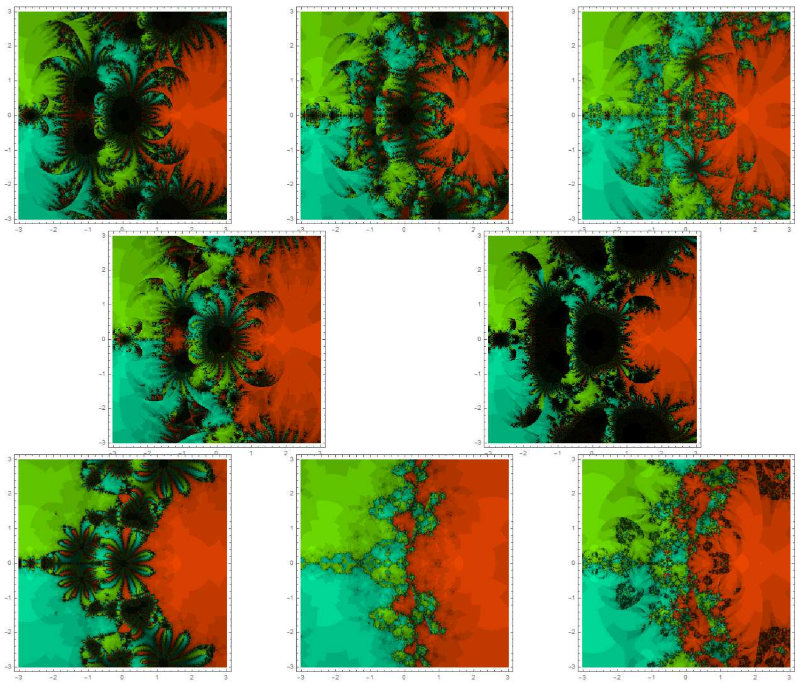

FIGURE 2. Basins of attraction of different methods for polynomial p_2 (the first row: ZCJT(left), BAASA1(middle), BAASA2(right), the second row: KKSdA(left), SK(right), the third row: NM1(left), NM2(middle), NM3(right))

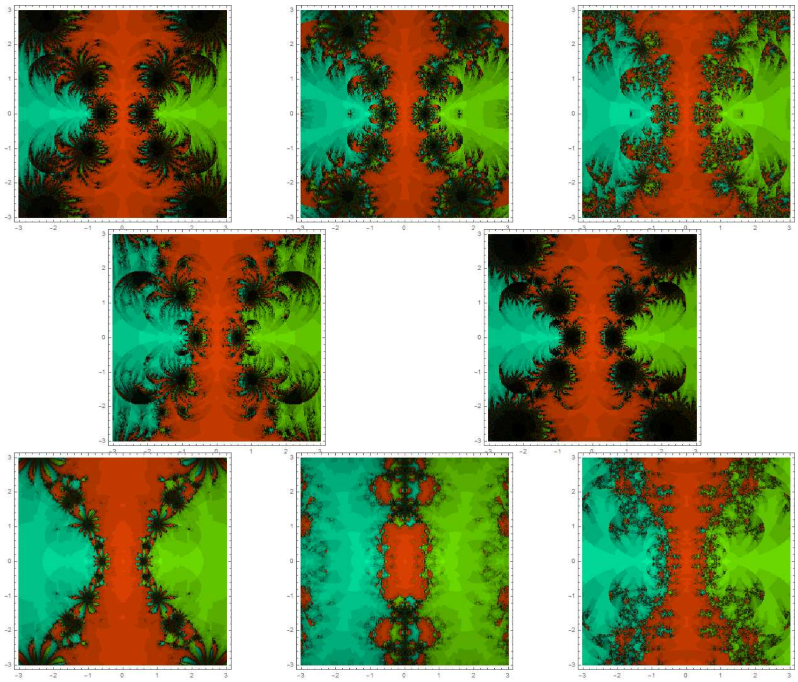

FIGURE 3. Basins of attraction of different methods for polynomial p_3 (the first row: ZCJT(left), BAASA1(middle), BAASA2(right), the second row: KKSdA(left), SK(right), the third row: NM1(left), NM2(middle), NM3(right))

does not take into account the number of iterations for inital points that do not reach the neighborhood of any root within 100 iterations. Table 10 presents the CPU time required for the depiction of each graph.

Table 9. The average number of iterations for p_1, p_2 and p_3

method	$p_1(z)$	$p_2(z)$	$p_3(z)$	total average
ZCJT	6.552	15.136	12.767	11.458
BAASA1	5.565	8.721	8.385	7.557
BAASA2	3.895	5.361	5.937	5.065
KKSdA	4.458	9.276	8.426	7.387
SK	11.179	20.128	15.625	15.644
NM1	6.609	7.794	5.353	6.585
NM2	3.568	3.959	5.177	4.235
NM3	3.577	5.347	5.166	4.697

Table 10. The CPU time (in seconds) for p_1, p_2 and p_3

method	$p_1(z)$	$p_2(z)$	$p_3(z)$	total average
ZCJT	1076.358	1527.498	1485.330	1363.062
BAASA1	1169.550	1317.174	1420.686	1302.470
BAASA2	1182.174	1258.020	1357.482	1265.892
KKSdA	1050.516	1402.170	1432.722	1295.136
SK	1204.392	1767.078	1632.984	1534.818
NM1	1032.780	1218.204	1298.514	1183.166
NM2	1021.500	1148.262	1288.422	1152.728
NM3	1051.200	1252.530	1310.922	1204.884

According to these results, all new methods are very competitive with the previously developed methods. For example, methods NM2 and NM3 are the best performers in terms of the total average of iterations, followed by BAASA2 and NM1. Note that NM1 and NM3 are the only methods without black initial points. Furthermore, the best CPU time results are associated with the new methods.

5. Conclusion

In this paper, we have considered a new optimal three-step iterative family of multiple root finders and compared some special members of the family to several recently published Newton-type optimal methods of eighth order. The construction of the new iterative scheme is based on one function and three derivative evaluations per iteration, which is a unique structure of the algorithm of the eighth convergence order. The eighth-order is empirically checked in the numerical section. The advantage of the proposed methods is their good dynamical performace. For some special cases of the new family, a dynamical analysis suggests better stability and wider basins of attraction compared to existing methods.

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