J. Korean Math. Soc. **59** (2022), No. 6, pp. 1139–1151
https://doi.org/10.4134/JKMS.j210669
pISSN: 0304-9914 / eISSN: 2234-3008

DIRICHLET EIGENVALUE PROBLEMS UNDER MUSIELAK-ORLICZ GROWTH

Allami Benyaiche and Ismail Khlifi

Abstract. This paper studies the eigenvalues of the $G(\cdot)$-Laplacian Dirichlet problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{g(x,|\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda\left(\dfrac{g(x,|u|)}{|u|}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and g is the density of a generalized Φ-function $G(\cdot)$. Using the Lusternik-Schnirelmann principle, we show the existence of a nondecreasing sequence of nonnegative eigenvalues.

1. Introduction

In the fields of partial differential equations and the calculus of variations, there has been much research on non-standard growth problems, such as the eigenvalue problems [5]. The study of eigenvalue problems relies on the Lusternik-Schnirelmann (L-S) theory of critical points for an even functional on a manifold. The presentations of this theory, in both finite and infinite-dimensional spaces, can be found in [1, 4, 14, 15].

A mathematical prototype for nonlinear elliptic eigenvalue problems is expressible by involving the p-Laplacian operator

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda\left(|u|^{p-2}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < \infty$ and Ω is a bounded domain of $\mathbb{R}^N$. The problem (1.1) has attracted much attention and has been extensively studied in the literature (see for examples [2, 7, 9]). One of the important consequences of the Lusternik-Schnirelmann principle is the existence, exactly as for the classical Laplace

Received November 5, 2021; Revised June 29, 2022; Accepted August 30, 2022.
2020 *Mathematics Subject Classification.* Primary 35P30, 47J10, 49R05.
Key words and phrases. Dirichlet eigenvalue problems, Lusternik-Schnirelmann principle, $G(\cdot)$-Laplacian, Musielak-Orlicz growth, generalized Φ-function, generalized Orlicz-Sobolev space.

operator ($p = 2$), of an increasing sequence of eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \quad \lambda_i \to \infty.$$

Later on, this result has been generalized to variable exponent and Orlicz cases

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda\left(|u|^{p(x)-2}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\text{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda\left(\dfrac{g(|u|)}{|u|}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $x \to p(x)$ is a continuous function on $\overline{\Omega}$ such that $1 < p(x) < \infty$ and $t \to g(t)$ is the density of a Φ-function G (see [6, 12, 13]).

One naturally asks whether a similar result holds in the Musielak-Orlicz case. For this, we consider the following eigenvalue problem under generalized Orlicz growth

$$\begin{cases} -\text{div}\left(\dfrac{g(x,|\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda\left(\dfrac{g(x,|u|)}{|u|}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g(x, \cdot)$ is the right-hand derivative of a Φ-function $G(x, \cdot)$. This situation covers not only the variable exponent $G(x,t) = t^{p(x)}$ and Orlicz case $G(x,t) = G(t)$, but also the variable exponent perturbation $G(x,t) = t^{p(x)}\ln(e+t)$, the double phase $G(x,t) = t^p + a(x)t^q$ and their various combinations (see [8]). Note that, some particular vector inequalities are helpful in the study of the eigenvalue problem for the p-Laplacian. In our situation, a lack of these inequalities and homogeneity are a major source of difficulties. To overcome these problems, we developed a method inspired by Lieberman's pioneering article [10], which allows us to apply the L-S principle for establish the existence of a nondecreasing sequence of nonnegative eigenvalue tending to infinity of the problem (1.2) (see Theorem 3.7).

2. Musielak-Orlicz-Sobolev spaces

To deal with the problem (1.2), we need Musielak-Orlicz-Sobolev spaces. Most of the results concerning these spaces are given in Musielak's monograph [11], hence the alternative name of Musielak-Orlicz spaces.

Definition 2.1. A function $G : \Omega \times [0,\infty) \longrightarrow [0,\infty]$ is called a generalized Φ-function, denoted by $G(\cdot) \in \Phi(\Omega)$, if the following conditions hold:

- For each $t \in [0,\infty)$, the function $G(\cdot, t)$ is measurable.
- For a.e. $x \in \Omega$, the function $G(x, \cdot)$ is a Φ-function, i.e.,
  (1) $G(x,0) = \lim_{t\to 0^+} G(x,t) = 0$ and $\lim_{t\to\infty} G(x,t) = \infty$;
  (2) $G(x, \cdot)$ is increasing and convex.

Note that, a generalized Φ-function can be represented as

$$G(x,t)=\int_0^t g(x,s)\,\mathrm{d}s,$$

where $g(x,\cdot)$ is the right-hand derivative of $G(x,\cdot)$. Furthermore for each $x\in\Omega$, the function $g(x,\cdot)$ is right-continuous and nondecreasing.

Assumptions. We say that $G(\cdot)$ satisfies
(SC) : If for a.e. $x\in\Omega$, the function $t\to g(x,t)$ is a $C^1(\mathbb{R}^+)$ and there exist two constants $g_0, g^0>0$ such that,

$$g_0\le \frac{tg'(x,t)}{g(x,t)}\le g^0.$$

(A_0) : If there exists a constant $c_0>1$ such that,

$$\frac{1}{c_0}\le G(x,1)\le c_0 \text{ for a.e. } x\in\Omega.$$

(A_1) : If there exists $C>0$ such that, for every $x,y\in B_R\subset\Omega$ with $R\le 1$, we have

$$G_{B_R}(x,t)\le CG_{B_R}(y,t) \quad \text{when } G^-_{B_R}(t)\in\left[1,\frac{1}{R^N}\right],$$

where $G^-_{B_R}(t):=\inf_{B_R}G(x,t)$.

Remark 2.2. 1) Note that (A_1) corresponds to local log-Holder continuity in the variable exponent case (see Proposition 7.1.2 in [8]). In the double phase case $G(x,t)=t^P+a(x)t^q$, condition (A_1) is equivalent to $a(y)-a(x)\le C|y-x|^{\frac{N}{p}(q-p)}$ (see Proposition 7.2.2 in [8]).

2) The condition (SC) implies

$$g_0+1\le\frac{tg(x,t)}{G(x,t)}\le g^0+1.$$

So, we have the following inequalities [3]

$$\sigma^{g_0+1}G(x,t)\le G(x,\sigma t)\le\sigma^{g^0+1}G(x,t) \text{ for } x\in\Omega,\ t\ge 0 \text{ and } \sigma\ge 1. \tag{2.1}$$

$$\sigma^{g^0+1}G(x,t)\le G(x,\sigma t)\le\sigma^{g_0+1}G(x,t) \text{ for } x\in\Omega,\ t\ge 0 \text{ and } \sigma\le 1. \tag{2.2}$$

Definition 2.3. We define $G^*(\cdot)$ the conjugate Φ-function of $G(\cdot)$, by

$$G^*(x,s):=\sup_{t\ge 0}(st-G(x,t)) \text{ for } x\in\Omega \text{ and } s\ge 0.$$

Note that $G^*(\cdot)$ is also a generalized Φ-function and can be represented as

$$G^*(x,t)=\int_0^t g^{-1}(x,s)\,\mathrm{d}s,$$

with $g^{-1}(x,s):=\sup\{t\ge 0: g(x,t)\le s\}$.

Remark 2.4. If $G(\cdot)$ satisfies (SC), then $G^*(\cdot)$ satisfies also (SC), as follows

$$\frac{g^0+1}{g^0} \leq \frac{tg^{-1}(x,t)}{G^*(x,t)} \leq \frac{g_0+1}{g_0}. \tag{2.3}$$

The functions $G(\cdot)$ and $G^*(\cdot)$ satisfies the following Young inequality

$$st \leq G(x,t) + G^*(x,s) \text{ for } x \in \Omega \text{ and } s,t \geq 0.$$

Further, we have the equality if $s = g(x,t)$ or $t = g^{-1}(x,s)$.

Definition 2.5. Let $G(\cdot) \in \Phi(\mathbb{R}^N)$, the generalized Orlicz space, also called Musielak-Orlicz space, is defined as the set

$$L^{G(\cdot)}(\Omega) := \{u \in L^0(\Omega) : \lim_{\lambda \to 0} \int_\Omega G(x, \lambda|u|)\,\mathrm{d}x = 0\},$$

where $L^0(\Omega)$ is the set of measurable functions in Ω. If $G(\cdot)$ satisfies (SC), then

$$L^{G(\cdot)}(\Omega) = \{u \in L^0(\Omega) : \int_\Omega G(x,|u|)\,\mathrm{d}x < \infty\}.$$

Remark 2.6. On the generalized Orlicz space, we define the following norms:

- Luxembourg norm: $||u||_{G(\cdot)} = \inf\{\lambda > 0 : \int_\Omega G(x, \frac{|u|}{\lambda})\,\mathrm{d}x \leq 1\}$.
- Orlicz norm: $||u||^0_{G(\cdot)} = \sup\{|\int_\Omega u(x)v(x)\,\mathrm{d}x| : \int_\Omega G^*(x,|v|)\,\mathrm{d}x \leq 1\}$.

These norms are equivalent, precisely, we have

$$||u||_{G(\cdot)} \leq ||u||^0_{G(\cdot)} \leq 2||u||_{G(\cdot)}.$$

The functions $G(\cdot)$ and $G^*(\cdot)$ satisfy the Hölder inequality

$$\left|\int_\Omega u(x)v(x)\,\mathrm{d}x\right| \leq 2||u||_{G(\cdot)}||v||_{G^*(\cdot)} \text{ for } u \in L^{G(\cdot)}(\Omega) \text{ and } v \in L^{G^*(\cdot)}(\Omega).$$

The following lemmas establish properties of convergent sequences in generalized Orlicz spaces (see [8]).

Lemma 2.7. *Let $G(\cdot) \in \Phi(\Omega)$. For any sequence $\{u_i\}_i$ in $L^{G(\cdot)}(\Omega)$, we have the following properties: If $G(\cdot)$ satisfies (SC), then*

$$||u_i||_{G(\cdot)} \to 0 \ (\textit{resp. } 1;\infty) \Longleftrightarrow \int_\Omega G(x,|u_i(x)|)\,\mathrm{d}x \to 0 \ (\textit{resp. } 1;\infty).$$

Lemma 2.8. *Let $G(\cdot) \in \Phi(\Omega)$ and $\{u_i\}_i$ be a sequence of measurable functions u_i. Assume the sequence $\{u_i\}_i$ converges almost everywhere to a measurable function u, and is dominated by a function $h \in L^{G(\cdot)}(\Omega)$. Then all u_i as well as u are in $L^{G(\cdot)}(\Omega)$ and the sequence $\{u_i\}_i$ converges to u in $L^{G(\cdot)}(\Omega)$.*

Lemma 2.9. *Let $G(\cdot) \in \Phi(\Omega)$ satisfies (SC) and $\{u_i\}_i$ be a sequence in $L^{G(\cdot)}(\Omega)$. If $u_i \to u$ in $L^{G(\cdot)}(\Omega)$, then there exist a subsequence $\{u_{i_j}\}_j$ and a function $h \in L^{G(\cdot)}(\Omega)$ such that to $u_{i_j} \to u$, for a.e. in Ω and for all j, $|u_{i_j}(x)| \leq h(x)$ for a.e. in Ω.*

Proof. Since $u_i \to u$ in $L^{G(\cdot)}(\Omega)$, then by Lemma 2.7, we have

$$\int_\Omega G(x, |u_i(x) - u(x)|)\, \mathrm{d}x \to 0.$$

So, for any $\delta > 0$, let $A_{\delta,i} := \{x \in \Omega : |u_i(x) - u(x)| > \delta\}$. Using the inequalities (2.1), (2.2) and the condition (A_0), we get

$$\begin{aligned}\frac{1}{c_0} \min(\delta^{g_0}, \delta^{g^0})|A_{\delta,i}| &\le \int_{A_{\delta,i}} G(x, \delta)\, \mathrm{d}x \\ &\le \int_{A_{\delta,i}} G(x, |u_i - u|)\, \mathrm{d}x \\ &\le \int_\Omega G(x, |u_i - u|)\, \mathrm{d}x.\end{aligned}$$

Hence, $|A_{\delta,i}| \to 0$, and $u_i \to u$ in measure. Therefore, there exists a subsequence (u_{i_j}) of (u_i) which converge to u, for a.e. in Ω.

Then we can define

$$h := \sum_{j=1}^{\infty} |u_{i_j} - u|$$

and it can be seen that $h \in L^{G(\cdot)}(\Omega)$ and h is a dominating function of (u_{i_j}) for all j. □

Definition 2.10. We define the generalized Orlicz-Sobolev space by

$$W^{1,G(\cdot)}(\Omega) := \{u \in L^{G(\cdot)}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega) : |\nabla u| \in L^{G(\cdot)}(\Omega) \text{ in the distribution sense}\},$$

equipped with the norm

$$||u||_{1,G(\cdot)} = ||u||_{G(\cdot)} + ||\nabla u||_{G(\cdot)}.$$

Definition 2.11. $W_0^{1,G(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,G(\cdot)}(\Omega)$.

Next, we recall the norm version of the Poincare inequality, which will be investigated in this work (see [8]).

Theorem 2.12. *Let Ω be a bounded set of $\mathbb{R}^N$ and $G(\cdot) \in \Phi(\Omega)$ satisfy (SC), (A_0) and (A_1). For every $u \in W_0^{1,G(\cdot)}(\Omega)$, we have*

$$||u||_{G(\cdot)} \le C||\nabla u||_{G(\cdot)}.$$

In particular, $||\nabla u||_{G(\cdot)}$ is a norm on $W_0^{1,G(\cdot)}(\Omega)$ and it is equivalent to the norm $||u||_{1,G(\cdot)}$.

The following compact embedding theorem for Musielak-Sobolev spaces is given by P. Hasto [8].

Theorem 2.13. *Let $G(\cdot) \in \Phi(\mathbb{R}^N)$ satisfy (SC), (A_0) and (A_1) and let Ω be bounded. Then*

$$W_0^{1,G(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{G(\cdot)}(\Omega).$$

3. Eigenvalue problems for the G(·)-Laplacian

3.1. Lusternik-Schnirelmann principle (L-S principle)

We recall here a version of the Lusternik-Schnirelmann principle, which Browder discussed in [4] and Zeidler in [14, 15].

Theorem 3.1. *Let X be a real reflexive Banach space. If A, B are two functionals on X satisfying the following properties*:

(LS_1): *$A, B : X \to \mathbb{R}$ are even functionals and that $A, B \in C^1(X, \mathbb{R})$ with $A(0) = B(0) = 0$.*

(LS_2): *A' is strongly continuous (i.e., $u_i \rightharpoonup u$ in X implies $A'(u_i) \to A'(u)$) and,*

$$\langle A'(u), u\rangle = 0, \ \ u \in \overline{coS_B} \ \ \textit{implies} \ \ A(u) = 0,$$
$$A(u) = 0, \ \ u \in \overline{coS_B} \ \ \textit{implies} \ \ u = 0,$$

where $\overline{coS_B}$ is the closed convex hull of $S_B := \{u \in X, \ B(u) = 1\}$.

(LS_3): *B' is continuous, bounded and satisfies (S_0), i.e., as $i \to \infty$,*

$$u_i \rightharpoonup u, \ B'(u_i) \rightharpoonup v, \ \langle B'(u_i), u_i\rangle \to \langle v, u\rangle \quad \textit{implies} \quad u_i \to u.$$

(LS_4): *The level set S_B is bounded and $u \not\equiv 0$ implies*

$$\langle B'(u), u\rangle > 0, \quad \lim_{t\to\infty} B(tu) = \infty, \quad \inf_{u\in S_B} \langle B'(u), u\rangle > 0.$$

Then the eigenvalue problem

$$A'(u) = \mu B'(u), \quad u \in S_B, \ \ \mu \in \mathbb{R}, \tag{3.1}$$

admits a sequence of eigenpairs $\{u_i, \mu_i\}$ such that $u_i \rightharpoonup u$, $\mu \to 0$ as $i \to \infty$ and $\mu_i \neq 0$ for all i.

3.2. Application of L-S principle in $W_0^{1,G(\cdot)}(\Omega)$

In the sequel, let $G(\cdot) \in \Phi(\mathbb{R}^N)$ satisfy (SC), (A_0), (A_1), $G(x, \cdot) \in C^1([0, \infty))$ for every $x \in \Omega$ and C a generic constant which may change from line to line. We consider the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{g(x, |\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda\left(\dfrac{g(x, |u|)}{|u|}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $g(x, \cdot)$ is the derivative of $G(x, \cdot)$. The existence of a weak solution to the Dirichlet-Sobolev problem associated of (3.2) has been studied in [3].

Definition 3.2. Let $\lambda \in \mathbb{R}$ and $u \in W_0^{1,G(\cdot)}(\Omega)$. (u, λ) is called a solution of the problem (3.2) if

$$\int_\Omega \frac{g(x, |\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega \frac{g(x, |u|)}{|u|} uv \, \mathrm{d}x, \ \ \forall v \in W_0^{1,G(\cdot)}(\Omega). \tag{3.3}$$

If (u, λ) is a solution of the problem (3.2) and $u \not\equiv 0$, as usual, we call λ and u an eigenvalue and an eigenfunction corresponding to λ of (3.2), respectively.

In what follows we use the previous version of the L-S principle in order to prove the existence of a sequence of eigenvalues for problem (3.2). For this reason, we define on $X = W_0^{1,G(\cdot)}(\Omega)$ the functionals

$$A(u) := \int_\Omega G(x, |u|)\,\mathrm{d}x, \tag{3.4}$$

$$B(u) := \int_\Omega G(x, |\nabla u|)\,\mathrm{d}x. \tag{3.5}$$

Lemma 3.3. *Let A and B be defined in* (3.4), (3.5). *Then A and B satisfies* (LS_1).

Proof. Let $G(\cdot) \in \Phi(\mathbb{R}^N)$. By definition of a generalized Φ-function, we have $G(x, 0) = 0$ which implies $A(0) = B(0) = 0$. The C^1-smooth regularity of the functionals A and B follows by computing the Gateaux derivatives of A and B at $u \in W_0^{1,G(\cdot)}(\Omega)$ in the direction $v \in W_0^{1,G(\cdot)}(\Omega)$. Precisely, for every $u, v \in W_0^{1,G(\cdot)}(\Omega)$, we have

$$\langle A'(u), v\rangle = \int_\Omega \frac{g(x, |u|)}{|u|} uv\,\mathrm{d}x, \tag{3.6}$$

$$\langle B'(u), v\rangle = \int_\Omega \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v\,\mathrm{d}x. \tag{3.7}$$

Then A and B satisfies (LS_1). □

Lemma 3.4. *Let A be defined in* (3.4). *Then A' satisfies* (LS_2).

Proof. Let $u \in W_0^{1,G(\cdot)}(\Omega)$, by the condition (SC), we have

$$A(u) = \int_\Omega G(x, |u|)\,\mathrm{d}x \le \frac{1}{g_0 + 1} \int_\Omega g(x, |u|)|u|\,\mathrm{d}x \le \langle A'(u), u\rangle.$$

Then, if $\langle A'(u), u\rangle = 0$ implies $A(u) = 0$.

Next, we assume $A(u) = 0$. By Proposition 2.2.7 in [8], there exists $\tilde{G}(\cdot) \in \Phi(\Omega)$ with $\tilde{G}(\cdot) \approx G(\cdot)$ which is a strictly increasing. So, we have

$$0 \le \int_\Omega \tilde{G}(x, |u|)\,\mathrm{d}x \le C \int_\Omega G(x, |u|)\,\mathrm{d}x = 0 \quad \text{implies} \quad u = 0.$$

To end the proof of Lemma 3.4, it remains for us to prove that A' is strongly continuous. Let $u_i \rightharpoonup u$ in $W_0^{1,G(\cdot)}(\Omega)$, we need to show that

$$A'(u_i) \longrightarrow A(u) \text{ in } W_0^{1,G(\cdot)}(\Omega)^*.$$

For every $v \in W_0^{1,G(\cdot)}(\Omega)$, using the Holder inequality, we get

$$\left|\int_\Omega \frac{g(x, |u_i|)}{|u_i|} u_i v\,\mathrm{d}x - \int_\Omega \frac{g(x, |u|)}{|u|} uv\,\mathrm{d}x\right| \tag{3.8}$$
$$\le \int_\Omega \left|\frac{g(x, |u_i|)}{|u_i|} u_i - \frac{g(x, |u|)}{|u|} u\right| |v|\,\mathrm{d}x$$

$$\leq 2 \left|\left| \frac{g(x,|u_i|)}{|u_i|}u_i - \frac{g(x,|u|)}{|u|}u \right|\right|_{G^*(\cdot)} ||v||_{G(\cdot)}.$$

Since $u_i \rightharpoonup u$ in $W_0^{1,G(\cdot)}(\Omega)$, then by the compact embedding Theorem 2.13, we have $u_i \to u$ in $L^{G(\cdot)}(\Omega)$. So, using the reverse dominate convergence theorem, Lemma 2.9, there are a subsequence $\{u_{i_j}\}_j$ and a function $h \in L^{G(\cdot)}(\Omega)$ such that $u_{i_j} \to u$ for a.e. in Ω and $|u_{i_j}| \leq h$.

As the function $t \to g(x,t)$ is continuous and increasing, we have

$$\frac{g(x,|u_{i_j}|)}{|u_{i_j}|}u_{i_j} \longrightarrow \frac{g(x,|u|)}{|u|}u \text{ for a.e. in } \Omega \text{ and } \frac{g(x,|u_{i_j}|)}{|u_{i_j}|}u_{i_j} \leq \frac{g(x,|h|)}{|h|}h.$$

Note that $\frac{g(x,|h|)}{|h|}h \in L^{G^*(\cdot)}(\Omega)$. Indeed, by the Young equality and the condition (SC), we have

$$\begin{aligned}\int_\Omega G^*(x, \left|\frac{g(x,|h|)}{|h|}h\right|)\,\mathrm{d}x &= \int_\Omega G^*(x, g(x,|h|))\,\mathrm{d}x \\ &= \int_\Omega g(x,|h|)|h| - G(x,|h|)\,\mathrm{d}x \\ &\leq g^0 \int_\Omega G(x,|h|)\,\mathrm{d}x \\ &< \infty.\end{aligned}$$

Then, by the dominate convergence theorem, Lemma 2.8, we have

$$\frac{g(x,|u_{i_j}|)}{|u_{i_j}|}u_{i_j} \longrightarrow \frac{g(x,|u|)}{|u|}u \text{ in } L^{G^*(\cdot)}(\Omega).$$

Hence, by inequality (3.8), we have

$$\int_\Omega \frac{g(x,|u_{i_j}|)}{|u_{i_j}|}u_{i_j} v\,\mathrm{d}x \longrightarrow \int_\Omega \frac{g(x,|u|)}{|u|}uv\,\mathrm{d}x.$$

Since the weak limit is independent of the choice of the subsequence, it follows that

$$\int_\Omega \frac{g(x,|u_i|)}{|u_i|}u_i v\,\mathrm{d}x \longrightarrow \int_\Omega \frac{g(x,|u|)}{|u|}uv\,\mathrm{d}x.$$

Therefore $A'(u_i) \to A(u)$ in $W_0^{1,G(\cdot)}(\Omega)^*$. □

Inspired by the proof of Theorem 1.7 of Lieberman [10], we have:

Lemma 3.5. *Let B be defined in* (3.5)*. Then B' satisfies* (LS_3).

Proof. Using the Hölder inequality, we have

$$\begin{aligned}||B'||_{\left(W_0^{1,G(\cdot)}(\Omega)\right)^*} &= \sup\{\langle B'(u), v\rangle;\ ||v||_{1,G(\cdot)} \leq 1\} \\ &\leq \sup\left|\int_\Omega \frac{g(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla v\,\mathrm{d}x\right| \\ &\leq 2||g(x,|\nabla u|)||_{G^*(\cdot)}||v||_{1,G(\cdot)}.\end{aligned} \tag{3.9}$$

Using the equivalent between Luxembourg norm and Orlicz norm, and the Young inequality, we have

$$||g(x,|\nabla u|)||_{G^*(\cdot)} \leq ||g(x,|\nabla u|)||^0_{G^*(\cdot)} \leq 1 + \int_\Omega G^*(x, g(x,|\nabla u|)\,\mathrm{d}x.$$

Then, by the Young equality and the condition (SC), we obtain

$$||g(x,|\nabla u|)||_{G^*(\cdot)} \leq g^0 \int_\Omega G(x,|\nabla u|)\,\mathrm{d}x.$$

So, by the inequality (3.9), we get

$$||B'||_{W_0^{1,G(\cdot)}(\Omega)^*} \leq C\left(g^0\int_\Omega G(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}x + 1\right)||v||_{1,G(\cdot)}.$$

Hence B' is bounded. Moreover, a similar argument to the one we used to prove (LS_2), we get the continuity of B'.

Complete the proof of Lemma 3.5, that is, prove that B satisfies condition (S_0). Let $\{u_i\}_i$ be a sequence in $W_0^{1,G(\cdot)}(\Omega)$ such that

$$u_i \rightharpoonup u\,,\ \ B'(u_i) \rightharpoonup v \ \text{ and } \ \langle B'(u_i), u_i\rangle \to \langle v, u\rangle$$

for some $v \in W_0^{1,G(\cdot)}(\Omega)^*$ and $u \in W_0^{1,G(\cdot)}(\Omega)$. Then, we have

$$\begin{aligned}(3.10)\qquad &\langle B'(u_i) - B'(u), u_i - u\rangle \\ &= \langle B'(u_i), u_i\rangle - \langle B'(u_i), u\rangle - \langle B'(u), u_i - u\rangle \to 0.\end{aligned}$$

On the other hand, from the condition (SC) and Cauchy-Schwarz inequality, we have for $\theta_t = tu + (1-t)u_i$, $t \in (0,1)$

$$\begin{aligned}&\left(\frac{g(x,|\nabla u|)}{|\nabla u|}\nabla u - \frac{g(x,|\nabla u_i|)}{|\nabla u_i|}\nabla u_i\right)\cdot(\nabla u - \nabla u_i)\\ =&\left(\int_0^1 \frac{\partial}{\partial t}\left(\frac{g(x,|\nabla\theta_t|)}{|\nabla\theta_t|}\nabla\theta_t\right)\mathrm{d}t\right)\cdot(\nabla u - \nabla u_i)\\ =&\ |\nabla u - \nabla u_i|^2\int_0^1 \frac{g(x,|\nabla\theta_t|)}{|\nabla\theta_t|}\,\mathrm{d}t\\ &+\int_0^1 g(x,|\nabla\theta_t|)\left(\frac{|\nabla\theta_t|g'(x,|\nabla\theta_t|)}{g(x,|\nabla\theta_t|)} - 1\right)\frac{(\nabla\theta_t\cdot(\nabla u - \nabla u_i))^2}{|\nabla\theta_t|^3}\,\mathrm{d}t\\ \geq&\ \min(1,g_0)|\nabla u - \nabla u_i|^2\int_0^1\frac{g(x,|\nabla\theta_t|)}{|\nabla\theta_t|}\,\mathrm{d}t.\end{aligned}$$

Which implies

$$\langle B'(u_i) - B'(u), u_i - u\rangle \geq \min(1,g_0)\int_\Omega\int_0^1\frac{g(x,|\nabla\theta_t|)}{|\nabla\theta_t|}|\nabla u - \nabla u_i|^2\,\mathrm{d}t\mathrm{d}x.$$

Now we write $S_1 = \{x \in \Omega, |\nabla u - \nabla u_i| \leq 2|\nabla u|\}$ and $S_2 = \{x \in \Omega, |\nabla u - \nabla u_i| > 2|\nabla u|\}$. Then $S_1 \cup S_2 = \Omega$ and

$$\begin{aligned} &\frac{1}{2}|\nabla u| \leq |\nabla \theta_t| \leq 3|\nabla u| && \text{on } S_1 \ \text{ for } t \geq \frac{3}{4}, \\ &\frac{1}{4}|\nabla u - \nabla u_i| \leq |\nabla \theta_t| \leq 3|\nabla u - \nabla u_i| && \text{on } S_2 \ \text{ for } t \leq \frac{1}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} &\langle B'(u_i) - B'(u), u_i - u\rangle \\ &\geq C\biggl(\int_{S_1} \frac{g(x,|\nabla u|)}{|\nabla u|} |\nabla u - \nabla u_i|^2 \,\mathrm{d}x + \int_{S_2} G(x, |\nabla u - \nabla u_i|)\,\mathrm{d}x \biggr). \end{aligned} \tag{3.11}$$

Hence

$$\int_{S_2} G(x, |\nabla u - \nabla u_i|)\,\mathrm{d}x \leq C\langle B'(u_i) - B'(u), u_i - u\rangle. \tag{3.12}$$

To estimate the integrals over S_1, using the condition (SC), $t \to g(x,t)$ is a nondecreasing function and the Hölder inequality in $L^2(S_1)$, we have

$$\begin{aligned} &\int_{S_1} G(x, |\nabla u - \nabla u_i|)\,\mathrm{d}x \\ &\leq C \int_{S_1} g(x, |\nabla u - \nabla u_i|)|\nabla u - \nabla u_i|\,\mathrm{d}x \\ &\leq C \int_{S_1} \left(\frac{g(x,|\nabla u|)}{|\nabla u|} |\nabla u| \right)^{\frac{1}{2}} |\nabla u - \nabla u_i| \left(g(x, |\nabla u - \nabla u_i|) \right)^{\frac{1}{2}} |\,\mathrm{d}x \\ &\leq C \left(\int_{S_1} \frac{g(x,|\nabla u|)}{|\nabla u|} |\nabla u - \nabla u_i|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{S_1} g(x, |\nabla u|)|\nabla u|\,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{S_1} \frac{g(x,|\nabla u|)}{|\nabla u|} |\nabla u - \nabla u_i|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{S_1} G(x, |\nabla u|)\,\mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, using the inequality (3.11), we have

$$\begin{aligned} &\int_{S_1} G(|\nabla u - \nabla u_i|)\,\mathrm{d}x \\ &\leq C \left(\langle B'(u_i) - B'(u), u_i - u\rangle \right)^{\frac{1}{2}} \left(\int_{\Omega} G(x, |\nabla u|)\,\mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

Collecting the inequalities (3.11), (3.12), (3.13), we have

$$\begin{aligned} &\int_{\Omega} G(|\nabla u - \nabla u_i|)\,\mathrm{d}x \\ &\leq C\biggl(\left(\langle B'(u_i) - B'(u), u_i - u\rangle \right)^{\frac{1}{2}} \left(\int_{\Omega} G(|\nabla u|) + G(x, |u|)\,\mathrm{d}x \right)^{\frac{1}{2}} \end{aligned}$$

$$+ \langle B'(u_i) - B'(u), u_i - u\rangle\Big).$$

Therefore, by the inequality (3.10), Theorem 2.12 and Lemma 2.7, we have $u_i \to u$ in $W_0^{1,G(\cdot)}(\Omega)$. □

Lemma 3.6. *Let B be defined in* (3.5)*. Then B and B' satisfies (LS_4).*

Proof. Let $u \neq 0$, by the inequalities (2.1) and (2.2), for all $t \in \mathbb{R}^+$ we have

$$B(tu) = \int_\Omega G(x, |\nabla tu|)\,\mathrm{d}x \leq \max(t^{g^0+1}, t^{g_0+1}) \int_\Omega G(x, |\nabla u|)\,\mathrm{d}x.$$

Then $\lim_{t\to+\infty} B(tu) = +\infty$.

Next, by the condition (SC) and Proposition 2.2.7 in [8], there exists $\tilde{G}(\cdot) \in \Phi(\Omega)$ with $\tilde{G}(\cdot) \approx G(\cdot)$ which is a strictly increasing. Then, we have

$$\begin{aligned}\langle B'(u), u\rangle &= \int_\Omega g(x, |\nabla u|)|\nabla u|\,\mathrm{d}x \\ &\geq (g_0+1)\int_\Omega G(x, |\nabla u|)\,\mathrm{d}x \\ &\geq C\int_\Omega \tilde{G}(x, |\nabla u|)\,\mathrm{d}x \\ &\geq 0.\end{aligned}$$

Note that, the last inequality result from the fact if $u \in W_0^{1,G(\cdot)}$ and $\nabla u = 0$ then $u = 0$.

So, if $u \in S_B$, then, by the previous inequality, we have $\langle B'(u), u\rangle \geq g_0 + 1$.

Therefore B and B' satisfies the hypothesis (LS_4). □

Theorem 3.7. *Let $G(\cdot) \in \Phi(\mathbb{R}^N)$ satisfy (SC), (A_0) and (A_1). Let A and B be the two functionals defined in* (3.4), (3.5)*. Then there exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_i\}_i$ of* (3.2) *such that $\lambda_i \to \infty$ as $i \to \infty$.*

Proof. By Lemmas 3.3-3.6 combined with Theorem 3.1, there exists a nonnegative nonincreasing sequence $\{\mu\}_i$ such that $\mu_i \to 0$ as $i \to \infty$ and each μ_i is an eigenvalue of $A'(u) = \mu B'(u)$ which means, for every $v \in W_0^{1,G(\cdot)}(\Omega)$, we have

$$\int_\Omega \frac{g(x,|u|)}{|u|} uv\,\mathrm{d}x = \mu \int_\Omega \frac{g(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla v\,\mathrm{d}x.$$

Which is equivalent

$$\int_\Omega \frac{g(x,|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla v\,\mathrm{d}x = \frac{1}{\mu}\int_\Omega \frac{g(x,|u|)}{|u|} uv\,\mathrm{d}x.$$

Then $\lambda_i := \frac{1}{\mu_i} \to \infty$ as $i \to \infty$ is a nondecreasing sequence of nonnegative eigenvalues of (3.2). □

References

[1] H. Amann, *Lusternik-Schnirelman theory and non-linear eigenvalue problems*, Math. Ann. **199** (1972), 55–72. https://doi.org/10.1007/BF01419576

[2] L. Barbu, G. Moroşanu, and C. Pintea, *A nonlinear elliptic eigenvalue-transmission problem with Neumann boundary condition*, Ann. Mat. Pura Appl. (4) **198** (2019), no. 3, 821–836. https://doi.org/10.1007/s10231-018-0801-5

[3] A. Benyaiche and I. Khlifi, *Sobolev-Dirichlet problem for quasilinear elliptic equations in generalized Orlicz-Sobolev spaces*, Positivity **25** (2021), no. 3, 819–841. https://doi.org/10.1007/s11117-020-00789-z

[4] F. E. Browder, *Existence theorems for nonlinear partial differential equations*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968), 1–60, Amer. Math. Soc., Providence, RI, 1970.

[5] R. Chiappinelli, *What do you mean by nonlinear eigenvalue problems?*, Axioms. **7** (2018), 39.

[6] X. Fan, Q. Zhang, and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302** (2005), no. 2, 306–317. https://doi.org/10.1016/j.jmaa.2003.11.020

[7] J. P. García Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations **12** (1987), no. 12, 1389–1430. https://doi.org/10.1080/03605308708820534

[8] P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, 2236, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-15100-3

[9] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), no. 5, 1057–1099. https://doi.org/10.1016/j.na.2005.05.056

[10] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361. https://doi.org/10.1080/03605309108820761

[11] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/BFb0072210

[12] V. D. Rădulescu and S. Saiedinezhad, *A nonlinear eigenvalue problem with $p(x)$-growth and generalized Robin boundary value condition*, Commun. Pure Appl. Anal. **17** (2018), no. 1, 39–52.

[13] M. Tienari, *Lusternik-Schnirelmann theorem for the generalized Laplacian*, J. Differential Equations **161** (2000), no. 1, 174–190. https://doi.org/10.1006/jdeq.2000.3712

[14] E. Zeidler, *The Lusternik-Schnirelmann theory for indefinite and not necessarily odd nonlinear operators and its applications*, Nonlinear Anal. **4** (1980), no. 3, 451–489. https://doi.org/10.1016/0362-546X(80)90085-1

[15] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. III*, translated from the German by Leo F. Boron, Springer-Verlag, New York, 1985. https://doi.org/10.1007/978-1-4612-5020-3

Allami Benyaiche
Ibn Tofail University
Department of Mathematics
B.P: 133
Kenitra 14000, Morocco
Email address: allami.benyaiche@uit.ac.ma

Ismail Khlifi
Ibn Tofail University
Department of Mathematics
B.P: 133
Kenitra 14000, Morocco
Email address: is.khlifi@gmail.com