KYUNGPOOK Math. J. 62(2022), 595-613
https://doi.org/10.5666/KMJ.2022.62.3.595
pISSN 1225-6951 eISSN 0454-8124

An Ideal-based Extended Zero-divisor Graph on Rings

Mohammad Ashraf
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
e-mail : mashraf80@hotmail.com

Mohit Kumar*
Department of Mathematics, Institute of Applied Sciences and Humanities, GLA University Mathura-281 406, Uttar Pradesh, India
e-mail : mohitkumaramu123@gmail.com

Abstract. Let R be a commutative ring with identity and let I be a proper ideal of R. In this paper, we study the ideal based extended zero-divisor graph $\Gamma_I^{'}(R)$ and prove that $\Gamma'_I(R)$ is connected with diameter at most two and if $\Gamma'_I(R)$ contains a cycle, then girth is at most four girth at most four. Furthermore, we study affinity the connection between the ideal based extended zero-divisor graph $\Gamma'_I(R)$ and the ideal-based zero-divisor graph $\Gamma_I(R)$ associated with the ideal I of R. Among the other things, for a radical ideal of a ring R, we show that the ideal-based extended zero-divisor graph $\Gamma'_I(R)$ is identical to the ideal-based zero-divisor graph $\Gamma_I(R)$ if and only if R has exactly two minimal prime-ideals which contain I.

1. Introduction

Throughout this paper let R be a commutative ring identity, I be a proper ideal of R which is not a prime ideal of R, $Z(R)$ be the set of zero-divisors of R, $Z^*(R) = Z(R) \setminus \{0\}$, $Z_I^*(R) = \{u \notin I \mid uv \in I \text{ for some } v \notin I\}$, $Z_I(R) = Z_I^*(R) \cup I$ and $N(R)$ be the set of nilpotent elements of R. Let B be a submodule of an R-module M and X be any subset of M. Then $(B : X) = \{r \in R \mid rx \in B \text{ for all } x \in X\}$. $Min_I(R)$ will denote the set of minimal prime ideals of R which contain I. Let $\beta(I) = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$ be a prime radical of I in R, then $\beta^*(I) = \beta(I) \setminus I$ and I is said to be radical ideal if $\beta(I) = I$. R/I denotes the quotient ring of R, and for any $x + I \in R/I$ we use the notation $[x]$. For any subset A of R, we have $A^* = A \setminus \{0\}$.

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the set of vertices and $E(G)$ be the set of edges of G. We say that G is connected if there exists a path between any two

---

* Corresponding Author.
Received June 10, 2019; revised April 13, 2020; accepted August 18, 2020.
2010 Mathematics Subject Classification: Primary 13A15; Secondary 13B99, 05C20, 05C69.
Key words and phrases: connected, diameter, girth, annihilator, prime ideal, Prime radical.

distinct vertices of G. For vertices a and b of G, $d(a,b)$ denotes the length of the shortest path from a to b. In particular, $d(a,a)=0$ and $d(a,b)=\infty$ if there exists no such path. The diameter of G, denoted by $diam(G)=sup\{d(a,b)\mid a,b\in V(G)\}$. A cycle in a graph G is a path that begins and ends at the same vertex. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G, ($gr(G)=\infty$ if G contains no cycle). A complete graph G is a graph where all distinct vertices are adjacent. The complete graph with $|V(G)|=n$ is denoted by K_n. A graph G is said to be complete k-partite if there exists a partition $\bigcup_{i=1}^{k} V_i = V(G)$, such that $u-v\in E(G)$ if and only if u and v are in different part of partition. If $|V_i|=n_i$, then G is denoted by $K_{n_1,n_2,\cdots,n_k}$ and in particular G is called complete bipartite if $k=2$. $K_{1,n}$ is said to be a star graph. $\overline{G}$ denotes the complement graph of G. A graph $H=(V(H),E(H))$ is said to be a subgraph of G, if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. Moreover, H is said to be induced subgraph of G if $V(H)\subseteq V(G)$ and $E(H)=\{u-v\in E(G)\mid u,v\in V(H)\}$ and is denoted by $G[V(H)]$. Let H_1 and H_2 be two disjoint graphs. The join of H_1 and H_2, denoted by $H_1\vee H_2$, is a graph with vertex set $V(H_1\vee H_2)=V(H_1)\cup V(H_2)$ and edge set $E(H_1\vee H_2)=E(H_1)\cup E(H_2)\cup\{u-v\mid u\in V(H_1),v\in V(H_2)\}$. Also G is called a null graph if $E(G)=\phi$. For a graph G, a complete subgraph of G is called a clique. The clique number, $\omega(G)$, is the greatest integer $n\geqslant 1$ such that $K_n\subseteq G$, and $\omega(G)=\infty$ if $K_n\subseteq G$ for all $n\geqslant 1$. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour all the vertices of G such that every two adjacent vertices get different colours. A graph G is perfect if $\chi(H)=\omega(H)$ for every induced subgraph H of G. For a connected graph G, $\delta(G)=\min\{deg(x)\mid x\in V(G)\}$, $V(\delta(G))=\{x\mid x\in V(G),deg(x)=\delta(G)\}$ and $\Delta(G)=\max\{deg(x)\mid x\in V(G)\}$, $V(\Delta(G))=\{x\mid x\in V(G),deg(x)=\Delta(G)\}$. A subset $D\subseteq V(G)$ is said to be a dominating set if every vertex in $V(G)\setminus D$ is adjacent to a vertex in D. A dominating set D is called a weak (or strong) dominating set if for every $u\in V(G)\setminus D$ there exists $v\in D$ with $deg(v)\leq deg(u)$ (or $deg(u)\leq deg(v)$) and u is adjacent to v. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set of G and such a dominating set of G is called a γ-set of G. In a similar way, we define the weak (or strong) domination number $\gamma_w(G)$ (or $\gamma_s(G)$) of G. A graph G is said to be excellent, if for every $u\in V(G)$, there exists a γ-set D containing u. Graph-theoretic terms are presented as they appear in Diestel [9].

The study of algebraic structure associated with graph is an active and interesting area of research. Several authors have done a lot of work in this area for instance, see [1–4,7,8,12]. The idea of a zero-divisor graph of a commutative ring R with identity was introduced by I. Beck in [8], who defined the graph on the vertex set R in which distinct vertices $u,v\in R$ are adjacent if and only if $uv=0$. He was mainly interested in coloring of rings. The first simplification of Beck's zero divisor graph was introduced by Anderson and Livingston in [2]. We recall from [2] that a zero-divisor graph $\Gamma(R)$ of R is the (undirected) graph with set of vertices $Z^*(R)$ and on the vertex set $Z^*(R)$, in which any two distinct vertices u and v of $\Gamma(R)$ are adjacent if and only if $uv=0$. In [13] Redmond introduced an ideal-based zero-divisor graph $\Gamma_I(R)$ with set of vertices $Z_I^*(R)$ and vertex set $Z_I^*(R)$, in which any two distinct vertices u and v of $\Gamma_I(R)$ are adjacent if and only if $uv\in I$. In [5] Bakhtyiari et al. introduced the extended zero-divisor graph $\Gamma'(R)$. The extended zero-divisor graph of R is an (undirected) graph $\Gamma'(R)$ with the vertex set $Z^*(R)$ and two distinct vertices u and v of $\Gamma'(R)$ are adjacent if and only if either $Ru\cap ann_R(v)\neq\{0\}$ or $Rv\cap ann_R(u)\neq\{0\}$.

In this paper we generalize the extended zero divisor graph $\Gamma'(R)$ to an ideal-based extended zero-divisor graph $\Gamma'_I(R)$. The ideal based extended zero-divisor graph $\Gamma'_I(R)$ is the (undirected) graph with the vertex set $Z_I^*(R)$, in which two distinct vertices u and v are adjacent if and only if either $(Ru+I)\cap(I:\{v\})\neq I$ or $(Rv+I)\cap(I:\{u\})\neq I$. If we take $I=(0)$, then $\Gamma'_I(R)=\Gamma'(R)$. It follows that the ideal-based zero-divisor graph $\Gamma_I(R)$ is a subgraph of $\Gamma'_I(R)$. We prove that $\Gamma'_I(R)$ is connected with diameter at most two, and if $\Gamma'_I(R)$ contain a cycle, then girth is at most four. Furthermore, we study the connection between the ideal based extended zero-divisor graph $\Gamma'_I(R)$ and the ideal-based zero-divisor graph $\Gamma_I(R)$ associated with the ideal I of a commutative ring R. Among the other things, for a radical ideal of a commutative ring R, we show that ideal-based extended zero-divisor graph $\Gamma'_I(R)$ is identical to the ideal-based zero-divisor graph $\Gamma_I(R)$ if and only if R has exactly two minimal prime-ideals which contain I.

2. Fundamental Properties of Ideal-based Extended Zero-divisor Graph

In this section, we generalize the notion of an extended zero-divisor graph $\Gamma'(R)$ to an ideal-based extended zero-divisor graph $\Gamma'_I(R)$ and study fundamental properties of $\Gamma'_I(R)$.

Definition 2.1. Let I be an ideal in a commutative ring R with unity. An ideal-based extended zero divisor graph $\Gamma'_I(R)$ is an undirected graph with the set of vertices $Z_I^*(R)$, where any two distinct vertices u, v of $\Gamma'_I(R)$ are adjacent if and only if either $(Ru+I)\cap(I:\{v\})\neq I$ or $(Rv+I)\cap(I:\{u\})\neq I$.

Proposition 2.2. *Let I be an ideal in a commutative ring R with unity. Then*
(i) $\Gamma_I(R)$ is a subgraph of $\Gamma'_I(R)$.
(ii) if $I=(0)$, then $\Gamma'_I(R)=\Gamma'(R)$ and $\Gamma(R)$ is a subgraph of $\Gamma'_I(R)$.

Proof. Let I be an ideal of a commutative ring R.
(i) Clearly, $V(\Gamma'_I(R))=V(\Gamma_I(R))$ and let u and v be any two adjacent vertices of $\Gamma_I(R)$. Then $uv\in I$ and $u\in(Ru+I)\cap(I:\{v\})$, $v\in(Rv+I)\cap(I:\{u\})$, i.e., $(Ru+I)\cap(I:\{v\})\neq I$, $(Rv+I)\cap(I:\{u\})\neq I$. Hence u and v also adjacent in $\Gamma'_I(R)$, and by definition $\Gamma_I(R)$ is a subgraph of $\Gamma'_I(R)$.

(ii) It trivially holds. □

Lemma 2.3. *Let I be a radical ideal in a commutative ring R which is not prime, and let $u\in Z_I^*(R)$. Then*

(i) $(I:\{u\})=(I:\{u^n\})$ *for each positive integer* $n\geqslant 2$,

(ii) $(Ru+I)\cap(I:\{u\})=I$.

Proof. Assume that I is a radical ideal of a ring R which is not prime and $u\in Z_I^*(R)$.
(i) Let $n\geq 2$. It is clear that $(I:\{u\})\subseteq(I:\{u^n\})$. If $v\in(I:\{u^n\})$, then $vu^n\in I$. Since I is a radical ideal, $vu\in I$ and $v\in(I:\{u\})$. Thus $(I:\{u^n\})=(I:\{u\})$.

(ii) This is clearly true. □

The following lemma gives several useful properties of $\Gamma'_I(R)$ and plays an important role in this section.

Lemma 2.4. *Let I be a proper ideal of a ring R.*

(i) If $u-v$ is not an edge of $\Gamma'_I(R)$ for some $u,v \in Z_I^*(R)$, then $(I:\{u\}) = (I:\{v\})$. If I is a radical ideal, then the converse is also true.

(ii) If $(I:\{u\}) \nsubseteq (I:\{v\})$ or $(I:\{v\}) \nsubseteq (I:\{u\})$ for some $u,v \in Z_I^*(R)$, then $u-v$ is an edge of $\Gamma'_I(R)$.

(iii) If $(Ru+I) \cap (I:\{u\}) \neq I$ for some $u \in Z_I^*(R)$, then u is adjacent to all other vertex in $\Gamma'_I(R)$. In particular if $u \in \beta^*(I)$, then u is adjacent to every other vertex of $\Gamma'_I(R)$.

(iv) $\Gamma'_I(R)[\beta^*(I)]$ is a complete subgraph of $\Gamma'_I(R)$.

Proof. Assume that I is an ideal of a ring R.
(i) If $u-v$ is not an edge of $\Gamma'_I(R)$ for some $u,v \in Z_I^*(R)$, then $(Ru+I) \cap (I:\{v\}) = I$ and $(Rv+I) \cap (I:\{u\}) = I$. Thus $(Ru+I)(I:\{v\}) \subseteq (Ru+I) \cap (I:\{v\}) = I$ and $(Rv+I)(I:\{u\}) \subseteq (Rv+I) \cap (I:\{u\}) = I$ and hence $(I:\{u\}) = (I:\{v\})$. If I is a radical ideal of R, then by Lemma 2.3(ii), $(Ru+I) \cap (I:\{v\}) = (Ru+I) \cap (I:\{u\}) = I$ and $(Rv+I) \cap (I:\{u\}) = (Rv+I) \cap (I:\{v\}) = I$. Thus $u-v$ is not an edge of $\Gamma'_I(R)$.

(ii) This is clear by part (i).

(iii) Assume that $(Ru+I) \cap (I:\{u\}) \neq I$ for some $u \in Z_I^*(R)$, and let v be another vertex of $\Gamma'_I(R)$. If u is not adjacent to v, then by part (i), $(I:\{u\}) = (I:\{v\})$ and hence $(Ru+I) \cap (I:\{u\}) = I$, a contradiction.

(iv) This is clearly true by (iii). □

Theorem 2.5. *Let I be an ideal of R. Then $\Gamma'_I(R)$ is connected and $dia(\Gamma'_I(R)) \leq 2$. Moreover if $\Gamma'_I(R)$ contains a cycle, then $gr(\Gamma'_I(R)) \leq 4$.*

Proof. By Lemma 2.2(i), $\Gamma_I(R)$ is a connected subgraph of $\Gamma'_I(R)$ such that $V(\Gamma_I(R)) = V(\Gamma'_I(R)$. Therefore $\Gamma'_I(R)$ is connected and $gr(\Gamma'_I(R)) \leq 4$. Now we prove that $dia(\Gamma'_I(R)) \leq 2$. If I is a non-radical ideal of R, then $\beta(I) \neq I$ and by Lemma 2.4(iii), $dia(\Gamma'_I(R)) \leq 2$. If I is a radical ideal of R, then $\beta(I) = I$. Let $u,v \in V(\Gamma'_I(R))$ such that $d(u,v) \neq 1$. Then by Lemma 2.4(i), $(I:\{u\}) = (I:\{v\})$. Since $\beta(I) = I$, by Lemma 2.3(ii), $(Rv+I) \cap (I:\{v\}) = I$. Therefore, for every $w \in (I:\{v\}) \setminus I$ both u,v are adjacent to w and $d(u,v) = 2$. Thus $diam(\Gamma'_I(R)) \leq 2$. This completes the proof. □

Lemma 2.6. *Let I be a proper ideal of a commutative ring R. Then $Z_I(R)$ is a union of prime ideals of R which contain I.*

Proof. Let us define a map $F: R \longrightarrow R/I$ by $F(x) = [x]$. Clearly, F is a homomorphism from R onto R/I. By [11, p. 3], $Z(R/I) = \bigcup P_i$ where P_i is a prime ideal in R/I. clearly, $Z_I(R) = \bigcup F^{-1}(P_i)$ where $F^{-1}(P_i)$ is a prime ideal in R which contains I. □

Corollary 2.7. *Let I be a radical ideal of a commutative ring R. Then $Z_I(R) = \bigcup P_i$, where $P_i \in Min_I(R)$.*

Proof. The corollary is immediate from Lemma 2.6 and [10, Corollory 2.4]. □

Theorem 2.8. *Let I be a proper ideal of a commutative ring R and let $\Gamma'_I(R)$ contain a cycle. Then $gr(\Gamma'_I(R)) = 4$ if and only if I is a radical ideal with $|Min_I(R)| = 2$.*

Proof. First assume that $gr(\Gamma'_I(R)) = 4$. If I is not a radical ideal, then $\beta(I) \neq I$ and by Lemma 2.4(iii) $gr(\Gamma'_I(R)) = 3$, a contradiction. Hence I must be a radical ideal of R. Let $u \in Z^*_I(R)$. We will prove that $(I : \{u\})$ is a prime ideal of R. Suppose that $ab \in (I : \{u\})$ such that $a, b \notin (I : \{u\})$ but $aubu \in I$. Hence for every $c \in (I : \{u\}) \setminus I$, it is easy to see that $c - au - bu - c$ is a triangle, a contradiction. Hence $(I : \{u\})$ is a prime ideal. Since I is a radical ideal and by Lemma 2.3(ii) together with [10, Theorem 2.1] implies that $(I : \{u\})$ is a minimal prime ideal which contains I. i.e., $(I : \{u\}) \in Min_I(R)$. By similar arguments $(I : \{v\}) \in Min_I(R)$, for each $v \in (I : \{u\}) \setminus I$. Now we prove that $Min_I(R) = \{(I : \{u\}), (I : \{v\})\}$. It is sufficient to show that $(I : \{u\}) \cap (I : \{v\}) = I$. Assume on contrary $(I : \{u\}) \cap (I : \{v\}) \neq I$ and $a \in (I : \{u\}) \cap (I : \{v\}) \setminus I$. Then $a - u - v - a$ is a triangle as $uv \in I$, a contradiction. Hence $Min_I(R) = \{(I : \{u\}), (I : \{v\})\}$.
Conversely, assume that I is a radical ideal of R and $|Min_I(R)| = 2$. Let $Q_1, Q_2 \in Min_I(R)$. Since I is a radical ideal, we have $Z_I(R) = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = I$, by Corollary 2.7. It is not difficult to check that $\Gamma'_I(R) = K_{|Q^*_1|, |Q^*_2|}$, where $|Q^*_1| = |Q_1 \setminus I|$ and $|Q^*_2| = |Q_2 \setminus I|$. Since $\Gamma'_I(R)$ contains a cycle, $gr(\Gamma'_I(R)) = 4$. □

Example 2.9. For $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $I = (0) \times \mathbb{Z}_3$, it may be observed that $Q_1 = (3) \times \mathbb{Z}_3$ and $Q_2 = (2) \times \mathbb{Z}_3$ are the only two minimal prime ideals of R, which contain radical ideal I, where $Z_I(R) = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = I$. Since $|Q^*_1| = 3$ and $|Q^*_2| = 6$, it can be easily seen in the following Figure 2.1 that $\Gamma'_I(R) = \Gamma_I(R) = K_{|Q^*_1|, |Q^*_2|} = K_{3,6}$ and $gr(\Gamma'_I(R)) = 4$.

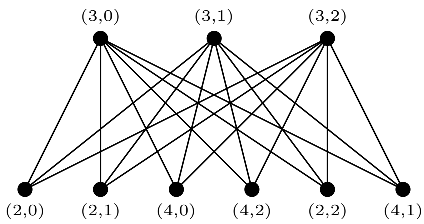


Figure 2.1

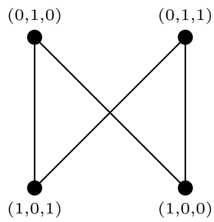
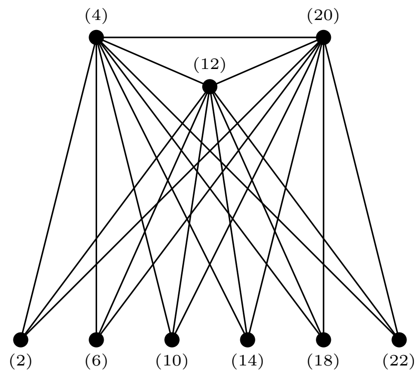


Figure 2.2

Example 2.10. For $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = (0) \times (0) \times \mathbb{Z}_2$, it can be easily seen in the above Figure 2.2, $K_{2,2}$ is realizable as $\Gamma'_I(R)$, which is not realizable as $\Gamma'(R)$.

Corollary 2.11. *Let I be a proper ideal of a commutative ring R. Then $\Gamma'_I(R)$ is $K_{2,2}$ if and only if I is a radical ideal of R with $|Min_I(R)| = 2$ and each element of $Min_I(R)$ contains exactly two elements other than I.*

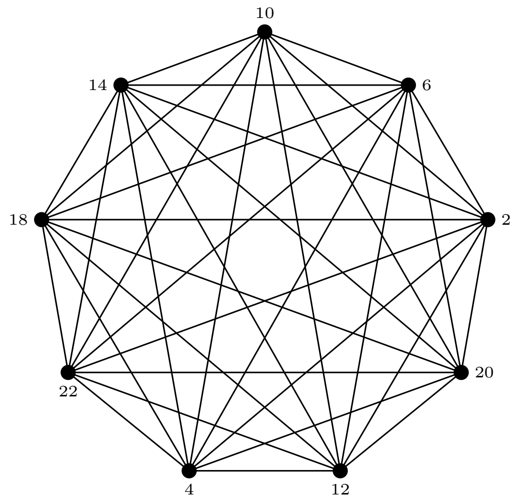
Example 2.12. For $R = \mathbb{Z}_{24}$ and $I = (8)$, it can be easily seen from the following Figures 2.3 and 2.4 that the ideal-based extended zero divisor graph $\Gamma'_I(R) = K_9$ is different from ideal-based zero divisor graph $\Gamma_I(R)$ and $\Gamma_I(R)$ is a subgraph of $\Gamma'_I(R) = K_9$.



$\Gamma_I(R)$

$R = \mathbb{Z}_{24}$ and $I = (8)$

Figure 2.3



$\Gamma'_I(R) = K_9$

$R = \mathbb{Z}_{24}$ and $I = (8)$

Figure 2.4

3. When Ideal-based Extended Zero Divisor Graph $\Gamma'_I(R)$ and Ideal-based Zero Divisor Graph $\Gamma_I(R)$ are Identical?

As we have seen in the previous section, ideal-based extended zero divisor graphs and ideal-based zero-divisor graphs are close to each other, it would be interesting to characterize ideals of a ring whose ideal-based extended zero-divisor graph and ideal-based zero divisor graph are identical. We first study the case when I is a radical ideal of R.

Theorem 3.1. *Let I be a radical ideal of a commutative ring R with $|Min_I(R)| = k \geq 2$. Then $k = 2$ if and only if $\Gamma'_I(R) = \Gamma_I(R)$.*

Proof. First assume that $\Gamma'_I(R) = \Gamma_I(R)$. To prove that $k = 2$, assume on the contrary Q_1, Q_2, Q_3 are distinct minimal prime ideals of R which contain I. Let $u \in Q_1 \setminus Q_2 \cup Q_3$. Thus $Q_2 \cup Q_3 \nsubseteq (I : \{u\})$ as $(I : \{u\}) \subseteq Q_2 \cap Q_3$. So one may choose $uv \notin I$, for some $v \in Q_2 \cup Q_3 \setminus Q_1$. Without loss of generality, assume that $v \in Q_2 \setminus Q_1$. Obviously, $(I : \{v\}) \subseteq Q_1$. Also, it follows from [10, Theorem 2.1], there exists an element $w \in (I : \{u\})$ such that $w \notin Q_1$. Therefore, $(I : \{u\}) \neq (I : \{v\})$ and by Theorem 2.4(ii), $u - v$ is an edge of $\Gamma'_I(R)$, a contradiction.

Conversely, assume that Q_1 and Q_2 are only two distinct minimal prime ideals of R which contain I. It is not difficult to check that $\Gamma_I(R) = \Gamma'_I(R) = K_{|Q_1^*|, |Q_2^*|}$. Where $Q_1^* = Q_1 \setminus I$ and $Q_2^* = Q_2 \setminus I$. □

The following corollary follows from Theorem 3.1.

Corollary 3.2. *Let I be a radical ideal of a commutative ring R, which is not a prime ideal. Then the following statements are equivalent:*

(i) $gr(\Gamma'_I(R)) = 4$.

(ii) $\Gamma'_I(R) = \Gamma_I(R)$ *and* $gr(\Gamma_I(R)) = 4$.

(iii) $|Min_I(R)| = 2$ *and each minimal prime ideal of* $Min_I(R)$ *has at least two different elements other then elements of* I.

(iv) $\Gamma'_I(R) = K_{m,n}$ *for some* $m, n \in \mathbb{N}$ *and* $m, n \geqslant 2$.

In the rest of this section we study the case that I is a non radical ideal of R

Theorem 3.3. *Let I be a non radical ideal of a commutative ring R. Then the following statements are equivalent.*

(i) $\Gamma'_I(R) = \Gamma_I(R)$.

(ii) *If* $uv \notin I$ *for some* $u, v \in Z_I^*(R)$, *then* $(I : \{u\}) = (I : \{v\})$ *and* $(I : \{u\})$ *is a prime ideal of* R.

Proof. (i) $\Rightarrow$ (ii) Assume that $uv \notin I$, for some $u, v \in Z_I^*(R)$. Since $\Gamma'_I(R) = \Gamma_I(R)$, we deduce that $(I : \{u\}) = (I : \{v\})$, by Lemma 2.4($i$). We now show that $(I : \{u\})$ is a prime ideal of R. Let $ab \in (I : \{u\})$, $a \notin (I : \{u\})$ and $b \notin (I : \{u\})$. Then $au \notin I$ and $bu \notin I$, $a, b \in Z_I^*(R)$. By Lemma 2.4(iii), $u, v \notin \beta(I)$ and hence $u \neq a$ or $u \neq b$. Without loss of generality, one may assume that $u \neq b$. But since $au \in (Ru + I) \cap (I : \{v\})$, we find that $ub \in I$, a contradiction. Therefore, $(I : \{u\})$ is a prime ideal of R, as desired.

(ii) $\Rightarrow$ (i) If $uv \in I$ for all $u, v \in Z_I^*(R)$, then $\Gamma_I(R)$ is complete and by Proposition 2.2(i), $\Gamma'_I(R)$ is complete. i.e., $\Gamma'_I(R) = \Gamma_I(R)$. To complete the proof, we prove that if $uv \notin I$. Then $(Ru + I) \cap (I : \{v\}) = I$ and $(Rv + I) \cap (I : \{v\}) = I$. Since $(I : \{u\}) = (I : \{v\})$ If $u \in (I : \{u\})$, then $u \in (I : \{v\})$ and hence $uv \in I$, a contradiction. Thus $u \notin (I : \{u\})$. Also, if $(Ru + I) \cap (I : \{u\}) \neq I$, then there exists $r \in R$ such that $ru \notin I$ and $ru^2 \in I$. Since $u^2 \notin (I : \{u\})$ as $(I : \{u\})$ is a prime ideal of R, $r \in (I : \{u\})$, a contradiction. Hence $(Ru + I) \cap (I : \{u\}) = I$. Similarly, $(Rv + I) \cap (I : \{v\}) = I$. □

Corollary 3.4. *Let I be a non radical ideal of a commutative ring R and $\Gamma'_I(R) = \Gamma_I(R)$. Then the following hold.*

(i) $Z_I(R)$ *is an ideal of* R.

(ii) $\beta(I)^2 \subseteq I$.

(iii) $(I : Z_I(R)) = \beta(I)$.

Proof. Assume that I is not a radical ideal of R.
(i) Since I is a non radical ideal of R, $\beta^*(I) \neq \phi$. Let $u \in \beta^*(I)$. Then by Lemma 2.4 (iii) u is adjacent to every other vertex of $\Gamma'_I(R)$. Since $\Gamma'_I(R) = \Gamma_I(R)$, u is adjacent to every other vertex of $\Gamma_I(R)$, and hence by [13, Theorem 2.5(b)] $[u]$, is adjacent to every other vertex of $\Gamma(R/(I)$ and by [2, Theorem 2.5], we find that $Z(R/I)$ is an annihilator ideal, i.e., $Z(R/I) = ann_{R/I}([u])$. Since $Z(R/I) = ann_{R/I}([u])$, we find that $(I : \{u\}) = Z_I(R)$ and thus $Z_I(R)$ is an ideal of R.

(ii) By the first part, clearly $\beta(I)^2 \subseteq I$.

(iii) By the first part, clearly $(I : Z_I(R)) = \beta(I)$. □

Corollary 3.5. *Let I be a non radical ideal of a commutative ring R. Then $\Gamma'_I(R) = \Gamma_I(R) = K_p \vee \overline{K_q}$ if and only if $(I : Z_I(R))$ is a prime ideal.*

Proof. First assume that $\Gamma_I(R) = \Gamma'_I(R) = K_p \vee \overline{K_q}$. Hence every vertex of K_p is adjacent to all the other vertices. But there is no adjacency between any two vertices of $\overline{K_q}$. This implies that $(I : Z_I(R)) = V(K_p) \cup I$, thus $uv \notin I$, for every $u, v \in V(\overline{K_q})$, and hence $(I : \{u\}) = (I : \{v\}) = (I : Z_I(R))$. By Theorem 3.3 $(I : Z_I(R))$ is a prime ideal of R.
Conversely since $(I : Z_I(R))$ is a prime ideal of R, we find that $uv \in I$, for all $u, v \in (I : Z_I(R))$ and $uv \notin I$ for all $u, v \in Z_I(R) \setminus (I : Z_I(R))$. Now it is enough to show that $\Gamma_I(R)[(I : Z_I^*(R))]$ is complete, $\Gamma_I(R)[Z_I(R) \setminus (I : Z_I(R))]$ is null graph and $\Gamma_I(R) = \Gamma_I(R)[(I : Z_I^*(R))] \vee \Gamma_I(R)[Z_I(R) \setminus (I : Z_I(R))]$. We finally show that $\Gamma_I(R) = \Gamma'_I(R)$. Obviously, $uv \notin I$ if and only if $u, v \in Z_I(R) \setminus (I : Z_I(R))$. This together with $(I : Z_I(R))$ is a prime ideal, imply that if $uv \notin I$, then $(I : \{u\}) = (I : \{v\}) = (I : Z_I(R))$. Thus $(I : \{u\})$ is a prime ideal of R. Now by Theorem 3.3, $\Gamma_I(R) = \Gamma'_I(R)$. □

Corollary 3.6. *Let I be a non trivial non-radical ideal of a commutative ring R. Then the following statements are equivalent.*

(i) $\Gamma'_I(R)$ *is a star graph.*

(ii) $gr(\Gamma'_I(R)) = \infty$.

(iii) $\Gamma_I(R) = \Gamma'_I(R)$ *and* $gr(\Gamma_I(R)) = \infty$.

(iv) $(I : Z_I(R))$ *is a prime ideal of* R, $|I| = |\beta^*(I)| = |Z_I^*(R)| = 2$.

(v) $\Gamma'_I(R) = K_{1,1}$.

(vi) $\Gamma_I(R) = K_{1,1}$.

Proof. $(i) \Rightarrow (ii)$ It is clear.

$(ii) \Rightarrow (iii)$ If $a \in \beta^*(I)$, then a is adjacent to every other vertex in $\Gamma'_I(R)$. Since $gr(\Gamma'_I(R)) = \infty$ and $\Gamma_I(R)$ is a connected subgraph of $\Gamma'_I(R)$, we conclude that $\Gamma'_I(R) = \Gamma_I(R)$, and hence $gr(\Gamma_I(R)) = \infty$.

$(iii) \Rightarrow (iv)$ Since I is a non trivial non radical ideal of R, it can be easily seen that $\Gamma'_I(R)$ is a star graph and $\Gamma'_I(R) = \Gamma_I(R)$. Therefore by Corollary 3.5, $(I : Z_I(R))$ is a prime ideal of R. Since I is a nontrivial non radical ideal of R, $|I| \geq 2$ and $|\beta(I)| \geq 4$. If $|I| = m > 2$, then $\beta(I)| = n \geq 6$ and we can assume that u, v, $w \in \beta^*(I)$ such that by Lemma 2.4 (iii),

$u - v - w - u$ is a triangle and $\Gamma'_I(R)$ is not a star graph. Thus $|I| = 2$. If $|I| = 2$, then $|\beta(I)| = 4$, otherwise by Lemma 2.4 (iii), $\Gamma'_I(R)$ is not a star graph. Thus $|I| = |\beta^*(I)| = 2$. If $|Z^*_I(R)| \geq 3$, the we can assume that $\beta_1, \beta_2 \in \beta^*(I)$ and $z \in Z^*_I(R) \setminus \beta^*(I)$ such that by Lemma 2.4 (iii), $\beta_1 - \beta_2 - z - \beta_1$ forms a triangle. Hence $|I| = |\beta^*(I)| = |Z^*_I(R)| = 2$.

$(iv) \Rightarrow (v)$ It is clear by Corollary 3.5.

$(v) \Rightarrow (vi)$ It is clear.

$(vi) \Rightarrow (i)$ It is clear. □

4. Results on Relationship Between $\Gamma'_I(R)$ and $\Gamma'(R/I)$

In this section, we study the graph theoretical relationship between $\Gamma'_I(R)$ and $\Gamma'(R/I)$ under certain parameters like clique number, max (or min) degree, vertex chromatic number, also determine a necessary and sufficient condition for $\Gamma'_I(R)$ to be regular and Eulerian.

Theorem 4.1. *Let I be an ideal of a commutative ring R and let $u, v \in Z^*_I(R)$. Then*

(i) if $[u]$ is adjacent to $[v]$ in $\Gamma'(R/I)$, then u is adjacent to v in $\Gamma'_I(R)$,

(ii) if u is adjacent to v in $\Gamma'_I(R)$ and $[u] \neq [v]$, then $[u]$ is adjacent to $[v]$ in $\Gamma'(R/I)$,

(iii) if u adjacent to v in $\Gamma'_I(R)$ and $[u] = [v]$, then there exists $r \in Z^*_I(R)$ such that $ru \notin I$ and $rv \notin I$, but $ru^2 \in I$ and $rv^2 \in I$,

(iv) if u is adjacent to v in $\Gamma'_I(R)$, then all (distinct) elements of $[u]$ and $[v]$ are adjacent in $\Gamma'_I(R)$. If there exists $r \in R$ such that $ru \notin I$ and $ru^2 \notin I$, then all the distinct elements of $[u]$ are adjacent in $\Gamma'_I(R)$.

Proof. (i) If $[u]$ is adjacent to $[v]$ in $\Gamma'(R/I)$, then either $(R/I)[u] \cap ann_{R/I}([v]) \neq \{I\}$ or $(R/I)[v] \cap ann_{R/I}([u]) \neq \{I\}$. This implies that either $(Ru + I) \cap (I : \{v\}) \neq I$ or $(Rv + I) \cap (I : \{u\}) \neq I$. By definition u is adjacent to v in $\Gamma'_I(R)$.

(ii)If u is adjacent to v in $\Gamma'_I(R)$ then either $(Ru+I) \cap (I : \{v\}) \neq I$ or $(Rv+I) \cap (I : \{u\}) \neq I$. Since $[u] \neq [v]$, either $(R/I)[u] \cap ann_{R/I}([v]) \neq \{I\}$ or $(R/I)[v] \cap ann_{R/I}([u]) \neq \{I\}$. By definition $[u]$ is adjacent to $[v]$ in $\Gamma'(R/I)$.

(iii) If u is adjacent to v in $\Gamma'_I(R)$, then either $(Ru + I) \cap (I : \{v\}) \neq I$ or $(Ru + I) \cap (I : \{v\}) \neq I$. i.e., either $(Ru + I) \cap (I : \{v\}) \setminus I \neq \phi$ or $(Ru + I) \cap (I : \{v\}) \setminus I \neq \phi$. Suppose that $(Ru + I) \cap (I : \{v\}) \setminus I \neq \phi$. Then there exists $\alpha \in (Ru + I) \cap (I : \{v\}) \setminus I$ such that $\alpha = ru + i$ for some $r \in R \setminus I, i \in I$. Clearly $ruv \in I$. Since $[u] = [v]$, $u = v + j$ for some $j \in I$, we find that $ru^2 = ruu = ru(v + j) = ruv + ruj \in I$. Similarly $rv^2 \in I$. Now if $(Ru + I) \cap (I : \{v\}) \setminus I \neq \phi$, then by the similar proof there exists $r' \in R \setminus I$ such that $r'u^2, r'v^2 \in I$.

(iv)If u is adjacent to v in $\Gamma'_I(R)$, then either $(Ru + I) \cap (I : \{v\}) \neq I$ or $(Rv + I) \cap (I : \{u\}) \neq I$. Let $u + i \in [u]$, $v + j \in [v]$. Then $(R(u + i) + I) \cap (I : \{v + j\}) \neq I$ or $(R(v + j) + I) \cap (I : \{u + i\}) \neq I$. By definition $u + i$ is adjacent to $v + j$ in $\Gamma'_I(R)$. □

Proposition 4.2. *Let I be an ideal of a ring R. Then $\Gamma'_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma'(R/I)$.*

Proof. Let $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq Z_I^*(R)$ be a set of coset representative vertices of $\Gamma'(R/I)$,i.e., $V(\Gamma'(R/I)) = \{[a_\lambda] : \lambda \in \Lambda\}$ and for each $\alpha \in I$, define a graph $G_\alpha = (V_\alpha, E_\alpha)$ with $V_\alpha = \{a_\lambda + \alpha : \lambda \in \Lambda\}$, where $a_\gamma + \alpha$ is adjacent to $a_\delta + \alpha$ in G_α whenever, $[a_\gamma]$ is adjacent to $[a_\delta]$ in $\Gamma'(R/I)$. i.e., either $(R/I)[a_\gamma] \cap ann_{(R/I)}([a_\delta]) \neq \{I\}$ or $(R/I)[a_\delta] \cap ann_{(R/I)}([a_\gamma]) \neq \{I\}$. By Theorem 4.1 G_α is a subgraph of $\Gamma'_I(R)$. Also each $G_\alpha \backsimeq \Gamma'_I(R/I)$, and $G_\alpha \cap G_\beta$ are disjoint if $\alpha \neq \beta$ because if $\alpha \neq \beta$ then $V(G_\alpha) \cap V(G_\beta) = \phi$. □

There is a strong relation between $\Gamma'_I(R)$ and $\Gamma'(R/I)$. Next theorem shows that how one can construct $\Gamma'_I(R)$ from $\Gamma'_I(R/I)$.

Theorem 4.3. *Let $\Gamma'_I(R)$ be an ideal based extended zero-divisor graph of a ring R. Then we can always construct $\Gamma'_I(R)$ from $\Gamma'(R/I)$.*

Proof. Let $\{[a_\lambda] \mid \lambda \in \Lambda\}$ be a set of coset representative vertices of $\Gamma'(R/I)$, i.e., $V(\Gamma'(R/I)) = \{[a_\lambda] : \lambda \in \Lambda\}$ and for each $\alpha \in I$, define a graph $G_\alpha = (V_\alpha, E_\alpha)$ with $V_\alpha = \{a_\lambda + \alpha : \lambda \in \Lambda\}$, where $a_\gamma + \alpha$ is adjacent to $a_\delta + \alpha$ in G_α whenever, $[a_\gamma]$ is adjacent to $[a_\delta]$ in $\Gamma'(R/I)$, i.e., either $(R/I)[a_\gamma] \cap ann_{(R/I)}([a_\delta]) \neq \{I\}$ or $(R/I)[a_\delta] \cap ann_{(R/I)}([a_\gamma]) \neq \{I\}$. Define a graph $H = (V(H), E(H))$ where $V(H) = \bigcup_{\alpha \in I} V(G_\alpha)$ and $E(H)$ is:

(i) all edge contained in G_α for each $\alpha \in I$.

(ii) For distinct $\gamma, \delta \in \Lambda$ and for any $\alpha, \beta \in I$, $a_\gamma + \alpha$ is adjacent to $a_\delta + \beta$ if and only if $[a_\gamma]$ is adjacent to $[a_\delta]$ in $(\Gamma'(R/I))$.

(iii) For $\gamma \in \Lambda$ and distinct $\alpha, \beta \in I$, $a_\gamma + \alpha$ is adjacent to $a_\gamma + \beta$ if and only if there exists a $r \in R$ such that $ra_\gamma \notin I$, but $ra_\gamma^2 \in I$.

Clearly, $V(H) \subseteq V(\Gamma'_I(R))$. Note that if $u \in V(\Gamma'_I(R))$, then by Theorem 4.1 $[u] \in V(\Gamma'(R/I))$ and therefore, $V(\Gamma'_I(R)) \subseteq V(H))$. So $V(H) = V(\Gamma'_I(R))$. By Theorem 4.1, all edges which are defined above by (i) and (ii) are also edges in $\Gamma'_I(R)$. If $a_\gamma + \alpha$ is adjacent to $a_\gamma + \beta$ for distinct $\alpha, \beta \in I$, then there exists $r \in R$ such that $ra_\gamma \notin I$, but $ra_\gamma^2 \in I$. Therefore, $(R(a_\gamma + \beta) + I) \cap (I : \{a_\gamma + \alpha\}) \neq I$ and $(R(a_\gamma + \gamma) + I) \cap (I : \{a_\gamma + \beta)\} \neq I$. Thus, the edges which are defined above by (iii) are also edge of $\Gamma'_I(R)$. Let u and v be distinct adjacent vertices of $\Gamma'_I(R)$. Then there exist $\alpha, \beta \in I$ and $\gamma, \delta \in \Lambda$ such that $u = a_\gamma + \alpha$ and $v = a_\delta + \beta$. If $\gamma \neq \delta$ and u adjacent to v in $\Gamma'_I(R)$. Hence by Theorem 4.1, $[a_\gamma]$ is adjacent to $[a_\delta]$ in $\Gamma'(R/I)$. Hence, the edge $u - v$ corresponds to an edge of type (i) or (ii) of H. If $\gamma = \delta$, then there exists $r \in R$ such that $ra_\gamma \notin I$, but $ra_\gamma^2 \in I$ and the edge $u - v$ corresponds to an edge of type (iii) of H. □

Proposition 4.4. *Let I be an ideal of a ring R. If $\Gamma'(R/I)$ is infinite, then $\Gamma'_I(R)$ is infinite. If $\Gamma'(R/I)$ is a graph with n vertices, then $\Gamma'_I(R)$ is a graph with $n|I|$ vertices.*

Proof. This is immediate from Theorem 4.3. □

Definition 4.5. Let $\{[a_\lambda] \mid \lambda \in \Lambda\}$ be a set of coset representative vertices of $\Gamma'(R/I)$. $[a_\lambda]$ is said to be a row of $\Gamma'_I(R)$, and if there exists $r \in R$ such that $ra_\lambda \notin I$ and $ra_\lambda^2 \in I$, then we call $[a_\lambda]$ connected row of $\Gamma'_I(R)$ and ξ_n denote the n connected row which is contained in a maximal complete subgraph of $\Gamma'(R/I)$.

Remark 4.6. Let I be an ideal in a commutative ring R with unity. Then every connected column of $\Gamma_I(R)$ defined in [13] is a connected row of $\Gamma'_I(R)$. By Example 2.12 and Figures 2.2 and 2.4 we observe that $[2] = \{2, 10, 18\}$ is a connected row of $\Gamma'_I(R)$ which is not a connected column of $\Gamma_I(R)$.

Theorem 4.7. *Let I be a ideal in a commutative ring R. Then* $\omega(\Gamma'_I(R)) = \xi_n|I| + \omega(\Gamma'(R/I)) - n$.

Proof. Suppose that $\omega(\Gamma'(R/I)) = k$ and $A = \{[a_1], [a_2], \cdots, [a_k]\} \subseteq V(\Gamma'(R/I))$ such that $\Gamma'(R/I)[A]$ is an induced maximal complete subgraph of $\Gamma'(R/I)$. Let $B = \bigcup [a_i]$ where $[a_i]$ is a connected row and $[a_i] \in A$, $C = \{a_i \mid [a_i] \text{ is a non-connected row}, [a_i] \in A\}$. Then by Theorem 4.1, $\Gamma'_I(R)[B \cup C]$ is a complete subgraph in $\Gamma'_I(R)$. If $B \cup C \cup \{u\}$ is a complete subgraph in $\Gamma'_I(R)$, then $\{[u]\} \cup A$ forms a clique of size $k+1$, a contradiction. Thus $\Gamma'_I(R)[B \cup C]$ is a maximal complete subgraph. Consequently, $\omega(\Gamma'_I(R)) = |B \cup C| = \xi_n|I| + \omega(\Gamma'(R/I)) - n$. □

Theorem 4.8. *Let I be an ideal of a commutative ring R such that $\Gamma'_I(R)$ has no connected row. Then*

(i) $\omega(\Gamma'_I(R)) = \omega(\Gamma'(R/I))$,

(ii) $\chi(\Gamma'_I(R)) = \chi(\Gamma'(R/I))$.

Proof. (i) Clearly, we observe that $\omega(\Gamma'(R/I)) \leq \omega(\Gamma'_I(R))$. Consider the case, when $\omega(\Gamma'(R/I)) = k < \infty$, and suppose that H is a complete subgraph of $\Gamma'_I(R)$ with the set of (distinct) vertices $u_1, u_2, \cdots, u_{k+1}$. Since H is complete, we get a complete subgraph of $\Gamma'_I(R)$ with the set of vertices $[u_1], [u_2], \cdots, [u_{k+1}]$. Now $\omega(\Gamma'(R/I)) = k$ implies that $[u_l] = [u_m]$ for some $l \neq m$ and hence $u_l = u_m + i$ for some $i \in I$. Since H is complete, u_l adjacent to u_m in $\Gamma'_I(R)$. Then we get $r \in R$ such that $ra_l \notin I$, but $ra_l^2 \in I$ and $[u_l]$ is a connected row $\Gamma'_I(R)$, a contradiction. Hence $\omega(\Gamma'_I(R)) = k$.

(ii) By Corollary 4.2, $\Gamma'(R/I$ is isomorphic to a subgraph of $\Gamma'_I(R)$ and hence $\chi(\Gamma'(R/I)) \leq \chi(\Gamma'_I(R)$. Suppose that $\chi(\Gamma'(R/I)) = n$ and $C_1, C_2, \cdots, C_n$ are distinct color classes of $\Gamma'(R/I)$. Consider the set $S_j = \bigcup_{[a] \in C_j} [a]$. Since $\Gamma'_I(R)$ has no connected row, each S_j is an independent set of $\Gamma'_I(R)$ and $V(\Gamma'_I(R)) = \bigcup_{j=1}^{n} S_j$. Thus $S_1, S_2, \cdots, S_n$ are distinct color classes for $\Gamma'_I(R)$ and the graph $\Gamma'_I(R)$ colored by n distinct proper colors, and therefore $\chi(\Gamma'_I(R) \leq n$. Hence $\chi(\Gamma'(R/I)) = \chi(\Gamma'_I(R)$. □

Corollary 4.9. *Let I be a radical ideal of a commutative ring R . Then*

(i) $\omega(\Gamma'_I(R)) = \omega(\Gamma'(R/I))$.

(ii) $\chi(\Gamma'_I(R)) = \chi(\Gamma'(R/I))$.

Theorem 4.10. *Let I be an ideal in a commutative ring R. If* $\omega(\Gamma'(R/I)) = \chi(\Gamma'(R/I))$, *then* $\omega(\Gamma'_I(R)) = \chi(\Gamma'_I(R))$.

Proof. Suppose that $\omega(\Gamma'(R/I)) = \chi(\Gamma'(R/I)) = n$. Let $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq Z_I^*(R)$ be a set of coset representative vertices of $\Gamma'(R/I)$, i.e., $V(\Gamma'(R/I)) = \{[a_\lambda] : \lambda \in \Lambda\}$ and $C_1, C_2, \cdots, C_n$ are distinct color classes of $\Gamma'(R/I)$. Since $\omega(\Gamma'(R/I)) = n$, there exists $[a_1], [a_2], \cdots, [a_n] \in V(\Gamma'(R/I))$ such that any two of them lies in distinct color classes. Without loss of generality, assume that $[a_j] \in C_j$, for all $j \in \{1, 2 \cdots, n\}$. $A = \{[a_1], [a_2], \cdots, [a_n]\}$. Then $\Gamma'(R/I)[A]$ is a maximal complete subgraph of $\Gamma'(R/I)$. Let $B = \{a_j \mid [a_j] \in A\} \cup \{a_j + i \mid [a_j] \in A,\ ra_j \notin I \text{ and } ra_j^2 \in I \text{ for some } r \in R, i \in I^*\}$. Since $\Gamma'(R/I)[A]$ is a maximal complete subgraph of $\Gamma'(R/I)$, $\Gamma'_I(R)[B]$ is a maximal complete subgraph of $\Gamma'_I(R)$, and therefore $|B| \leq \omega(\Gamma'_I(R))$. Hence we color the vertices of

$\Gamma'_I(R)$ with $|B|$ distinct colours. Clearly $[a]$, an induced independent set of $\Gamma'_I(R)$ when there does not exists any $r \in R$ such that $ra \notin I$ and $ra^2 \in I$ with $[a] \in A$ and color the vertices $a + i \in [a]$ with the colour of a for all $i \in I$. Let $U = \{a \; : \; [a] \in A\}$. Then U have distinct colors. For each $y \notin U, [y] = [a_t]$ such that $t \notin \{1, 2, \cdots, n\}$. Since $[a_t] \in S_j$ and $S'_j s$ are independent, for each $i \in I$ color the vertices $a_t + i$ with the color of $a_j + i$. Hence color the vertices of $C = V(\Gamma_I(R)) \setminus U$ in this way, and this coloring is proper, therefore $\chi(\Gamma_I(R)) \leq |B|$. Since $\omega(\Gamma_I(R)) \leq \chi(\Gamma_I(R))$, $\chi(\Gamma_I(R)) = \omega(\Gamma_I(R))$. This completes the proof. □

Lemma 4.11. *Let I be an ideal of a ring R and $a \in V(\Gamma'_I(R))$. Then*
$$deg(a) = \begin{cases} |I|deg_{\Gamma'}([a]), & if \;\; [a] \; is \; a \; non-connected \; row, \\ |I|deg_{\Gamma'}([a]) + |I| - 1, & if \;\; [a] \; is \; a \; connected \; row. \end{cases}$$

Proof. Clearly, $deg(a) \geq |I|deg_{\Gamma'}([a])$. If $[a]$ is connected row, then $\Gamma'_I(R)[[a]]$ is a complete subgraph of $\Gamma'_I(R)$. Thus $deg(a) = |I|deg_{\Gamma'}([a]) + |I| - 1$. If $[a]$ is non-connected row, then $deg(a) = |I|deg_{\Gamma'}([a])$. □

Lemma 4.12. *Let I be an ideal of a ring R Then*
$$\delta(\Gamma'_I(R)) = \begin{cases} |I|\delta(\Gamma'(R/I)) + |I| - 1, & if \;\; each \; [a] \in V(\delta(\Gamma'(R/I)) \; is \; a \; connected \; row, \\ |I|\delta(\Gamma'(R/I)), & otherwise. \end{cases}$$

Proof. If $[a] \in V(\delta(\Gamma'(R/I))$ is a connected row, then $deg(a) \leq deg(b)$ for all $b \in V(\Gamma'_I(R)$ and by Lemma 4.11, $deg(a) = |I|deg_{\Gamma'}([a]) + |I| - 1$ (or $deg(a) = |I|\delta(\Gamma'(R/I)) + |I| - 1$). Thus $\delta(\Gamma'_I(R)) = |I|\delta(\Gamma'(R/I)) + |I| - 1$. Otherwise, $deg(a) \leq deg(b)$ for all $b \in V(\Gamma'_I(R)$ and by Lemma 4.11, $deg(a) = |I|deg_{\Gamma'}([a])$ (or $deg(a) = |I|\delta(\Gamma'(R/I))$. Thus $\delta(\Gamma'_I(R)) = |I|\delta(\Gamma'(R/I))$. □

Lemma 4.13. *Let I be an ideal of a ring R Then*
$$\Delta(\Gamma'_I(R)) = \begin{cases} |I|\Delta(\Gamma'(R/I)) + |I| - 1, & if \;\; each[a] \in V(\Delta(\Gamma'(R/I)) \; is \; a \; non \; connected \; row, \\ |I|\Delta(\Gamma'(R/I)), & otherwise. \end{cases}$$

Proof. If $[a] \in V(\Delta(\Gamma'(R/I))$ is a non-connected row, then $deg(b) \leq deg(a)$ for all $b \in V(\Gamma'_I(R)$ and by Lemma 4.11, $deg(a) = |I|deg_{\Gamma'}([a])$ (or $deg(a) = |I|\Delta(\Gamma'(R/I))$). Thus $\Delta(\Gamma'_I(R)) = |I|\delta(\Gamma'(R/I))$. Otherwise, $deg(b) \leq deg(a)$ for all $b \in V(\Gamma'_I(R)$ by Lemma 4.11, $deg(u) = |I|deg_{\Gamma'}([a] + |I| - 1)$ (or $deg(a) = |I|\Delta(\Gamma'(R/I) + |I| - 1)$. Thus $\Delta(\Gamma'_I(R)) = |I|\Delta(\Gamma'(R/I)) + |I| - 1$. □

Theorem 4.14. *Let I be an ideal in a commutative ring R. If $\Gamma'_I(R)$ has no connected row, then $\Gamma'_I(R)$ is Eulerian if and only if $|I|$ is even or $\Gamma'(R/I)$ is Eulerian.*

Proof. Suppose that $\Gamma'_I(R)$ is Eulerian. Then $deg(a)$ is even for all $a \in V(\Gamma'_I(R))$. Since $\Gamma'_I(R)$ has no connected row, $deg(a) = |I|deg_{\Gamma'}([a])$ is even for all $[a] \in V(\Gamma'(R/I))$. Hence either $|I|$ is even or $deg_{\Gamma'}([a])$ is even for all $[a] \in V(\Gamma'(R/I))$, i.e., $\Gamma'(R/I)$ is Eulerian.

Conversely, assume that $\Gamma'(R/I)$ is Eulerian. Hence $deg_{\Gamma'}([a])$ is even for all $[a] \in V(\Gamma'(R/I))$. Since $\Gamma'_I(R)$ has no connected row, $deg(a) = |I|deg_{\Gamma'}([a])$ is even for all $a \in V(\Gamma'_I(R)$. i.e., $\Gamma'_I(R)$ is Eulerian. If $|I|$ is even, then $\Gamma'_I(R)$ is Eulerian. □

Theorem 4.15. *Let I be an ideal in a commutative ring R. If $\Gamma'_I(R)$ has a connected row, then $\Gamma'_I(R)$ is Eulerian if and only if $|I|$ is odd and $\Gamma'(R/I)$ is Eulerian.*

Proof. Suppose that $\Gamma'_I(R)$ is Eulerian. Since $\Gamma'_I(R)$ has a connected row, there exists $x \in V(\Gamma'_I(R))$ such that $[x]$ is a connected row in $\Gamma'_I(R)$ and by Lemma 4.11, $deg(x) = |I|deg_{\Gamma'}[x] + |I| - 1$ is even. Thus we have the following cases:

Case(a) $|I|deg_{\Gamma'}[x]$ and $|I| - 1$ are odd. Then $|I|$ is even. Since $|I|deg_{\Gamma'}[x]$ is odd and $|I|$ is even. Since $|I|$ is even, $|I|deg_{\Gamma'}[x]$ can not be odd, and this case is not possible.
Case(b) $|I|deg_{\Gamma'}[x]$ and $|I| - 1$ are even. Thus $|I|deg_{\Gamma'}[x]$ is even for all $[x] \in V(\Gamma'(R/I))$. i.e., $deg_{\Gamma'}[x]$ is even for all $[x] \in V(\Gamma'(R/I))$. Therefore $\Gamma'(R/I)$ is Eulerian and $|I|$ is odd.

Conversely, assume that $\Gamma'(R/I)$ is Eulerian, $|I|$ is odd and $x \in V(\Gamma'_I(R))$. If $[x]$ is a connected row, then $deg(x) = |I|deg_{\Gamma'}[x] + |I| - 1$ is even and if $[x]$ is a non-connected row, then $deg(x) = |I|deg_{\Gamma'}[x]$ is also even. Hence $\Gamma'_I(R)$ is Eulerian. □

Theorem 4.16. *Let I be an ideal in a commutative ring R. If $\Gamma'_I(R)$ has no connected row. Then $\Gamma'_I(R)$ is regular if and only if $\Gamma'(R/I)$ is regular.*

Proof. Suppose that $\Gamma'_I(R)$ is regular graph, $deg(x) = n$ for all $x \in V(\Gamma'_I(R))$. Since $\Gamma'_I(R)$ has no connected row, by Lemma 4.11, $deg(x) = |I|deg_{\Gamma'}[x] = n$ for all $[x] \in V(\Gamma'(R/I))$. Therefore $deg_{\Gamma'}[x] = n/|I|$ for all $[x] \in V(\Gamma'(R/I))$. Clearly, if n is prime, then $\Gamma'(R/I) \approxeq K_2$. Otherwise $\Gamma'(R/I)$ is a $\frac{n}{|I|}$-regular.

Conversely, suppose that $\Gamma'(R/I)$ is a regular graph. Then $deg_{\Gamma'}[x] = n \ \forall\ [x] \in V(\Gamma'(R/I))$. Since $\Gamma'_I(R)$ has no connected row, by Lemma 4.11, for all $x \in V(\Gamma'_I(R))$ $deg(x) = |I|deg_{\Gamma'}[x] = n|I|$. Therefore $\Gamma'_I(R)$ is $n|I|$-regular. □

Theorem 4.17. *Let I be an ideal in a commutative ring R and each row is connected. Then $\Gamma'_I(R)$ is n-regular, where $n \neq |I| - 1$ if and only if $\Gamma'(R/I)$ is regular.*

Proof. Assume that $\Gamma'_I(R)$ is a n-regular graph. Then $deg(x) = n$ for all $x \in V(\Gamma'_I(R))$. Since each row is connected, by Lemma 4.11, $deg(x) = |I|deg_{\Gamma'}[x] + |I| - 1$, for all $x \in V(\Gamma'_I(R))$ and hence $deg_{\Gamma'}[x] = \frac{n-|I|+1}{|I|}$ for all $[x] \in V(\Gamma'(R/I))$. Since $deg_{\Gamma'}[x] \neq 0$ and $n \neq |I| - 1$, $\Gamma'(R/I)$ is a $(\frac{n-|I|+1}{|I|})$-regular graph.

Conversely, suppose that $\Gamma'(R/I)$ is a regular graph. Then $deg_{\Gamma'}[x] = p$ for all $[x] \in V(\Gamma'(R/I)$. Since each row is connected, by Lemma 4.11, $deg(x) = p|I| + |I| - 1$ for all $x \in V(\Gamma'_I(R)$. Thus $\Gamma'_I(R$ is a n-regular. □

Theorem 4.18. *Let I be an ideal of a ring R. Then $1 \leq \chi(\Gamma'(R/I)) \leq \chi(\Gamma'_I(R)) \leq |I|\chi(\Gamma'(R/I))$.*

Proof. Clearly, $1 \leq \chi(\Gamma'(R/I))$. Since $\Gamma'(R/I)$ is isomorphic to a subgraph of $\Gamma'_I(R)$, $\chi(\Gamma'(R/I)) \leq \chi(\Gamma'_I(R))$. Let $\chi(\Gamma'(R/I)) = n$, and $C_1, C_2, \cdots, C_n$ be distinct color classes for $\Gamma'(R/I)$. Assume that each row is connected. Now for each $1 \leq j \leq n$, and $i \in I$ define a set $D_{ji} = \{x + j \ : \ [x] \in C_j\}$. Since C_j's are independent, D_{ji} are independent. Also $\bigcup\limits_{1 \leq j \leq n} (\bigcup\limits_{i \in I}) D_{ji} = V(\Gamma'_I(R)$. Thus $\{D_{ji} \ : \ 1 \leq j \leq n, i \in I\}$ are distinct color classes for $\Gamma'_I(R)$. $|I|n$ colors are required for colouring and this colouring is proper. Hence $\chi(\Gamma'_I(R)) \leq |I|\chi(\Gamma'(R/I))$. □

Proposition 4.19. *Let I be a proper ideal of a commutative ring R. If $\Gamma'_I(R)$ has a connected row, then $|I| \leq \omega(\Gamma'_I(R))$.*

Proof. Assume that $[u]$ is a connected row in $\Gamma'_I(R)$. Then there exists $r \in R$ such that $ru \notin I$ and $ru^2 \in I$. If $u_1, u_1 \in [u]$, then $(Ru_1 + I) \cap (I : \{u_2\}) \neq I$ and by definition u_1 is adjacent to u_2 in $\Gamma'_I(R)$. i.e., $K^{|I|}$ is a subgraph of $\Gamma'_I(R)$, and hence $|I| \leq \omega(\Gamma'_I(R))$. □

Corollary 4.20. *Let I be a proper ideal of a commutative ring R such that $|I| = \infty$. If $\Gamma'_I(R)$ has a connected row, then $\omega(\Gamma'_I(R)) = \infty$.*

Corollary 4.21. *Let I be a proper ideal of a commutative ring R such that $|V(\Gamma'_I(R))| \geq 2$. If $\Gamma'_I(R)$ has a connected row, then $|I| + 1 \leq \omega(\Gamma'_I(R))$.*

Lemma 4.22. *Let I be an ideal of a commutative ring R. Then $gr(\Gamma'_I(R)) \leq gr(\Gamma'(R/I))$.*

Proof. If $gr(\Gamma'_I(R)) = \infty$, then our result holds. Now suppose that $gr(\Gamma'(R/I)) = k < \infty$. Let $[a_1] - [a_2]-, \cdots, -[a_k] - [a_1]$ be a cycle in $\Gamma'_I(R)$ with k distinct vertices. Then $a_1 - a_2-, \cdots, -a_k - a_1$ is also a cycle in $\Gamma'_I(R)$ of length k. Hence $gr(\Gamma'_I(R)) \leq k$. □

5. When $\Gamma'_I(R)$ is Weakly Perfect and Planar?

In this section, our aim is to study the planarity of ideal based extended zero-divisor graph $\Gamma'_I(R)$ and explore the condition under which $\Gamma'_I(R)$ is planar. For a radical ideal I of an Artinian ring R, we show that $\Gamma'_I(R)$ is weakly perfect.

Theorem 5.1. *Let I be an ideal of a commutative ring R. Then $\Gamma'_I(R)$ is a complete n-partite graph if and only if $\Gamma'(R/I)$ is a complete n-partite graph.*

Proof. Suppose that $\Gamma'_I(R) = K_{|W_1|,|W_2|,\cdots,|W_n|}$ where $V(\Gamma'_I(R)) = \bigcup_{i=1}^{n} W_i$ and $W_j \cap W_k = \phi$ for $j \neq k$. Define a map $F : R \longrightarrow R/I$ by $F(x) = [x]$. Clearly F is a homomorphism from R onto R/I. It is easy to check that $\Gamma'(R/I) = K_{|F(W_1)|,|F(W_2)|,\cdots,|F(W_n)|}$ is a complete n-partite graph.

Conversely, suppose that $\Gamma'(R/I) = K_{|L_1|,|L_2|,\cdots,|L_n|}$ where $V(\Gamma'(R/I)) = \bigcup_{i=1}^{n} L_i$ and $L_j \cap L_k = \phi$ for $j \neq k$. Define a map $S : R \longrightarrow R/I$ by $S(y) = [y]$. Clearly S is a homomorphism from R onto R/I. It is easy check that $\Gamma'_I(R) = K_{|S^{-1}(L_1)|,|S^{-1}(L_2)|,\cdots,|S^{-1}(L_n)|}$ is a complete n-partite graph. □

Lemma 5.2. *Let I be an ideal of R such that $R/I \approxeq D_1 \times D_2 \times \cdots \times D_k$, where $k \geqslant 2$ is a positive integer and D_j is an integral domain, for every $1 \leqslant j \leqslant k$. Then $\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite.*

Proof. Given $R/I \approxeq D_1 \times D_2 \times \cdots \times D_k$. Then by [6, Lemma 2.1], $\Gamma'(R/I)$ is a complete $(2^k - 2)$-partite and by Theorem 5.1, $\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite hence proved. □

Proposition 5.3. *Let I be a radical ideal of a commutative ring R with $|Min_I(R)| < \infty$ and suppose that P, Q are coprime, for every two distinct $P, Q \in Min_I(R)$. Then the following statements are equivalent.*

(i) $|Min_I(R)| = k$.

(ii) *$\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $|Min_I(R)| = k$ and define a map $F : R \longrightarrow R/I$ by $F(x) = [x]$. Clearly, $F(\{Min_I(R)\}) = Min(R/I)$ and $|Min(R/I)| = k$. Then by [6, Corollory 2.2], $\Gamma'(R/I)$ is a complete $(2^k - 2)$-partite and by Theorem 5.1, $\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite.

$(ii) \Rightarrow (i)$ Assume that $\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite. Then by Theorem 5.1, $\Gamma'(R/I)$ is a complete $(2^k - 2)$-partite and by [6, Corollary 2.2], $|Min(R/I)| = k$. Let us define a map $S : R \longrightarrow R/I$ by $S(x) = [x]$. Clearly, $S^{-1}(\{Min(R/I\})) = Min_I(R)$ and $|Min_I(R)| = k$. □

Proposition 5.4. *Let I be an ideal in a ring R such that $R/I \approxeq D_1 \times D_2 \times \cdots \times D_k$, where $k \geqslant 2$ is a positive integer and D_j is an integral domain for each $j \in \{1, 2, \cdots, n\}$. Then $\omega(\Gamma'(R/I)) = \chi(\Gamma'_I(R)) = \chi(\Gamma'(R/I)) = \omega(\Gamma'_I(R)) = (2^k - 2)$.*

Proof. Given $R/I \approxeq D_1 \times D_2 \times \cdots \times D_k$ where $k \geqslant 2$ be a positive integer and D_j is an integral domain for each $j \in \{1, 2, \cdots, k\}$. Then by [6, Lemma 2.1], $\Gamma'(R/I)$ is a $(2^k - 2)$-partite graph and by Lemma 5.2, $\Gamma'_I(R)$ is a $(2^k - 2)$-partite graph. Hence $\omega(\Gamma'(R/I))$ $= \chi(\Gamma'_I(R)) = \chi(\Gamma'(R/I)) = \omega(\Gamma'_I(R)) = (2^k - 2)$. □

Corollary 5.5. *let I be a radical ideal in a commutative ring R with unity such that R/I is an Artinian ring. Then $\omega(\Gamma'(R/I)) = \chi(\Gamma'_I(R)) = \chi(\Gamma'(R/I)) = \omega(\Gamma'_I(R)) = 2^{|Max(R/I)|} - 2$.*

Corollary 5.6. *let I be a radical ideal in an Artinian ring R. Then $\omega(\Gamma'(R/I)) = \chi(\Gamma'_I(R))$ $= \chi(\Gamma'(R/I)) = \omega(\Gamma'_I(R)) = 2^{|Max(R/I)|} - 2$.*

In order to achieve the goal, we need a celebrated Kuratowski's theorem from Graph Theory [14, Theorem 6.2.2].

Theorem 5.7. *(Kuratowski's Theorem) A Graph G is planar if and only if it contains no subdivision of either $K_{3,3}$ or K_5.*

Proposition 5.8. *Let I be a proper ideal of R. If $\Gamma'_I(R)$ is a planar graph. Then $\Gamma'(R/I)$ is also a planar graph but the converse need not be true in general.*

Proof. Suppose that $\Gamma'_I(R)$ is a planar graph. Since $\Gamma'(R/I)$ is isomorphic to a sub graph of $\Gamma'_I(R)$. By Theorem 5.7, $\Gamma'(R/I)$ is a planar graph. For the converse with the help of Example 2.12, we note that in the Figure 2.4, $\Gamma'_I(R) = K_9$ is not planar, but $R/I = \mathbb{Z}_8$ and $\Gamma'(R/I) = K_3$ a planar graph. □

Theorem 5.9. *Let I be a radical ideal of a commutative ring R. Then the following statements are equivalent.*

(i) $\Gamma'_I(R)$ *is planar.*

(ii) $|Min_I(R)| = 2$ *and one element of $Min_I(R)$ has at most two elements different from I.*

Proof. $(i) \Rightarrow (ii)$ Assume that $\Gamma'_I(R)$ is planar. Suppose on the contrary that $|Min_I(R)| \geq 3$. Let us define a map $F : R \longrightarrow R/I$ by $F(x) = [x]$. Clearly, $F(Min_I(R)) = Min(R/I)$ and $|Min(R/I)| \geq 3$. By [6, Theorem 3.4], $\Gamma'(R/I)$ is not planar and by Lemma 5.8, $\Gamma'_I(R)$ is not planar, a contradiction. Therefore, $|Min_I(R)| = 2$ and by Theorem 3.1,

$\Gamma'_I(R) = \Gamma_I(R)$. Let $P_I, Q_I \in Min_I(R)$ such that $|P_I \setminus I| \geq 3, |Q_I \setminus I| \geq 3$. Then $K_{3,3}$ is a subgraph of $\Gamma'_I(R)$ which is not Planar, a contradiction. Thus one element of $Min_I(R)$ has at most two elements different from I.

$(ii) \Rightarrow (i)$ Suppose that $|Min_I(R)| = 2$ and one element of $Min_I(R)$ has at most two elements different from I. Then by Theorem 3.1, $\Gamma'_I(R) = \Gamma_I(R)$. Without loss of generality, we may assume that $P_I, Q_I \in Min_I(R)$ such that $|P_I \setminus I| = m$, where $1 \leq m \leq 2$ and $|Q_I \setminus I| = n$. Thus $\Gamma'_I(R) = K_{m,n}$, which is Planar. □

Proposition 5.10. *Let I be an ideal of a commutative ring R. Then $\Gamma'_I(R)$ is not planar if one of the following statements hold.*

(i) $|I| \geq 5$.

(ii) $|\beta^*(I)| > 4$.

(iii) *I is a radical ideal of R and* $|I| \geq 3$.

Proof. Directly follows from Theorem 5.7. □

Remark 5.11. It can be easily observed that if R is a commutative ring with unity, then $|Z(R)| = 2$ if and only if R is ring-isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$.

Theorem 5.12. *Let I be a non-radical ideal of a commutative ring R such that $|I| = 2$. Then $\Gamma'_I(R)$ is planar if and only if one of the following statements hold.*

(i) *R/I is ring-isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$.*

(ii) *$(I : Z_I(R))$ is a prime ideal of R and $|(I : Z_I(R))| = 4$.*

(iii) *$Z_I(R) = \beta(I)$ and $|\beta(I)| = 6$.*

Proof. Assume that $\Gamma'_I(R)$ is planar. If $|\beta(I)| = \infty$, then by Lemma 2.4 (iv), $\Gamma'_I(R)[\beta^*(I)]$ is not planar. Thus $\Gamma'_I(R)$ is not planar and we find that $|\beta(I)| < \infty$. Since I is a proper additive subgroup of $\beta(I)$, $|I|$ divides $|\beta(I)|$ and $|\beta(I)| = 2k$, where $k \in \mathbb{N} \setminus \{1\}$. Then the following cases arises:

Case(1) $k = 2$, i.e., $|\beta(I)| = 4$. Then $|Nil(R/I)| = 2$.
Subcase(i) If $|Z_I(R)| < \infty$, then $|Z(R/I)| < \infty$. If $|Z(R/I)| = 2$, then by Remark 5.11, R/I is isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$. If $2 \neq |Z(R/I)| < \infty$, then by [6, Theorem 3.6(1)], R/I is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{(x^2)}$. If R/I is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, then there exists an isomorphism $g : R/I \to \mathbb{Z}_2 \times \mathbb{Z}_4$.
Notice that there exist α_1, α_2, α_3, α_4 $\in$ $R \setminus I$ such that $[\alpha_1]$, $[\alpha_2]$, $[\alpha_3]$, $[\alpha_4]$ $\in$ R/I and $g([\alpha_1]) = (0, 1)$, $g([\alpha_2]) = (0, 3)$, $g([\alpha_3]) = (1, 0)$, $g([\alpha_4]) = (1, 2)$. Since $\Gamma'(R/I)[\{[\alpha_1], [\alpha_2][\alpha_3][\alpha_4]\}] = K_{2,2} \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4)[\{(0, 1), (0, 3), (1, 0), (1, 2)\}]$, without loss of generality, we may assume that $\alpha_1, \alpha_1 + i, \alpha_2, \alpha_3, \alpha_3 + i, \alpha_4 \in R \setminus I$, where $i \in I^*$ and by Theorem 4.1 (i), $\Gamma'_I(R)$ $[\{\alpha_1,\ \alpha_1 + i,\ \alpha_2,\ \alpha_3,\ \alpha_3 + i,\ \alpha_4\}] = K_{3,3}$, which is not planar, a contradiction. If R/I is isomorphic to $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{(x^2)}$, then there exists an isomorphism $f : R/I \to \mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{(x^2)}$. Notice that there exist β_1, β_2, β_3, $\beta_4 \in R \setminus I$ such that $[\beta_1]$, $[\beta_2]$, $[\beta_3]$, $[\beta_4]$ $\in$ R/I and $g([\beta_1]) = (0, (x^2))$, $g([\beta_2]) = (0, 1 + (x^2))$, $g([\beta_3]) = (1, (x^2))$, $g([\beta_4]) = (1, x + (x^2))$. Since $\Gamma'(R/I)[\{[\beta_1],\ [\beta_2]\ [\beta_3]\ [\beta_4]\}] = K_{2,2} \cong \Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4)[\{(0, (x^2)),\ (0, 1 + (x^2)),\ (1, (x^2)),\ (1, x + (x^2))\}]$, without loss of generality,

we may assume that $\beta_1, \beta_1 + i, \beta_2, \beta_3, \beta_3 + i, \beta_4 \in R \setminus I$, where $i \in I^*$ and by Theorem 4.1 (i), $\Gamma'_I(R)$ $[\{\beta_1, \beta_1 + i, \beta_2, \beta_3, \beta_3 + i, \beta_4\}] = K_{3,3}$, which is not planar, again we get a contradiction.

Subcase(ii) $|Z_I(R)| = \infty$. Since $|I| = 2 < \infty$, $|Z(R/I)| = \infty$. Hence by [6, Theorem 3.6(2)], $Ann(Z(R/I))$ is a prime ideal of R/I. This implies that $(I : Z_I(R))$ is a prime ideal of R and by Corollary 3.5, $\Gamma'_I(R) = \Gamma_I(R) = K_p \vee \overline{K_q}$, where $p = |\beta^*(I)|$, $q = |Z_I(R) \setminus \beta(I)| = \infty$ and by Corollary 3.4 (iii), $(I : Z_I(R)) = \beta(I)$. Thus if we take $|\beta^*(I)| = \ell > 4$, then $\Gamma'_I(R) = \Gamma_I(R) = K_\ell \vee \overline{K_\infty}$ and $\Gamma'_I(R) = \Gamma_I(R) = K_\ell \vee \overline{K_\infty}$ contain $K_{3,3}$ as a subgraph, and hence $\Gamma'_I(R)$ is not planar. If $|\beta(I)| = 4$, then $\Gamma'_I(R) = \Gamma_I(R) = K_2 \vee \overline{K_\infty}$, which is planar. Hence $|\beta(I)| = |(I : Z_I(R))| = 4$.

Case(2) $k = 3$, i.e., $|\beta(I)| = 6$. Then $|Nil(R/I)| = 3$ and by [6, Theorem 3.8], $Ann(Z(R/I))$ is a prime ideal of R/I. This implies that $(I : Z_I(R))$ is a prime ideal of R and by Corollary 3.5, $\Gamma'_I(R) = \Gamma_I(R) = K_p \vee \overline{K_q}$, where $p = |\beta(I)^*|$, $q = |Z_I(R) \setminus \beta(I)|$. If $Z_I(R) \neq \beta(I)$, then $K_5 = K_4 \vee \overline{K_1}$ is a subgraph of $K_4 \vee \overline{K_q}$, which is not planar. Hence $\beta(I) = Z_I(R)$ and by Lemma 2.4 (iv), $\Gamma'_I(R) = K_4$, which is Planar.

Case(3) $k \geq 3$, i.e., $|\beta(I)| \geq 8$. Then $|\beta^*(I)| > 4$ and by Proposition 5.10 (ii), $\Gamma'_I(R)$ is not Planar. Hence $|\beta(I)| \leq 6$.

Converse part holds trivially. □

Theorem 5.13. *Let I be a non-radical ideal of a commutative ring R and $|I| = 3$. Then $\Gamma'_I(R)$ is planar if and only if R/I is ring-isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$.*

Proof. Assume $\Gamma'_I(R)$ is planar. Since $|I| = 3$, $|\beta(I)| = 6$, and $|Nil(R/I)| = 2$. If $|Z(R/I)| > 2$, then $K_{3,3}$ is a subgraph of $\Gamma'_I(R)$. By Theorem 5.7, $\Gamma'_I(R)$ is not planar, a contradiction. Hence $|Z(R/I)| = 2$, then by Remark 5.11, R/I is isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$. Converse part holds trivially. □

Proposition 5.14. *Let I be a non-radical ideal of a commutative ring R and $|I| = 4$. Then $\Gamma'_I(R)$ is planar if and only if R/I is isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$.*

Proof. Assume that $\Gamma_I(R)$ is planar. Since $|I| = 4$, $|\beta(I)| = 8$. If $\beta(I) \neq Z_I(R)$, then there exists $\alpha \in Z_I(R) \setminus \beta(I)$ and by Lemma 2.4 (iv), $\Gamma_I(R)[\{\alpha\} \cup \beta^*(I)]$ forms K_5, which is not planar. Hence $\beta(I) = Z_I(R)$, $|Z(R/I)| = |Nil(R/I)| = 2$, and by Remark 5.11, R/I is isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$. Converse part holds trivially. □

Proposition 5.15. *Let I be non-radical ideal of a commutative ring R. Then $\gamma(\Gamma'_I(R))$ $= \gamma_s(\Gamma'_I(R)) = 1$.*

Proof. Let $x \in \beta^*(I)$. Then by Lemma 2.4, x is adjacent to every other vertex and $\deg(x) \geq \deg(y)$, for every y in $V(\Gamma'_I(R))$. Thus $\{x\}$ is a γ-set of $\Gamma'_I(R)$ and $\gamma(\Gamma'_I(R))$ $= \gamma_s(\Gamma'_I(R)) = 1$. □

Proposition 5.16. *Let I be a radical ideal of a commutative ring R. Then $\gamma(\Gamma'_I(R)) = 2$ and $\Gamma'_I(R)$ is excellent graph if one of the following statements hold.*

(i) *$R/I \cong D_1 \times D_2 \times \cdots \times D_k$ where $k \geqslant 2$ be a positive integer and D_j is an integral domain for each $j \in \{1, 2, \cdots, k\}$.*

(ii) $|Min_I(R)| = k$.

Proof. (i) Clearly by Lemma 5.2, $\Gamma'_I(R)$ is a complete (2^k-2)-partite. Assume that $\Gamma'_I(R)$ $= K_{|V_1|,|V_2|,\cdots,|V_k|}$. Clearly $\{x_1, x_2\}$ is a γ-set, where $x_1 \in V_1$ and $x_2 \in V_2$. Since $|I| \geq 2$, $|V_1| \geq 2$ and $|V_2| \geq 2$. Clearly $\{y_1, y_2\}$ is a γ-set, where $y_1 \in V_1 \setminus \{x_1\}$ and $y_2 \in V_2 \setminus \{x_2\}$. Therefore $\gamma(\Gamma'_I(R)) = 2$.

(ii) Clearly by Proposition 5.3, $\Gamma'_I(R)$ is a complete $(2^k - 2)$-partite any by part (i) $\gamma(\Gamma'_I(R)) = 2$. □

6. Ordering on the Vertices of $\Gamma'_{\mathscr{I}}(\mathscr{R})$

In this section, we study the ordering on the vertices of $\Gamma'_I(R)$.

Definition 6.1. Given a graph H with vertices u and v, we define the relations $\leq, \sim$ and $\perp$ on H as follows.

(i) $u \leq v$ if every vertex adjacent to v is also adjacent to u.

(ii) $u \sim v$ if $u \leq v$ and $v \leq u$.

(iii) $u \perp v$ if u and v are adjacent and no other vertex of H is adjacent to both u and v.

Remark 6.2. Graphs $\Gamma'_I(R)$ and $\Gamma'(R/I)$ are simple, so any vertex of these graphs is never considered to be self adjacent. Hence, if $u \leq v$, then $u - v$ not an edge (otherwise v is self adjacent).

Proposition 6.3. *Let I be an ideal of a commutative ring R. Let $u, v \in Z^*_I(R)$ such that $[u]$ and $[v]$ are nonconnected row of $\Gamma'_I(R)$. Then $[u] \leq [v]$ in $\Gamma'(R/I)$ if and only if $u \leq v$ in $\Gamma'_I(R)$.*

Proof. Assume $[u] \leq [v]$ in $\Gamma'(R/I)$. Let $z \in Z^*_I(R)$ be adjacent to v. Since $[v]$ is nonconnected, $[v] \neq [z]$ (otherwise, $[v]$ is connected row). Thus, by Theorem 4.1, $[z]$ is adjacent to $[v]$, since $[u] \leq [v]$ implies that $[z]$ is adjacent to $[u]$. Hence, By Theorem 4.1, u is adjacent to z.
Conversely, assume $u \leq v$ in $\Gamma'_I(R)$. Let $[z] \in Z^*(R/I)$ be adjacent to $[v]$ in $\Gamma'_I(R)$. Then, by Theorem 4.1, z is adjacent to v in $\Gamma'_I(R)$. Since $u \leq v$ implies that z is adjacent to u in $\Gamma'_I(R)$. Since $[u]$ is nonconnected row implies that $[z] \neq [u]$ and by Theorem 4.1, $[u]$ is adjacent to $[z]$ in $\Gamma'(R/I)$. □

Corollary 6.4. *Let I be a proper ideal of a commutative ring R, and let $u, v \in Z^*_I(R)$ such that $[u]$ and $[v]$ are nonconnected row of $\Gamma'_I(R)$. Then $[u] \sim [v]$ in $\Gamma'(R/I)$ if and only if $u \sim v$ in $\Gamma'_I(R)$.*

Corollary 6.5. *Let I be a proper ideal of a commutative ring R, and let $u, v \in Z^*_I(R)$ such that $u, v \in [z]$, where $[z]$ is a nonconnected row of $\Gamma'_I(R)$. Then $u \sim v$ in $\Gamma'_I(R)$.*

Remark 6.6. In case of connected row, the conclusion of the above result fails, because in case of connected row we find a self adjacent vertices, as mention in the previous remark.

Proposition 6.7. *Let I be an ideal of a commutative ring R such that $|V(\Gamma'_I(R))| \geqslant 3$. Suppose that $u, v \in Z^*_I(R)$ such that $[u] \neq [v]$ and both are nonconnected row of $\Gamma'_I(R)$. Then $[u] \perp [v]$ in $\Gamma'(R/I)$ if and only if $u \perp v$ in $\Gamma'_I(R)$.*

Proof. Assume $u \perp v$ in $\Gamma'_I(R)$. Then $u - v$ is an edge of $\Gamma'_I(R)$ and by Theorem 4.1, $[u] - [v]$ is an edge of $\Gamma'_I(R)$. If $[z] \in Z^*(R/I)$ such that $[u] - [z]$ and $[v] - [z]$ are edges in $\Gamma'(R/I)$, then by Theorem 4.1, $u - z$ and $v - z$ are edges in $\Gamma'_I(R)$, a contradiction. Hence $[u] \perp [v]$ in $\Gamma'(R/I)$.

Conversely suppose that $[u] \perp [v]$ in $\Gamma'(R/I)$. Then $u - v$ is an edge in $\Gamma'_I(R)$. Assume that $z \in Z^*_I(R)$ such that $u - z$ and $v - z$ are edges in $\Gamma'_I(R)$. Then there exists $r \in R$ such that either $ru \notin I$ or $rz \notin I$ but $ruz \in I$. Similarly, there exists $s \in R$ such that either $sv \notin I$ or $sz \notin I$, but $svz \in I$. Since $[u]$ and $[v]$ are non connected, $[u] \neq [z] \neq [v]$. Therefore, $[u] - [z]$ and $[v] - [z]$ are edges in $\Gamma'(R/I)$, which contradicts, $[u] \perp [v]$, and hence $u \perp v$ in $\Gamma'_I(R)$. □

Acknowledgements. The authors are indebted to the learned referee for several useful suggestions which have improved immensely.

References

[1] D. F. Anderson and A. Badawi, *On the zero-divisor graph of a commutative ring,* Comm. Algebra, **36(8)**(2008), 3073–3092.

[2] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring,* J. Algebra, **217**(1999), 434–447.

[3] M. Ashraf, M. Kumar and G. Mohammad, *A subspace based subspace inclusion graph on vector space,* Contrib. Discrete Math., **15(2)**(2020), 73–83.

[4] M. Ashraf, M. Kumar and A. Jabeen, *Subspace-based subspace sum graph on vector spaces,* Soft Computing, **15**(2021), 11429–11438.

[5] M. Bakhtyiari, M. J. Nikmehr and R. Nikandish, *The extended zero-divisor graph of a commutative ring I,* Hakkaido Mathematical Journal, **46**(2017), 381–393.

[6] M. Bakhtyiari, M. J. Nikmehr and R. Nikandish, *The extended zero-divisor graph of a commutative ring II,* Hakkaido Mathematical Journal, **46**(2017), 395–406.

[7] A. Badawi, *On the annihilator graph of a commutative ring* Comm. Algebra, **42(1)**(2014), 108–121.

[8] I. Beck, *Colouring of commutative rings,* J. Algebra, **116**(1988), 208–226.

[9] R. Diestel, *Graph Theory,* Springer-Verlag, New York(1997).

[10] J. A. Huckaba, *Commutative ring with zero-divisor,* Marcel Dekker, Inc., New York(1998).

[11] I. Kaplansky, *Commutative rings* rev. ed., Univ. of Chicago press, Chicago(1974).

[12] S. Kiani, H. R. Maimani and R. Nikandish, *Some results on the domination number of a zero divisor graph,* Canad. Math. Bull., **57(3)**(2014), 573–578.

[13] S. P. Redmond, *An ideal-based zero-divisor graph of commutative ring,* Comm. Algebra, **31(9)**(2003), 4425–4443.

[14] D. W. West, *Introduction to Graph Theory,* 2nd ed., Prentice Hall, Upper Saddle River(2001).