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Halpern Subgradient Method for Pseudomonotone Equilibrium Problems in Hilbert Space

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Abstract. In this paper, we introduce a new algorithm for finding a solution of an equilibrium problem in a real Hilbert space. Our paper extends the single projection method to pseudomonotone variational inequalities, from a 2018 paper of Shehu et. al., to pseudomonotone equilibrium problems in a real Hilbert space. On the basis of the given algorithm for the equilibrium problem, we develop a new algorithm for finding a common solution of a equilibrium problem and fixed point problem. The strong convergence of the algorithm is established under mild assumptions. Several of fundamental experiments in finite (infinite) spaces are provided to illustrate the numerical behavior of the algorithm for the equilibrium problem and to compare it with other algorithms.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. The paper is concerned with a method for finding solutions to equilibrium problems, stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in C, \tag{1.1}$$

where $f : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, is a function such that $f(x, .)$ is convex and subdifferentiable on $\mathcal{H}$ for every fixed $x \in C$. From now on, we denote the solution set of Problem (1.1) by $Sol(C, f)$. Problem (1.1) is a general model in the sense that it unifies, in a simple form, numerous known models of optimization problems, nonlinear complementary problems, and variational inequalites [3, 10, 12, 13]. Equilibrium problems have many direct applications in other fields, such as transportation, electricity markets, and network problems [7, 13, 15, 19, 24, 26, 27]. This may explain why the problem has become an attractive field and has received a lot of attention by many authors.

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Some notable methods for solving it have been proposed, to highlight a few, see [1, 2, 3, 4, 6, 7, 17, 28, 29, 30, 32, 33, 36].

In the special case, $f(x,y) = \langle F(x), y-x\rangle$, where $F : C \to \mathcal{H}$, Problem (1.1) is equivalent to the following variational inequality problem:

$$\text{Find } x^* \in C \ \text{ such that } \langle F(x^*), x - x^*\rangle \geq 0 \ \ \forall y \in C. \tag{1.2}$$

In order to solve the variational inequality problems, many iterative methods have been proposed. If F is strongly monotone, the problems can be solved by the Newton method [34] or by single projection methods [9, 16]. Under the assumption that F is monotone and L-Lipschitz continuous, the extragradient method for solving Problem (1.2) is introduced by Korpelevich in [20]. In this method, two projections onto the feasible set C are used at each iteration:

$$\begin{cases} x^0 \in C, \\ y^k = Pr_C(x^k - \lambda_k F(x^k)), \\ x^{k+1} = Pr_C(x^k - \lambda_k F(y^k)), \end{cases}$$

where $\lambda_k \in (0, \frac{1}{L})$ and Pr_C denotes Euclidean projection onto C. Korpelevich showed that the iterative sequence generated by the algorithm is convergent in Euclidean spaces. His extragradient method has since been expanded and improved by many mathematicians in different ways [11, 22]. Recently, in [33], the authors introduced a single projection method which combines a projection method with the Halpern iteration technique. This method requires only one projection onto the feasible set C at each iteration and the iterative process is given by

$$\begin{cases} x^0 \in C, \\ y^k = Pr_C(x^k - \lambda_k F(x^k)), \\ d^k := x^k - y^k - \lambda_k(F(x^k) - F(y^k)), \\ x^{k+1} = \alpha_k x^0 + (1-\alpha_k)(x^k - \gamma\rho_k d^k), \end{cases} \tag{1.3}$$

where $\gamma \in (0,2)$, $\alpha_k \in (0,1)$, $\lim_{k\to\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = +\infty$, $\lambda_k \in (0,\infty)$ and

$$\rho_k = \begin{cases} \frac{\langle x^k - y^k, d^k\rangle}{\|d^k\|^2} & \text{if } d^k \neq 0 \\ 0 & \text{if } d^k = 0. \end{cases}$$

Recall that the strong convergence result of the iterative sequence generated by the proposed method is only established in real Hilbert spaces when F is pseudomonotone and L-Lipschitz-continuous. When F is a multivalued mapping from C to $\mathcal{H}$, then Problem (1.1) becomes the following multivalued variational inequality problem

$$\text{Find } (x^*, w^*) \in C \times F(x^*) \text{ such that } \langle w^*, x - x^*\rangle \geq 0 \ \ \forall x \in C. \tag{1.4}$$

Recall that the Hausdorff distance $\rho(\mathcal{A},\mathcal{B})$ between two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$ is defined by:

$$\rho(\mathcal{A},\mathcal{B}) := \max\{d(\mathcal{A},\mathcal{B}), d(\mathcal{B},\mathcal{A})\},$$

where $d(\mathcal{A},\mathcal{B}) := \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} \|a-b\|$. A multivalued mapping is said to be Lipschitz continuous on C with constant L if

$$\rho(F(x), F(y)) \le L\|x-y\|^2, \forall x,\ y \in C.$$

In paper [8], by replacing a projection onto C in (1.3) at each iteration by an approximate projection or a proximal operator, we improved the method in [33] and proposed a new algorithm for solving Problem (1.4). We proved that the algorithm is strongly convergent under the assumption of the pseudomonotonicity and Lipschitz continuity of cost mappings.

In this paper, we extend the single projection method to pseudomonotone variational inequality in [33] (Algorithm (1.3)) for solving Problem (1.1) in a real Hilbert space. Under the assumptions that the equilibrium bifunction $f(x,y)$ is pseudomonotone and

$$\rho\left(\partial_2 f(x,\cdot)(y), \partial_2 f(y,\cdot)(y)\right) \le L\|x-y\|,\ \forall x \in \mathcal{H}, y \in C, \tag{1.5}$$

we have proved that the sequence $\{x^k\}$ generated by the algorithm is strongly convergent to a solution of the problem. Note that in many other methods, the assumption of Lipschitz-type continuity with the constants $c_1,\ c_2$ of bifunction $f(x,y)$ is necessary for obtaining the convergence theorem of the algorithm [1, 17, 18, 30]. In our algorithm, the condition (1.5) is considered to be an alternative to the condition that $f(x,y)$ is Lipschitz-type continuous. On the basis of the given algorithm for equilibrium problem we developed a new algorithm for finding a common solution of Problem (1.1) and of fixed point problems. The strong convergence of the algorithm is established under mild assumptions.

The paper is organized as follows. In Section 2 we review some necessary concepts and lemmas that will be used in proving the main results of the paper. Im Section 3 we give then Halpern subgradient method for solving Problem (1.1) and prove its convergence. In section 4, we develop the algorithm forom Section 3 for problem of finding a common solution of Problem (1.1) and fixed point problems. In the last section, several fundamental experiments are provided to illustrate the convergence of the algorithm from Section 3, and to compare it to other algorithms.

2. Preliminaries

Throughout this paper, unless otherwise mentioned, let $\mathcal{H}$ denote a Hilbert space with inner product $\langle .,.\rangle$ and the induced norm $\|.\|$.

Definition 2.1. Let C be a nonempty closed convex subset in $\mathcal{H}$. The metric projection from $\mathcal{H}$ onto C is denoted by Pr_C and

$$Pr_C(x) = argmin\{\|x-y\| :\ y \in C\}, \forall x \in \mathcal{H}.$$

It is well known that the metric projection $Pr_C(\cdot)$ has the following basic property:

$$\langle x - Pr_C(x), y - Pr_C(x)\rangle \leq 0,\ \forall x \in \mathcal{H},\ y \in C.$$

Definition 2.2. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A bifunction $f : C \times C \to \mathcal{H}$ is called

(i) *β-strongly monotone on C*, if

$$f(x,y) + f(y,x) \leq -\beta\|x-y\|^2 \ \ \forall x, y \in C;$$

(ii) *monotone on C*, if

$$f(x,y) + f(y,x) \leq 0 \ \ \forall x, y \in C;$$

(iii) *pseudomonotone on C*, if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0 \ \forall x, y \in C.$$

Definition 2.3. Let $C \subset \mathcal{H}$ be a nonempty subset. An operator $S : C \to \mathcal{H}$ is called

(i) *β-demicontractive* on C, if $Fix(S)$ is nonempty and there exists $\beta \in [0,1)$ such that

$$\|Sx - p\|^2 \leq \|x-p\|^2 + \beta\|x - Sx\|^2 \ \forall x \in C,\ \forall p \in Fix(S); \tag{2.1}$$

(ii) *demiclosed*, if for any sequence $\{x^k\} \subset C$, $x^k \rightharpoonup z \in C$, $(I-S)(x^k) \rightharpoonup 0$ implies $z \in Fix(S)$.

It is well known that if S is β-demicontractive on C then S is demiclosed and (2.1) is equivalent to (see [25])

$$\langle x - Sx, x - p\rangle \geq \frac{1}{2}(1-\beta)\|x - Sx\|^2 \ \forall x \in C,\ \forall p \in Fix(S). \tag{2.2}$$

To prove the main result in Section 3 and 4, we shall use the following lemmas in the sequel.

Lemma 2.4. For every $x,\ y \in \mathcal{H}$, we have the following assertions.

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.5. Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition:

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k\delta_k + \beta_k, \ \forall k \geq 1,$$

where $\{\alpha_k\} \subset [0,1]$, $\sum_{k=0}^{\infty} \alpha_k = +\infty$, $\limsup_{k\to\infty} \delta_k \leq 0$ and $\beta_k \geq 0$, $\sum_{n=1}^{\infty} \beta_k < \infty$. Then, $\lim\limits_{k\to\infty} a_k = 0$.

The subdifferential of a convex function $g : C \to \mathcal{R} \cup \{+\infty\}$ is defined by

$$\partial g(x) = \{u \in \mathcal{H} : \langle u, y-x\rangle \leq g(y) - g(x) \ \ \forall y \in C\}.$$

In convex programming, we have the following result.

Lemma 2.6. ([31]) Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, x^* is a solution to the following convex problem:

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where ∂g denotes the subdifferential of g and $N_C(x^*)$ is the outer normal cone of C at $x^* \in C$, that is, $N_C(x^*) = \{u \in \mathcal{H} : \langle u, y-x^*\rangle \leq 0 \ \ \forall y \in C\}$.

3. Halpern subgradient method

3.1. Assumption

In this article, in order to find a point in $Sol(f,C)$, we assume that the bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfies the following conditions:

A_1. $f(x,x) = 0$ for all $x \in C$, f is pseudomonotone and weakly continuous on $\mathcal{H}$, i.e., $x^k \rightharpoonup \bar{x}$ and $y^k \rightharpoonup \bar{y} \Rightarrow f(x^k, y^k) \to f(\bar{x}, \bar{y})$;

A_2. There exists a real nonnegative number L such that

$$\rho\left(\partial_2 f(x,\cdot)(y), \partial_2 f(y,\cdot)(y)\right) \leq L\|x-y\|, \ \forall x \in \mathcal{H}, y \in C;$$

A_3. $Sol(C,f)$ is nonempty.

A_4. $f(x,\cdot)$ is convex and subdifferentiable on $\mathcal{H}$.

Remark 3.1. (a) Let $g : \mathcal{H} \to \mathcal{H}$ be a convex subdifferentiable and weakly continuous on $\mathcal{H}$. Clearly, $f(x, y) := g(y) - g(x)$ satisfies conditions $A_1 - A_2$. We well known that the following optimization problem

$$\min g(x) \text{ such that } x \in C,$$

is equivalent to Problem (1.1).

(b) Let $F(x)$ be a Lipschitz continuous and weakly continuous function on $\mathcal{H}$, i.e., $x^k \rightharpoonup \bar{x} \Rightarrow F(x^k) \to F(\bar{x})$. Setting $f(x, y) := \langle F(x), y - x\rangle$, it is easy to see that $f(x, y)$ satisfies conditions A_1 and A_2.

3.2. Algorithm

Algorithm 3.2. Choose starting point $x^0 \in \mathcal{H}$, $\mathcal{L} > L$, sequences $\{\alpha_k\}, \{\lambda_k\}$ and $\{\epsilon_k\}$ such that

$$\begin{cases} \{\alpha_k\} \subset (0,1), \lim_{k\to\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = +\infty, \\ 0 < \rho_k\epsilon_k \le \alpha_k^3, \sum_{k=0}^{\infty} (\rho_k\epsilon_k)^{\frac{1}{2}} < \infty, \\ \{\lambda_k\} \subset [a, b] \subset \left(0, \frac{1}{\mathcal{L}}\right) \subset (0, \infty). \end{cases} \tag{3.1}$$

Step 1. $(k = 0, 1, ...)$ Find $y^k \in \mathcal{H}$ such that

$$y^k = \operatorname{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}.$$

If $x^k - y^k = 0$ then STOP.

Step 2. Take $u^k \in \partial_2 f(x^k, y^k)$ satisfying $\langle x^k - y^k - \lambda_k u^k, y^k - x\rangle \ge -\epsilon_k,\ \forall x \in C$ and

$$v^k \in B(u^k, \mathcal{L}\|x^k - y^k\|) \cap \partial_2 f(y^k, y^k),$$

where $B\left(u^k, \mathcal{L}\|x^k - y^k\|\right) := \{x \in \mathcal{H} : \|x - u^k\| \le \mathcal{L}\|x^k - y^k\|\}$.

Step 3. Compute $x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)z^k$, where $z^k := x^k - \rho_k d^k$ and

$$d^k := x^k - y^k - \lambda_k(u^k - v^k),\ \rho_k = \frac{\langle x^k - y^k, d^k\rangle}{\|d^k\|^2}.$$

Step 4. Set $k := k + 1$, and go to Step 1.

Remark 3.3. (a) By Step 1 and Lemma 2.6, we have

$$0 \in \lambda_k\partial_2 f(x^k, y^k) + y^k - x^k + N_C(y^k),$$

it follows that there is a $u^k \in \partial_2 f(x^k, y^k)$ such that

$$\langle x^k - y^k - \lambda_k u^k, y^k - x\rangle \ge 0,\ \forall x \in C,$$

i.e., the u^k in Step 2 always exists.
(b) If $d^k = 0$ then

$$\|x^k - y^k\| = \lambda_k \|u^k - v^k\| \leq \lambda_k \mathcal{L} \|x^k - y^k\|.$$

Since $\lambda_k \leq \frac{1}{\mathcal{L}}$ for all k we have

$$(1 - \lambda_k \mathcal{L}) \|x^k - y^k\| \leq 0 \Rightarrow x^k = y^k.$$

Thus, we observe that $d^k \neq 0$ and ρ_k of Step 3 is defined.

3.3. Convergence analysis of algorithm

In this section, we show that the algorithm proposed is strongly convergent if assumptions $A_1 - A_4$ satisfy.

Lemma 3.4. *Let sequence $\{x^k\}$ be generated by Algorithm 3.2 and $x^* \in Sol(C, f)$. Then,*

(i) $\|z^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|z^k - x^k\|^2 + 2\rho_k \epsilon_k$;

(ii) sequences $\{x^k\}$ and $\{z^k\}$ are bounded.

Proof. Using Step 2, we have

$$\langle x^k - y^k - \lambda_k u^k, y^k - x \rangle \geq -\epsilon_k \ \forall x \in C.$$

Replace x by $x^* \in C$ in the last inequality, we have

$$\langle x^k - y^k - \lambda_k u^k, y^k - x^* \rangle \geq -\epsilon_k.$$

Using $x^* \in Sol(C, f)$, $f(y^k, y^k) = 0$, $v^k \in \partial_2 f(y^k, y^k)$ and the pseudomonotone assumption of f, we get

$$\lambda_k \langle v^k, y^k - x^* \rangle \geq \lambda_k [f(y^k, y^k) - f(y^k, x^*)] \geq 0.$$

Adding two last inequalities, it follows that

$$-\epsilon_k \leq \langle y^k - x^*, x^k - y^k - \lambda_k u^k + \lambda_k v^k \rangle = \langle y^k - x^*, d^k \rangle.$$

Using the above inequality and the definition of z^k, we have

$$\begin{aligned}
\|z^k - x^*\|^2 =& \|x^k - \rho_k d^k - x^*\|^2 \\
=& \|x^k - x^*\|^2 - 2\rho_k \langle x^k - x^*, d^k \rangle + \rho_k^2 \|d^k\|^2 \\
\leq& \|x^k - x^*\|^2 - 2\rho_k \langle x^k - y^k, d^k \rangle + \rho_k^2 \|d^k\|^2 + 2\rho_k \epsilon_k \\
=& \|x^k - x^*\|^2 - 2\rho_k \langle x^k - y^k, d^k \rangle + \rho_k \langle x^k - y^k, d^k \rangle + 2\rho_k \epsilon_k \\
=& \|x^k - x^*\|^2 - \rho_k \langle x^k - y^k, d^k \rangle + 2\rho_k \epsilon_k,
\end{aligned}$$

and $\rho_k\langle x^k - y^k, d^k\rangle = \|z^k - x^k\|^2$. Therefore

$$\begin{aligned}\|z^k - x^*\|^2 \le& \|x^k - x^*\|^2 - \rho_k\langle x^k - y^k, d^k\rangle + 2\rho_k\epsilon_k\\ =& \|x^k - x^*\|^2 - \|z^k - x^k\|^2 + 2\rho_k\epsilon_k.\end{aligned}$$

This follows (i). We now prove (ii). From (i) and condition (3.1), it follows that

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|\alpha_k x^0 + (1-\alpha_k)z^k - x^*\|\\ &\le \alpha_k\|x^0 - x^*\| + (1-\alpha_k)\|z^k - x^*\|\\ &\le \alpha_k\|x^0 - x^*\| + (1-\alpha_k)\sqrt{\|x^k - x^*\|^2 + 2\rho_k\epsilon_k}\\ &\le \alpha_k\|x^0 - x^*\| + (1-\alpha_k)(\|x^k - x^*\| + \sqrt{2\rho_k\epsilon_k})\\ &\le \max\left\{\|x^0 - x^*\|, \|x^k - x^*\| + \sqrt{2\rho_k\epsilon_k}\right\}\\ &\ldots\\ &\le \max\left\{\|x^0 - x^*\|, \|x^0 - x^*\| + \sqrt{2}\sum_{i=0}^{k}\sqrt{\rho_i\epsilon_i}\right\}\\ &\le \|x^0 - x^*\| + \sqrt{2}\sum_{i=0}^{\infty}\sqrt{\rho_i\epsilon_i} < +\infty. \qquad (3.2)\end{aligned}$$

It implies that $\{x^k\}$ is bounded. Using again (i), we have that the sequence $\{z^k\}$ is bounded. □

Lemma 3.5. *Let $x^* \in Sol(C, f)$. Set $a_k = \|x^k - x^*\|^2$, $b_k = 2\langle x^0 - x^*, x^{k+1} - x^*\rangle$ and $\beta_k = \sqrt{2\rho_k\epsilon_k}$. Then,*

- *(i)* $a_{k+1} \le (1-\alpha_k)a_k + \alpha_k b_k + \beta_k$;
- *(ii)* $\beta_k \ge 0$, $\sum_{k=1}^{\infty}\beta_k < \infty$;
- *(iii)* $\lim_{k\to\infty}\frac{\beta_k}{\alpha_k} = 0$;
- *(iv)* $-1 \le \limsup_{k\to\infty} b_k < \infty$.

Proof. Using Lemma 2.4 (ii), we get

$$\begin{aligned}\|x^{k+1} - x^*\|^2 =& \|\alpha_k(x^0 - x^*) + (1-\alpha_k)(z^k - x^*)\|^2\\ \le& (1-\alpha_k)^2\|z^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle x^0 - x^*, x^{k+1} - x^*\rangle\\ \le& (1-\alpha_k)\|z^k - x^*\|^2 + 2\alpha_k\langle x^0 - x^*, x^{k+1} - x^*\rangle.\end{aligned}$$

Combining the last inequality and Lemma 3.4 (i), we get

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \le& (1-\alpha_k)\|x^k - x^*\|^2 + 2\alpha_k\langle x^0 - x^*, x^{k+1} - x^*\rangle + (1-\alpha_k)\sqrt{2\rho_k\epsilon_k}\\ \le& (1-\alpha_k)\|x^k - x^*\|^2 + 2\alpha_k\langle x^0 - x^*, x^{k+1} - x^*\rangle + \sqrt{2\rho_k\epsilon_k}.\end{aligned}$$

This follows (i). Assumptions (ii) and (iii) are directly inferred from condition (3.1).

Since $\{x^k\}$ is bounded, we have $b_k \leq 2\|x^0 - x^*\|\|x^{k+1} - x^*\| < \infty$, and so $\limsup_{k\to\infty} b_k < \infty$. We now assume contradiction that $\limsup_{k\to\infty} b_k < -1$. There exists $k_0 \in \mathcal{N}$ such that $b_k < -1$ for all $k \geq k_0$. It follows from (i) that

$$\begin{aligned}
a_{k+1} \leq &(1-\alpha_k)a_k + \alpha_k b_k + \beta_k \\
<&(1-\alpha_k)a_k - \alpha_k + \beta_k \\
=&a_k - (a_k+1)\alpha_k + \beta_k \\
\leq &a_k - \alpha_k + \beta_k. \\
\cdots& \\
\leq &a_{k_0} - \sum_{i=k_0}^{k} \alpha_i + \sum_{i=k_0}^{k} \beta_i \quad \forall k \geq k_0.
\end{aligned}$$

Using this and the result (ii), one have

$$\limsup_{k\to\infty} a_k \leq a_{k_0} - \sum_{i=k_0}^{+\infty} \alpha_i + \sum_{i=k_0}^{+\infty} \beta_i = -\infty.$$

This contradicts the fact that $a_k \geq 0$ for all $k \in \mathcal{N}$.
Therefore, $\limsup_{k\to\infty} b_k \geq -1$. □

Theorem 3.6. *Let bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be satisfying the assumptions $A_1 - A_4$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.2 converges strongly to a solution $z \in Sol(C, f)$, where $z = Pr_{Sol(C,f)}(x^0)$.*

Proof. Set $a_k := \|x^k - z\|$. In oder to prove this theorem, we consider two following cases.

Case 1. Suppose that there exists $k_0 \in \mathcal{N}$ such that $a_{k+1} \leq a_k$ for all $k \geq k_0$. Then, there exists the limit $\lim_{k\to\infty} a_k \in [0, \infty)$. From Step 3, Lemma 3.4 (i) and Lemma 2.4 (ii), it follows that

$$\begin{aligned}
\|x^{k+1} - z\|^2 =&\|(1-\alpha_k)(z^k - z) + \alpha_k(x^0 - z)\|^2 \\
\leq&(1-\alpha_k)^2\|z^k - z\|^2 + 2\alpha_k\langle x^0 - z, x^{k+1} - z\rangle \\
\leq&\|z^k - z\|^2 + 2\alpha_k\langle x^0 - z, x^{k+1} - z\rangle \\
\leq&\|x^k - z\|^2 - \|z^k - x^k\|^2 + 2\alpha_k\langle x^0 - z, x^{k+1} - z\rangle + \sqrt{2\rho_k\epsilon_k} \\
\leq&\|x^k - z\|^2 - \|z^k - x^k\|^2 + \sqrt{2\rho_k\epsilon_k} + \alpha_k Q_0,
\end{aligned}$$

where $Q_0 := \sup\{2\langle x^0 - z, x^{k+1} - z\rangle : k = 0, 1, ...\} < \infty$. This implies that

$$a_{k+1} - a_k + \|z^k - x^k\|^2 \leq +\sqrt{2\rho_k\epsilon_k} + \alpha_k Q_0 \ \forall k \geq 0. \tag{3.3}$$

Taking $k \to \infty$ in the last inequality and using the assumptions $\lim_{k\to\infty} \alpha_k = 0$, $\lim_{k\to\infty} \sqrt{2\rho_k \epsilon_k} = 0$, we have $\lim_{k\to\infty} \|z^k - x^k\| = 0$. From $v^k \in B(u^k, \mathcal{L}\|x^k - y^k\|)$ for all k, it follows

$$\begin{aligned}\langle x^k - y^k, d^k\rangle &= \|x^k - y^k\|^2 - \lambda_k\langle x^k - y^k, u^k - v^k\rangle \\ &\geq \|x^k - y^k\|^2 - \lambda_k\|x^k - y^k\|.\|u^k - v^k\| \\ &\geq (1 - b\mathcal{L})\|x^k - y^k\|^2, \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\|d^k\| &= \|x^k - y^k - \lambda_k(u^k - v^k)\| \\ &\leq \|x^k - y^k\| + \lambda_k\|u^k - v^k\| \\ &\leq (1 + \lambda_k\mathcal{L})\|x^k - y^k\| \\ &\leq (1 + b\mathcal{L})\|x^k - y^k\|. \qquad (3.5)\end{aligned}$$

Using Step 3, (3.4) and (3.5), we get $\rho_k \geq \frac{1-b\mathcal{L}}{(1+b\mathcal{L})^2}$ and

$$\begin{aligned}\|x^k - y^k\|^2 \leq& \frac{1}{(1 - b\mathcal{L})}\langle x^k - y^k, d^k\rangle \\ =& \frac{1}{(1 - b\mathcal{L})\rho_k}\|z^k - x^k\|^2 \\ \leq& \frac{(1 + b\mathcal{L})^2}{(1 - b\mathcal{L})^2}\|z^k - x^k\|^2.\end{aligned}$$

From the above inequality and $\lim_{k\to\infty} \|z^k - x^k\| = 0$, it follows that

$$\lim_{k\to\infty} \|x^k - y^k\| = 0.$$

Using this and $\lim_{k\to\infty} \|z^k - x^k\| = 0$, we obtain $\lim_{k\to\infty} \|z^k - y^k\| = 0$. By the definition of x^{k+1} and Lemma 3.4 (ii), we have

$$\|x^{k+1} - z^k\| = \alpha_k\|x^0 - z^k\| \leq \alpha_k Q_1 \to 0 \ as \ k \to \infty,$$

where $Q_1 = \sup\{\|x^0 - z^k\| : \ k = 0, 1, ...\} < +\infty$. This together with $\lim_{k\to\infty} \|z^k - x^k\| = 0$ implies that

$$\|x^{k+1} - x^k\| \leq \|x^{k+1} - z^k\| + \|z^k - x^k\| \to 0 \ as \ k \to \infty.$$

Since sequence $\{x^k\}$ is bounded, so there exists a subsequence $\{x^{k_i+1}\}$ such that $x^{k_i+1} \rightharpoonup p$ as $i \to \infty$, and

$$\limsup_{k\to\infty}\langle x^0 - z, x^{k+1} - z\rangle = \lim_{i\to\infty}\langle x^0 - z, x^{k_i+1} - z\rangle. \qquad (3.6)$$

We will show that $p \in Sol(C, f)$. Indeed, by Step 2, one has

$$\langle x^{k_i+1} - y^{k_i} - \lambda_{k_i+1}u^{k_i+1}, y^{k_i+1} - x\rangle \geq 0 \ \forall x \in C.$$

Combining this inequality and $u^{k_i+1} \in \partial_2 f(x^{k_i+1}, y^{k_i+1})$, we get

$$\begin{aligned}\langle x^{k_i+1} - y^{k_i+1}, x - y^{k_i+1}\rangle \leq& \lambda_{k_i+1}\langle u^{k_i+1}, x - y^{k_i+1}\rangle \\ \leq& \lambda_{k_i+1}[f(x^{k_i+1}, x) - f(x^{k_i+1}, y^{k_i+1})].\end{aligned}$$

Since $\|x^k - y^k\| \to 0$ as $k \to \infty$, $\{x^{k_i+1}\}$ is bounded and converges weakly to p as $i \to \infty$, $\{y^{k_i+1}\}$ also is bounded and $y^{k_i+1} \rightharpoonup p$. For each fixed point $x \in C$, take the limit as $i \to \infty$, using $\lim_{i\to\infty} \|x^{k_i+1} - y^{k_i+1}\| = 0$ and weakly continuity of bifunction $f(x, y)$, we get

$$f(p, x) \geq 0 \quad \forall x \in C.$$

Using $p \in Sol(C, f)$ and (3.6), we have

$$\begin{aligned}\limsup_{k\to\infty} b_k &= \limsup_{k\to\infty}\langle x^0 - z, x^{k+1} - z\rangle, \\ &= 2\lim_{k\to\infty}\langle x^0 - z, x^{k_i+1} - z\rangle \\ &= 2\langle x^0 - z, p - z\rangle \leq 0.\end{aligned}$$

Applying Lemma 2.5 for Lemma 3.5 (i) and using the last inequality, we deduce

$$\lim_{k\to\infty} \|x^k - z\| = 0.$$

Thus, $\{x^k\}$ converges strongly to the solution $z = Pr_{Sol(C,f)}(x^0)$.

Case 2. We now assume that there is not $\bar{k} \in \mathbb{N}$ such that $\{a_k\}_{k=\bar{k}}^{\infty}$ is monotonically decreasing. So, there exists an integer $k_0 \geq \bar{k}$ such that $a_{k_0} \leq a_{k_0+1}$. Then, there exists a subsequence $\{a_{\tau(k)}\}$ of $\{a_k\}$ such that (see Remark 4.4, [21])

$$0 \leq a_k \leq a_{\tau(k)+1}, a_{\tau(k)} \leq a_{\tau(k)+1} \quad \forall k \geq k_0,$$

where $\tau(k) = \max\{i \in \mathbb{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}$. Using $a_{\tau(k)} \leq a_{\tau(k)+1}$, $\forall k \geq k_0$ and (3.3), one has

$$\begin{aligned}0 &\leq \|w^{\tau(k)} - x^{\tau(k)}\| \\ &\leq a_{\tau(k)+1} - a_{\tau(k)} + \|w^{\tau(k)} - x^{\tau(k)}\| \\ &\leq \alpha_{\tau(k)}Q_0 + \sqrt{2\rho_k\epsilon_k} \to 0 \text{ as } k \to \infty,\end{aligned}$$

and so $\lim_{k\to\infty} \|w^{\tau(k)} - x^{\tau(k)}\| = 0$. By a similar way as in *Case 1*, we can show that

$$\lim_{n\to\infty} \|x^{\tau(k)+1} - x^{\tau(k)}\| = \lim_{n\to\infty} \|x^{\tau(k)} - y^{\tau(k)}\| = \lim_{n\to\infty} \|w^{\tau(k)} - y^{\tau(k)}\| = 0. \tag{3.7}$$

Since $\{x^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x^{\tau(k)}\}$, still denoted by $\{x^{\tau(k)}\}$, which converges weakly to p. Arguing similarly as in *Case 1*, we can prove that $p \in Sol(C, f)$ and

$$\limsup_{k\to\infty} b_{\tau(k)} \leq 0. \tag{3.8}$$

From Lemma 3.5 (i) and $a_{\tau(k)} \leq a_{\tau(k)+1},\ \forall k \geq k_0$, it follows that

$$\alpha_{\tau(k)} a_{\tau(k)} \leq a_{\tau(k)} - a_{\tau(k)+1} + \alpha_{\tau(k)} b_{\tau(k)} + \beta_{\tau(k)} \leq \alpha_{\tau(k)} b_{\tau(k)} + \beta_{\tau(k)}.$$

It is equivalent to

$$a_{\tau(k)} \leq b_{\tau(k)} + \frac{\beta_{\tau(k)}}{\alpha_{\tau(k)}}.$$

By Lemma 3.5 (iii), (3.8) and the last inequality, we get

$$\limsup_{k\to\infty} a_{\tau(k)} \leq \limsup_{k\to\infty} b_{\tau(k)} \leq 0.$$

Therefore, $\lim_{k\to\infty} a_{\tau(k)} = 0$. As a consequence, we get

$$\begin{aligned} \sqrt{a_{\tau(k)+1}} &= \|x^{\tau(k)+1} - z\| \\ &\leq \|x^{\tau(k)+1} - x^{\tau(k)}\| + a_{\tau(k)} \to 0,\ k \to \infty. \end{aligned}$$

It follows that $\lim_{k\to\infty} a_{\tau(k)+1} = 0$. Furthermore, $0 \leq a_k \leq a_{\tau(k)+1}$ for all $k \geq k_0$. Hence, $\lim\limits_{k\to\infty} a_k = 0$, i.e., $x^k \to z$, as $k \to \infty$. □

4. Find a Common Solution of Equilibrium Problem and Fix Point Problems

Let a finite system of mappings S_j $(j \in J := \{1, 2, ..., r\})$ of $\mathcal{H}$ into itself. Denote the fixed point set of S_j by

$$Fix(S_j) := \{x \in \mathcal{H} : S_j x = x\}.$$

We consider the problem which finds a common element of the solution set of Problem (1.1) and the set of fixed points of a finite system of mappings S_j $(j \in J)$, namely:

$$\text{Find } x^* \in \cap_{j\in J} Fix(S_j) \cap Sol(C, f).$$

4.1. Assumption

In this section, we assume that the bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and the mappings S_j $(j \in J)$ satisfy the following conditions:

B_1. $f(x, x) = 0$ for all $x \in C$, f is pseudomonotone and weakly continuous on $\mathcal{H}$ and $f(x, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$;

B_2. There exists a real nonnegative number L such that

$$\rho\left(\partial_2 f(x, \cdot)(y), \partial_2 f(y, \cdot)(y)\right) \leq L\|x - y\|,\ \forall x \in \mathcal{H}, y \in C;$$

B_3. $\cap_{j\in J} Fix(S_j) \cap Sol(C, f) \neq \emptyset$;

B_4. $S_j : \mathcal{H} \to \mathcal{H}$ is β_j-demicontractive for every $j \in J$.

4.2. Algorithm

Algorithm 4.1. Choose starting point $x^0 \in \mathcal{H}$, $\mathcal{L} > L$, sequences $\{\alpha_k\}, \{\lambda_k\}$ and $\{\epsilon_k\}$ such that

$$\begin{cases} \{\alpha_k\} \subset (0,1), \lim_{k\to\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = +\infty, \\ 0 < \rho_k \epsilon_k \leq \alpha_k^3, \sum_{k=0}^{\infty} (\rho_k\epsilon_k)^{\frac{1}{2}} < \infty, \\ \{\lambda_k\} \subset [a,b] \subset \left(0, \frac{1}{\mathcal{L}}\right) \subset (0,\infty). \end{cases} \tag{4.1}$$

Step 1*. $(k = 0, 1, ...)$ Find $y^k \in \mathcal{H}$ such that

$$y^k = \text{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}.$$

If $x^k - y^k = 0$ then STOP.

Step 2*. Take $u^k \in \partial_2 f(x^k, y^k)$ satisfying $\langle x^k - y^k - \lambda_k u^k, y^k - x\rangle \geq -\epsilon_k,\ \forall x \in C$ and

$$v^k \in B(u^k, \mathcal{L}\|x^k - y^k\|) \cap \partial_2 f(y^k, y^k),$$

where $B\left(u^k, \mathcal{L}\|x^k - y^k\|\right) := \{x \in \mathcal{H} : \|x - u^k\| \leq \mathcal{L}\|x^k - y^k\|\}$. Set $d^k := x^k - y^k - \lambda_k(u^k - v^k)$ and $z^k := x^k - \rho_k d^k$ with

$$d^k := x^k - y^k - \lambda_k(u^k - v^k),\ \rho_k = \frac{\langle x^k - y^k, d^k\rangle}{\|d^k\|^2}.$$

Step 3*. Compute

$$p^k = \alpha_k x^0 + (1 - \alpha_k) z^k,$$
$$q_j^k = (1 - \omega)p^k + \omega S_j p^k,\ 0 < \omega < \frac{1 - \beta_j}{2}\ \forall j \in J,$$

$$x^{k+1} = q_{j_0}^k,\ j_0 = argmax\{||q_j^k - p^k||,\ j \in J\},\ k \geq 1. \tag{4.2}$$

Step 4*. Set $k := k + 1$, and go to Step 1*.

Lemma 4.2. *The sequences $\{p^k\}, \{x^k\}, \{z^k\}$ and $\{y^k\}$ are bounded.*

Proof. Let $x^* \in \cap_{j\in J} Fix(S_j) \cap Sol(C, f)$. Using Step 3* and the β_j demicontractive assumption of $S_j,\ j = 1, 2, ...,$ we get

$$\begin{aligned} ||x^{k+1} - x^*||^2 &= ||(1-\omega)p^k + \omega S_{j_0} p^k - x^*||^2 \\ &= ||(p^k - x^*) + \omega(S_{j_0} p^k - p^k)||^2 \\ &\leq ||p^k - x^*||^2 + 2\omega\langle p^k - x^*, S_{j_0}p^k - p^k\rangle + \omega^2||S_{j_0}p^k - p^k||^2 \\ &\leq ||p^k - x^*||^2 + \omega(\omega + \beta_{j_0} - 1)||S_{j_0}p^k - p^k||^2 \\ &\leq ||p^k - x^*||^2. \end{aligned} \tag{4.3}$$

From Lemma 3.4 (i) and the last inequality, it follows that

$$||z^{k+1} - x^*|| \le ||p^k - x^*|| + \sqrt{2}\sqrt{\rho_{k+1}\epsilon_{k+1}}. \tag{4.4}$$

Using Step 3*, condition (4.1) and (4.4), we have

$$\begin{aligned}
\|p^{k+1} - x^*\| &= ||\alpha_{k+1}(x^0 - x^*) + (1-\alpha_{k+1})(z^{k+1} - x^*)|| \\
&\le \alpha_{k+1}||x^0 - x^*|| + (1-\alpha_{k+1})||z^{k+1} - x^*|| \\
&\le \alpha_{k+1}||x^0 - x^*|| + (1-\alpha_{k+1})(||p^k - x^*|| + \sqrt{2}\sqrt{\rho_{k+1}\epsilon_{k+1}}) \\
&\le \max\{||p^k - x^*|| + \sqrt{2}\sqrt{\rho_{k+1}\epsilon_{k+1}}\} \\
&\dots \\
&\le \max\{||p^0 - x^*|| + \sum_{i=1}^{k+1}\sqrt{2}\sqrt{\rho_{k+1}\epsilon_{k+1}},\ ||x^0 - x^*||\} < +\infty.
\end{aligned}$$

So, the sequence $\{p^k\}$ is bounded. From (4.3) and (4.4), it follows that the sequences $\{x^k\}$ and $\{z^k\}$ are bounded. □

Lemma 4.3. *Let $x^* \in \cap_{j\in J}Fix(S_j) \cap Sol(C,f)$. Set $a_k = \|x^k - x^*\|^2$, $\beta_k = 2\rho_k\epsilon_k$ and $b_k = 2\langle x^0 - x^*, p^k - x^*\rangle$. Then,*

(i) $a_{k+1} \le (1-\alpha_k)a_k + \alpha_k b_k + \beta_k$;

(ii) $\beta_k \ge 0,\ \sum_{n=1}^{\infty}\beta_k < \infty$;

(iii) $\lim\limits_{k\to\infty}\frac{\beta_k}{\alpha_k} = 0$;

(iv) $-1 \le \limsup\limits_{n\to\infty} b_k < \infty$.

Proof. Using Lemma 2.4 (ii), Lemma 3.4 (i) and Step 3*, we get

$$\begin{aligned}
||p^k - x^*||^2 &= ||\alpha_k(x^0 - x^*) + (1-\alpha_k)(z^k - x^*)||^2 \\
&\le (1-\alpha_k)||z^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle \\
&\le (1-\alpha_k)||x^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle + 2\rho_k\epsilon_k(1-\alpha_k) \\
&\le (1-\alpha_k)||x^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle + 2\rho_k\epsilon_k. \qquad (4.5)
\end{aligned}$$

Using last inequality and (4.3), we have

$$||x^{k+1} - x^*||^2 \le (1-\alpha_k)||x^k - x^*|| + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle + 2\rho_k\epsilon_k.$$

We have (i). By arguing similarly as in the proof of Lemma 3.5, we obtain (ii), (iii) and (iv). □

Theorem 4.4. *Suppose that conditions $B_1 - B_4$ are satisfied. Let $\{x^k\}$ be a sequence generated by Algorithm 4.1. Then, the sequence $\{x^k\}$ converges strongly to a solution*

$$z \in \cap_{j\in J}Fix(S_j) \cap Sol(C,f),$$

where $z = Pr_{\cap_{j\in J} Fix(S_j)\cap Sol(C,f)}(x^0)$.

Proof. Set $a_k := \|x^k - z\|$. In oder to prove this theorem, we consider two following cases.

Case 1. Suppose that there exists $k_0 \in \mathcal{N}$ such that $a_{k+1} \le a_k$ for all $k \ge k_0$. Then, there exists the limit $\lim_{k\to\infty} a_k \in [0,\infty)$.

Using Step 3*, Lemma 2.1 (ii), Lemma 3.4 (i) and (2.2), we obtain

$$\begin{aligned}
\|x^{k+1} - z\|^2 =&\|(1-\omega)p^k + \omega S_{j_0}p^k - z\|^2\\
=&\|p^k - z\|^2 - 2\omega\langle p^k - z, p^k - S_{j_0}p^k\rangle + \omega^2\|p^k - S_{j_0}p^k\|^2\\
\le&\|p^k - z\|^2 - \omega(1-\beta_{j_0}-\omega)\|p^k - S_{j_0}p^k\|^2\\
=&||\alpha_k(x^0 - z) + (1-\alpha_k)(z^k - z)||^2 - \frac{1}{\omega}(1-\beta_{j_0}-\omega)\|x^{k+1} - p^k\|^2\\
\le&(1-\alpha_k)||z^k - z||^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle - \|x^{k+1} - p^k\|^2\\
\le&||z^k - z||^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle - \|x^{k+1} - p^k\|^2\\
\le&\|x^k - z\|^2 - \|z^k - x^k\|^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle - \|x^{k+1} - p^k\|^2 + 2\rho_k\epsilon_k\\
\le&\|x^k - z\|^2 - \|z^k - x^k\|^2 + \alpha_k M_0 - \|x^{k+1} - p^k\|^2 + 2\rho_k\epsilon_k, \qquad (4.6)
\end{aligned}$$

where $M_0 := \sup\{2\langle x^0 - z, p^k - z\rangle : k = 0, 1, ...\} < \infty$. It follows that

$$a_{k+1} - a_k + \|z^k - x^k\|^2 + \|x^{k+1} - p^k\|^2 \le \alpha_k M_0 + 2\rho_k\epsilon_k \quad \forall k \ge 0. \qquad (4.7)$$

Passing the limit as $k \to \infty$ and using the assumptions

$$\lim_{k\to\infty} \alpha_k = 0, \lim_{k\to\infty} 2\rho_k\epsilon_k = 0,$$

we have

$$\lim_{k\to\infty} \|z^k - x^k\| = 0, \ \lim_{k\to\infty} \|x^{k+1} - p^k\| = 0.$$

By a similar way as in the proof of Theorem 3.6, we can show that

$$\lim_{k\to\infty} \|x^k - y^k\| = 0.$$

It follows that

$$\|z^k - y^k\| \le \|z^k - x^k\| + \|x^k - y^k\| \to 0, \text{ as } k \to \infty.$$

Using Step 3*, we have

$$\|p^k - z^k\| = \alpha_k\|x^0 - z^k\| \le \alpha_k M_1 \to 0, \text{ as } k \to \infty,$$

where $M_1 = \sup\{\|x^0 - z^k\| : \ k = 0, 1, ...\}0 < +\infty$. Therefore,

$$\|x^{k+1} - x^k\| \le \|x^{k+1} - p^k\| + \|p^k - z^k\| + \|z^k - x^k\| \to 0 \text{ as } k \to \infty.$$

From this and
$$\|x^k - p^k\| \le \|x^{k+1} - x^k\| + \|x^{k+1} - p^k\|,$$
it follows that $\lim_{k\to\infty} \|x^k - p^k\| = 0$. Since sequence $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ such that $x^{k_i} \rightharpoonup p \in \mathcal{H}$, $p^{k_i} \rightharpoonup p$ and

$$\limsup_{k\to\infty} \langle x^0 - z, p^k - z\rangle = \lim_{i\to\infty} \langle x^0 - z, p^{k_i} - z\rangle. \tag{4.8}$$

Now, we will show that
$$p \in \cap_{j\in J} Fix(S_j) \cap Sol(C, f).$$

By a similar way as in the proof of Theorem 3.6, we can prove that $p \in Sol(C, f)$. For each $j \in J$, using Step 3*, we have
$$||p^k - S_j p^k|| = \frac{1}{\omega}||p^k - q_j^k|| \le \frac{1}{\omega}||p^k - q_{j_0}^k|| = \frac{1}{\omega}||x^{k+1} - p^k||.$$

By $\lim_{k\to\infty} \|x^{k+1} - p^k\| = 0$ and the last inequality, we get
$$||p^k - S_j p^k|| \to 0,\ k \to \infty.$$

From $\lim_{k\to\infty} \|x^k - p^k\| = 0$ and $x^{k_i} \rightharpoonup p$, it follows that $p^{k_i} \rightharpoonup p$. Using this, $\lim_{k\to\infty} ||p^k - S_j p^k|| = 0$ and the demiclosedness of S_j, we have $p \in Fix(S_j)$. Therefore,
$$p \in \cap_{j\in J} Fix(S_j) \cap Sol(C, f).$$

This together with (4.8) implies that
$$\begin{aligned}\limsup_{k\to\infty} b_k &= 2\lim_{i\to\infty} \langle x^0 - z, p^{k_i} - z\rangle \\ &= 2\langle x^0 - z, p - z\rangle \le 0.\end{aligned}$$

Using this, Lemma 2.5 and Lemma 4.3, we obtain $\lim_{k\to\infty} \|x^k - z\| = 0$.

Case 2. We now assume that there is not $\bar{k} \in \mathcal{N}$ such that $\{a_k\}_{k=\bar{k}}^{\infty}$ is monotonically decreasing. So, there exists an integer $k_0 \ge \bar{k}$ such that $a_{k_0} \le a_{k_0+1}$. Then, there exists a subsequence $\{a_{\tau(k)}\}$ of $\{a_k\}$ such that (see Remark 4.4, [21])
$$0 \le a_k \le a_{\tau(k)+1}, a_{\tau(k)} \le a_{\tau(k)+1} \ \ \forall k \ge k_0,$$
where $\tau(k) = \max\{i \in \mathcal{N} : k_0 \le i \le k, a_i \le a_{i+1}\}$. Using $a_{\tau(k)} \le a_{\tau(k)+1}$, for all $k \ge k_0$ and (3.3), we get
$$\|z^{\tau(k)} - x^{\tau(k)}\| \to 0, \|x^{\tau(k)+1} - p^{\tau(k)}\| \to 0,\ k \to \infty.$$

By a similar way as in case 1, we can show that

$$\lim_{k\to\infty} \|x^{\tau(k)} - p^{\tau(k)}\| = \lim_{k\to\infty} \|x^{\tau(k)} - y^{\tau(k)}\| = \lim_{k\to\infty} \|z^{\tau(k)} - y^{\tau(k)}\| = 0. \tag{4.9}$$

Since $\{x^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x^{\tau(k)}\}$, still denoted by $\{x^{\tau(k)}\}$, which converges weakly to $p \in \mathcal{H}$. By a similar way as in case 1, we can prove that $p \in \cap_{j\in J} Fix(S_j) \cap Sol(C, f)$ and

$$\limsup_{k\to\infty} b_{\tau(k)} \leq 0.$$

Using Lemma 4.3 (i) and $a_{\tau(k)} \leq a_{\tau(k)+1},\ \forall k \geq k_0$, we have

$$\alpha_{\tau(k)} a_{\tau(k)} \leq a_{\tau(k)} - a_{\tau(k)+1} + \alpha_{\tau(k)} b_{\tau(k)} + \beta_{\tau(k)} \leq \alpha_{\tau(k)} b_{\tau(k)} + \beta_{\tau(k)}.$$

Since $\alpha_{\tau(k)} > 0$, we get

$$a_{\tau(k)} \leq b_{\tau(k)} + \frac{\beta_{\tau(k)}}{\alpha_{\tau(k)}}.$$

From Lemma 4.3 (iii) and last inequality, it follows that

$$\limsup_{k\to\infty} a_{\tau(k)} \leq \limsup_{k\to\infty} b_{\tau(k)} \leq 0.$$

Hence, $\lim_{k\to\infty} a_{\tau(k)} = 0$. It follows that

$$\begin{aligned} a_{\tau(k)+1} &= \|x^{\tau(k)+1} - z\|^2 \\ &\leq (\|x^{\tau(k)+1} - x^{\tau(k)}\| + \|x^{\tau(k)} - z\|)^2 \to 0,\ k \to \infty. \end{aligned}$$

Using $0 \leq a_k \leq a_{\tau(k)+1}$ for all $k \geq k_0$, we get $\lim\limits_{n\to\infty} a_k = 0$. Hence, $x^k \to z$ as $k \to \infty$. □

5. Computational Experiments

In this final section, we present some fundamental experiments in finite/infinite spaces to illustrate the numerical behavior of Algorithm 3.2 and to compare it with known algorithms. All programs are coded in Matlab R2018a and the program was run on a PC Intel(R) Core(TM) i7-6600X CPU @ 2.60GHz 16GB Ram.

Let $\mathcal{H} = \mathcal{R}^n$. Consider Problem (1.1), whose feasible region C is a polyhedral convex set given by

$$C = \{x \in \mathcal{R}^n : Ax \leq b\},$$

and the bifunction $f : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$ which is often found in Nash-Cournot equilibrium models of the form (see [29]):

$$f(x, y) = \langle Px + Qy + q, y - x\rangle, \tag{5.1}$$

where $b \in \mathcal{R}^m$, $q \in \mathcal{R}^n$, A is a $m \times n$ matrix, $P,\ Q$ are $n \times n$ matrices such that Q is symmetric positive semidefinite and $P - Q$ is negative semidefinite. The bifunction f satisfies the conditions $A_1,\ A_3$ and A_4 (see [29]). Now, we will show that the condition A_2 holds. Indeed, we have $\partial_2 f(x,\cdot)(y) = \{Px + 2Qy - Qx + q\}$ and $\partial_2 f(y,\cdot)(y) = \{Py + Qy + q\}$. It follows that

$$\rho\left(\partial_2 f(x,\cdot)(y), \partial_2 f(y,\cdot)(y)\right) = \|(P-Q)(x-y)\| \leq \|P-Q\|\|x-y\|, \ \forall x, y \in \mathcal{R}^n.$$

Test 1. Let $n = 5, m = 10$. We perform some experiments to show the numerical behavior of Algorithm 3.2, (the computation results are shown in **Fig. 1**), where $\mathcal{L} = \|P - Q\|$, $\lambda_k = \frac{1}{2\mathcal{L}}$, $\epsilon_k = 0$ for all k, $\alpha_k = \frac{1}{25k+1}$ and the matrices $P,\ Q,\ A,\ b$ are chosen as follow:

$$A = \begin{pmatrix} 1.1378 & -0.3305 & 1.0301 & 0.5701 & -1.9009 \\ -0.2146 & -0.9073 & 1.1676 & 1.8277 & -1.9109 \\ 1.6476 & -0.7412 & -0.4565 & -1.2547 & 1.1941 \\ -1.6537 & 0.3268 & -0.7278 & 1.6381 & -1.5688 \\ 1.9957 & 0.7574 & -0.0763 & 0.2750 & 0.7552 \\ 0.9103 & 1.4555 & -0.8860 & -1.2259 & 1.1001 \\ 0.2336 & -1.1566 & -1.3679 & 0.8170 & 1.1660 \\ -1.4287 & 0.0270 & -0.4184 & 0.6063 & 0.8379 \\ 1.3505 & -0.2998 & -0.8582 & -0.6724 & 1.7862 \\ 0.9010 & -1.0412 & -1.0278 & 0.0863 & -0.8079 \end{pmatrix}, \ b = \begin{pmatrix} 2.9916 \\ 2.1089 \\ 2.3122 \\ 2.8907 \\ 1.4493 \\ 2.963 \\ 1.3412 \\ 2.8821 \\ 2.9139 \\ 1.35 \end{pmatrix},$$

$$P = \begin{pmatrix} 6.0789 & 2.0000 & 0 & 0 & 0 \\ 2.0000 & 7.9330 & 0 & 0 & 0 \\ 0 & 0 & 8.0712 & 2.0000 & 0 \\ 0 & 0 & 2.0000 & 8.5923 & 0 \\ 0 & 0 & 0 & 0 & 6.5521 \end{pmatrix},$$

$$Q = \begin{pmatrix} 3.7329 & 1.0000 & 0 & 0 & 0 \\ 1.0000 & 3.5758 & 0 & 0 & 0 \\ 0 & 04.2547 & 1.0000 & 0 & \\ 0 & 0 & 1.0000 & 3.9077 & 0 \\ 0 & 0 & 0 & 0 & 3.4648 \end{pmatrix}.$$

Test 2. This test compares the computation results of Algorithm 3.2 (*Alg.* 3.2) and the Halpern subgradient extragradient method (*HSEM*) in [17] with the different initial points (**Table 1**). The data of the algorithms are as follows:

- A is a matrix of the size $m \times n$ with entries generated randomly in $[-2, 2]$ and elements of b generated randomly in $[1, 3]$.
- q is a zero vector and two matrices $P,\ Q$ are defined as follows:

$$P = \begin{pmatrix} 8.9487 & 2 & 0 & 0 & 0 \\ 2 & 9.8750 & 0 & 0 & 0 \\ 0 & 0 & 6.9455 & 2 & 0 \\ 0 & 0 & 2 & 8.7904 & 0 \\ 0 & 0 & 0 & 0 & 10.2969 \end{pmatrix},$$

$$Q = \begin{pmatrix} 4.8100 & 1 & 0 & 0 & 0 & \\ 1 & 4.4537 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 3.2502 & 1 & 0 \\ 0 & 0 & 1 & 4.0523 & 0 & \\ 0 & 0 & 0 & 0 & 5.1967 & \end{pmatrix}.$$

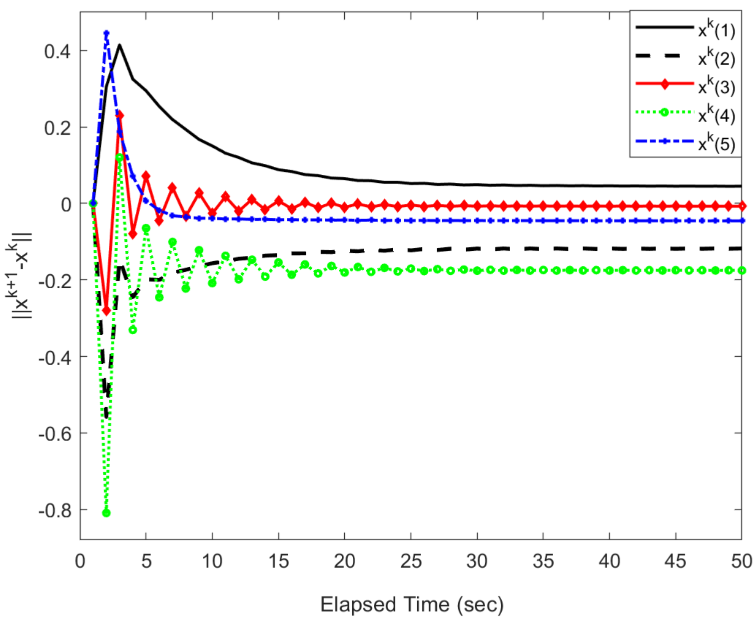

Figure 1: Convergence of Algorithm 3.2 with the tolerance $\epsilon = 10^{-3}$, the stopping criterion is $\|x^{k+1} - x^k\| \leq \epsilon$.

- *Alg.* 3.2: The parameters are the same in Test 1, the stopping criterion is $\|x^{k+1} - x^k\| \leq \epsilon$.
- $HSEM$: $\lambda_k = \frac{1}{2\mathcal{L}}$ for all k, $\alpha_k = \frac{1}{k+1}$, the stopping criterion is $\|x^{k+1} - x^k\| \leq \epsilon$.

Test 3. In this test, we perform some experiments to show the numerical behavior of Algorithms 3.2 and the Halpern subgradient extragradient method ($HSEM$) ([17], Algorithm 3.2) and the extragradient-viscosity method (EVM) ([36], Algorithm 1). Computational results are reported in **Table 2**. The data of the algorithms are as follows:

- A is a matrix of the size $m \times n$ with entries generated randomly in $[-2, 2]$, elements of b generated randomly in $[1, 3]$.
- q is a zero vector, the matrix $P = Q - T$ where the symmetric positive semidefinite matrix Q is made by using Q_1 and a random orthogonal matrix, the negative semidefinite T is made by Q_2 and another random orthogonal matrix; Q_1, Q_2 are random diagonal matrices with their diagonal elements in $[1, m]$ and $[-m, 0]$, respectively.

Init. point	*Alg.* 3.2		*HSEM*	
	Iter.	CPU-times	Iter.	CPU-times
(1,1,1,1,1)	14	1.2656	14	3.8438
(1,0,0,0,0)	17	1.4531	28	7.2031
(1,0,1,0,1)	57	4.5781	10	2.0938
(0,0,10,0,5)	530	38.3594	63	9.5938
(5,0,10,0,5)	536	38.7344	336	40.7188
(1,2,10,0,5)	502	35.7656	120	15.4531
(30,2,10,0,5)	642	44.7188	342	44.3438
(30,2,10,10,5)	526	36.375	579	95.2188
(3,2,1,1,5)	16	1.4531	30	4.0156
(30,2,1,1,50)	46	3.9375	1660	230.5469
(10,2,1,1,20)	28	2.3906	686	85

Table 1: The comparative results for different initial points, where $\epsilon = 10^{-3}$.

- Alg. 3.2: The parameters are the same in Test 2.
- $HSEM$: The parameters are the same in Test 2.
- EVM: $F(x) = x - x^0,\ x^0 \in C,\ \beta = \frac{1}{2},\ \beta_k = \frac{1}{2k+1},\ S = I$, where I is identify mapping.

Test 4. Consider the infinite dimensional Hilbert space $Ł^2([0,1])$ with the innner product and induced norm indicated as

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt\ \forall x, y \in Ł^2([0,1]),\ \|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}.$$

We consider Problem (1.1) with the feasible set $C = \{x \in Ł^2([0,1]) : \|x\| \leq 1\}$ and the bifunction

$$f(x, y) = \langle F(x), y - x\rangle,$$

where $F : Ł^2([0,1]) \to Ł^2([0,1])$ is of the form $F(x) = \max\{0, x(t)\}$. Obviously, $F(x)$ is monotone and 1-Lipschitz continuous and f satisfies the conditions $A_1 - A_4$. In this test, we compare the behavior of Algorithm 3.2 (Alg. 3.2) with the adaptive golden ratio algorithm ($AGRA$) of Malitsly in [23] and the subgradient extragradient algorithm (SEA) of Censor et al. ([11], Algorithm 4.1). In all three algorithms we take $x^0(t) = t^2, \mathcal{L} = 2$, the stopping criterion is $\|x^{k+1} - x^k\| \leq \epsilon$. The computation results of this test show in **Table 3**.

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	$m = 50$					
	Alg. 3.2		*HSEM*		*EVM*	
n	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times
2	35	45.2344	10	13.6719	9	34.7344
3	23	23	15	22.8281	10	34.1719
5	43	31.2656	71	63.75	11	18.0469
10	14	12.8438	114	62.7344	13	22.9063
20	80	27.3438	128	59.1563	14	10.7031
	$m = 100$					
	Alg. 3.2		*HSEM*		*EVM*	
n	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times
2	41	95.0469	15	1.75	9	38.8281
3	60	123.2188	75	5.8281	9	33.0313
5	21	31.1563	109	11.3594	10	25.4531
10	27	26.7656	264	80.8281	12	23.1563
20	304	29.0469	230	150.5469	14	15.3125

Table 2: The comparative results where tolerance parameter $\epsilon = 10^{-3}$, the stopping criterion is $\|x^{k+1} - x^k\| \leq \epsilon$.

Alg. 3.2			*SEA*			*AGRA*		
λ_k	Iter.	CPU-time	τ_k	Iter.	CPU-time	$\bar{\lambda}_k$	Iter.	CPU
$\frac{1}{500\mathcal{L}+1}$	1	0.4375	$\frac{1}{k+1}$	1	0.2344	$\frac{\varphi}{2k+3}$	39	18.3438
$\frac{1}{4\mathcal{L}+1}$	47	16.2969	$\frac{1}{5}$	26	5.875	$\frac{\varphi}{20k^{\frac{1}{5}}+50}$	32	16.4844
$\frac{1}{10\mathcal{L}+1}$	77	22.2969	$\frac{1}{50}$	111	26.0781	$\frac{\varphi}{30k^{\frac{1}{2}}+30}$	51	26.6406
$\frac{1}{\mathcal{L}+10}$	55	16.6563	$\frac{1}{100}$	151	61.8906	$\frac{\varphi}{5k^{\frac{1}{2}}+30}$	79	39.7969
$\frac{1}{\mathcal{L}+1}$	24	7.1719	$\frac{1}{100+k}$	12	43.5313	$\frac{\varphi}{k^{\frac{1}{2}}+3}$	35	16.8125
$\frac{1}{100\mathcal{L}+1}$	123	34.1719	$\frac{1}{50+k}$	101	35.4375	$\frac{\varphi}{5k+20}$	79	54.4688
$\frac{1}{2\mathcal{L}^2+1}$	47	14.7656	$\frac{1}{5+k}$	44	14.9219	$\frac{\varphi}{5k+1}$	68	38.8281
$\frac{1}{2\mathcal{L}^5+1}$	133	40.2031	$\frac{1}{5+k^2}$	15	5.6406	$\frac{\varphi}{3k^3+20}$	77	40.9063
$\frac{1}{20\mathcal{L}^{\frac{1}{5}}+1}$	83	28	$\frac{1}{20+k^5}$	5	1.4531	$\frac{\varphi}{4k^3+15}$	74	41.9844
$\frac{1}{\mathcal{L}^{\frac{1}{2}}+8}$	48	15.4844	$\frac{1}{20+k^{\frac{1}{2}}}$	72	25.5313	$\frac{\varphi}{(5k^3+1)^{\frac{1}{3}}}$	5	2.8125
$\frac{1}{\mathcal{L}^{\frac{1}{5}}+5}$	37	16.7188	$\frac{1}{20+k^{\frac{1}{5}}}$	68	21.4219	$\frac{\varphi}{(5k^{\frac{1}{2}}+30)^{\frac{1}{2}}}$	43	21.9219

Table 3: The comparative results for for different given parameters, where $\epsilon = 10^{-3}$.

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