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A New Analytical Series Solution with Convergence for Nonlinear Fractional Lienard's Equations with Caputo Fractional Derivative

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Abstract. Lienard's equations are important nonlinear differential equations with application in many areas of applied mathematics. In the present article, a new approach known as the modified fractional Taylor series method (MFTSM) is proposed to solve the nonlinear fractional Lienard equations with Caputo fractional derivatives, and the convergence of this method is established. Numerical examples are given to verify our theoretical results and to illustrate the accuracy and effectiveness of the method. The results obtained show the reliability and efficiency of the MFTSM, suggesting that it can be used to solve other types of nonlinear fractional differential equations that arise in modeling different physical problems.

1. Introduction

Nonlinear fractional differential equations (NFDEs) are an important tool in modeling many real-world problems that arise in fluid mechanics, elasticity, signal processing, chemical reactions, electromagnetism, biology, biomedical, biomathematics and so on. See for example [1, 2, 3, 8, 9, 14].

The difficulty of solving some NFDEs exactly, has necessitated the developement of efficient numerical methods to solve them. In recent years, solutions of NFDEs have been discussed by many researchers using various numerical techniques such as: the Laplace decomposition method [11], the homotopy perturbation transform method [12], the optimal homotopy analysis method [5], the fractional variational iteration method [13], the new iterative method [6], and the residual power series method [7].

The aim of this article is to use a new approach known as the modified fractional Taylor series method (MFTSM) to obtain an analytical series solution for the

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nonlinear Caputo fractional Lienard equation in the form

$$D^{2\eta}v(\zeta) + av(\zeta) + bv^3(\zeta) + cv^5(\zeta) = 0, \zeta > 0, \tag{1.1}$$

with

$$v(0) = \Psi_0, D^{\eta}v(0) = \Psi_1. \tag{1.2}$$

Here $D^{2\eta}$ denotes the fractional derivative operator, in the Caputo sense, of order 2η with $1/2 < \eta \leq 1$ and a, b, c, Ψ_0 and Ψ_1 are real numbers.

The MFTSM is an iterative algorithm. It is effective and makes it easy to obtain a power series solution for linear and nonlinear fractional differential equations without resorting to linearization, perturbation, or discretization. Unlike other series methods, the MFTSM does not require matching the coefficients of similar conditions, and no repeated connection is needed. The present method computes the coefficients of the power series by a bond of algebraic equations. In addition, the MFTSM does not need any transformation during the change from low order to higher order, thus it is possible to work with the present method directly on a given example by choosing an suitable initial estimate approximation.

The rest of the article is structured as follows. In Section 2, we provide some definitions and preliminary concepts of fractional calculus theory. Section 3 is devoted to the basic idea of the MFTSM. In Section 4, we apply the above-mentioned method to two numerical examples of a nonlinear Caputo fractional Lienard equation and discuss the applicability and reliability of the method through tables and graphs. Section 5 is devoted to the conclusion.

2. Preliminaries

In this section we recall the basic definitions and concepts of fractional calculus theory that are used in the present article.

Definition 2.1.([9]) Let $v : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continous function. The fractional integral in the Riemann-Liouville sense of order $\eta \geq 0$, is defined as

$$I^{\eta}v(\zeta) = \begin{cases} \dfrac{1}{\Gamma(\eta)} \int\limits_0^{\zeta} (\zeta - \mu)^{\eta - 1} v(\mu)d\mu, \eta > 0, \\ v(\zeta), \qquad \eta = 0. \end{cases} \tag{2.1}$$

Here, $\Gamma(.)$ denotes the gamma function.

Definition 2.2.([9]) Let $v^{(n)} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continous function. The fractional derivative in the Caputo sense of order $n - 1 < \eta \leq n$, $n \in \mathbb{N}^*$, is defined as

$$D^{\eta}v(\zeta) = \begin{cases} \dfrac{1}{\Gamma(n - \eta)} \int\limits_0^{\zeta} (\zeta - \mu)^{n - \eta - 1} v^{(n)}(\mu)d\mu, n - 1 < \eta < n, \\ v^{(n)}(\zeta), \qquad \eta = n. \end{cases} \tag{2.2}$$

Some properties of D^η are as follows

1)
$$D^\eta(\lambda) = 0, \text{ where } \lambda \in \mathbb{R}.$$

2)
$$D^\eta \zeta^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\eta+1)}\zeta^{\gamma-\eta}, \gamma > n-1, \\ 0, \qquad\qquad\qquad \gamma \leq n-1. \end{cases}$$

3)
$$D^\eta\left(v^n(\zeta)\right) = nv^{n-1}(\zeta)D^\eta v(\zeta).$$

3. Analysis of MFTSM for nonlinear Caputo fractional Lienard equation

Theorem 3.1. *Suppose we have the nonlinear Caputo fractional Lienard equation (1.1) with (1.2). Using MFTSM, the solution of (1.1)-(1.2) can be expressed as*

$$v(\zeta) = \sum_{i=0}^{\infty} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}, 0 < \eta \leq 1, 0 < \zeta < R. \tag{3.1}$$

Here, (3.1) is an infinite series which converges rapidly to the exact solution, Ψ_i *are real coefficients and* R *is the radius of convergence.*

Proof. To prove this result, we assume that the solution of equation (1.1) takes the following form

$$v(\zeta) = \sum_{i=0}^{\infty} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}. \tag{3.2}$$

Therefore, the $n^{th}-$order approximate solution of equation (1.1), can be written as

$$v_n(\zeta) = \sum_{i=0}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)} = \Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}. \tag{3.3}$$

Applying the operator $D^{2\eta}$ on equation (3.3), we get the following formula

$$D^{2\eta} v_n(\zeta) = \sum_{i=0}^{n-2} \Psi_{i+2} \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}. \tag{3.4}$$

Then, by replacing equations (3.3) and (3.4) in equation (1.1), we obtain the following iterative relation

$$\begin{aligned} 0 &= \sum_{i=0}^{n-2} \Psi_{i+2} \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)} + a\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right) \\ &+ b\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^3 \\ &+ c\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^5 . \end{aligned}$$

To determine the coefficient $\Psi_n, n = 2, 3, 4, ...,$ we follow the same methodology used to obtain the coefficients of the Taylor series. To achieve this, we must solve the following equation

$$D^{(n-2)\eta} \{F(\zeta, \eta, n)\} \downarrow_{\zeta=0} = 0,$$

where

$$\begin{aligned} F(\zeta, \eta, n) &= \sum_{i=0}^{n-2} \Psi_{i+2} \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)} + a\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right) \\ &+ b\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^3 \\ &+ c\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \sum_{i=2}^{n} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^5 . \end{aligned}$$

We now determine the terms of the sequence $\{\Psi_n\}_2^N$.
For $n = 2$ we have

$$\begin{aligned} F(\zeta, \eta, 2) &= \Psi_2 + a\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \Psi_2 \frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}\right) \\ &+ b\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \Psi_2 \frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}\right)^3 \\ &+ c\left(\Psi_0 + \Psi_1 \frac{\zeta^{\eta}}{\Gamma(\eta+1)} + \Psi_2 \frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}\right)^5 . \end{aligned}$$

Solving $F(0, \eta, 2) = 0$, yields

$$\Psi_2 = -(a\Psi_0 + b\Psi_0^3 + c\Psi_0^5).$$

To determine Ψ_3, we consider

$$\begin{aligned} F(\zeta,\eta,3) &= \Psi_2+\Psi_3\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+a\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\Psi_2\frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}+\Psi_3\frac{\zeta^{3\eta}}{\Gamma(3\eta+1)}\right)\\ &+b\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\Psi_2\frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}+\Psi_3\frac{\zeta^{3\eta}}{\Gamma(3\eta+1)}\right)^3\\ &+c\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\Psi_2\frac{\zeta^{2\eta}}{\Gamma(2\eta+1)}+\Psi_3\frac{\zeta^{3\eta}}{\Gamma(3\eta+1)}\right)^5. \end{aligned}$$

Then, we solve $D^{\eta}\{F(\zeta,\eta,3)\}\downarrow_{\zeta=0}=0$, to get

$$\Psi_3=-(a\Psi_1+3b\Psi_0^2\Psi_1+5c\Psi_0^4\Psi_1).$$

In general, to determine Ψ_r, we consider

$$\begin{aligned} F(\zeta,\eta,r) &= \sum_{i=0}^{r-2}\Psi_{i+2}\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}+a\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\sum_{i=2}^{r}\Psi_i\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)\\ &+b\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\sum_{i=2}^{r}\Psi_i\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^3\\ &+c\left(\Psi_0+\Psi_1\frac{\zeta^{\eta}}{\Gamma(\eta+1)}+\sum_{i=2}^{r}\Psi_i\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\right)^5. \end{aligned}$$

Then, we solve $D^{(r-2)\eta}\{F(\zeta,\eta,r)\}\downarrow_{\zeta=0}=0$, to get

$$\Psi_r=-\left(\begin{array}{c} a\Psi_{r-2}+b\sum\limits_{i=0}^{r-2}\sum\limits_{j=0}^{r-2-i}\frac{\Psi_i\Psi_j\Psi_{r-2-i-j}\Gamma((r-2)\eta+1)}{\Gamma(i\eta+1)\Gamma(j\eta+1)\Gamma((r-2-i-j)\eta+1)}\\ +c\sum\limits_{i=0}^{r-2}\sum\limits_{j=0}^{r-2-i}\sum\limits_{l=0}^{r-2-i-j}\sum\limits_{m=0}^{r-2-i-j-l}\frac{\Psi_i\Psi_j\Psi_l\Psi_m\Psi_{r-2-i-j-l-m}\Gamma((r-2)\eta+1)}{\Gamma(i\eta+1)\Gamma(j\eta+1)\Gamma(l\eta+1)\Gamma(m\eta+1)\Gamma((r-2-i-j-l-m)\eta+1)} \end{array}\right).$$

Therefore, the solution of equations (1.1)-(1.2) is

$$\begin{aligned} v(\zeta) &= \lim_{n\longrightarrow\infty}v_n(\zeta)\\ &= \lim_{n\longrightarrow\infty}\sum_{i=0}^{n}\Psi_i\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}\\ &= \sum_{n=0}^{\infty}\Psi_i\frac{\zeta^{i\eta}}{\Gamma(i\eta+1)}. \end{aligned}$$

□

Theorem 3.2. *The series solution given in equation (3.1) converges to the exact solution if there exists a constant* $0<\tau<1$ *such that*

$$\|v_{n+1}(\zeta)\|\leq\tau\|v_n(\zeta)\|,n\in\mathbb{N},0<\zeta<R.$$

Proof. For every $0 < \zeta < R$, we have

$$\begin{aligned}
\|v(\zeta) - v_n(\zeta)\| &= \left\| \sum_{r=n+1}^{\infty} v_r(\zeta) \right\| \leq \sum_{r=n+1}^{\infty} \|v_r(\zeta)\| \\
&\leq \sum_{r=n+1}^{\infty} \tau \|v_{r-1}(\zeta)\| \leq \sum_{k=n+1}^{\infty} \tau^2 \|v_{r-2}(\zeta)\| \\
&\leq ... \leq \|v_0\| \sum_{r=n+1}^{\infty} \tau^r \\
&= \frac{\tau^{n+1}}{1-\tau} \|v_0\| .
\end{aligned}$$

Because $0 < \tau < 1$ and v_0 is bounded, so we get

$$\lim_{n \longrightarrow \infty} \|v(\zeta) - v_n(\zeta)\| = 0.$$

□

4. Examples

In this section, we demonstrate the accurateness and effectiveness of the proposed method by presenting two different examples of nonlinear Caputo fractional Lienard equations.

Example 4.1. Let us take the following nonlinear Caputo fractional Lienard equation

$$D^{2\eta} v(\zeta) - v(\zeta) + 4v^3(\zeta) - 3v^5(\zeta) = 0, 1/2 < \alpha \leq 1, \zeta > 0, \tag{4.1}$$

with

$$v(0) = \frac{1}{\sqrt{2}}, D^{\eta} v(0) = \frac{1}{\sqrt{8}}. \tag{4.2}$$

Using the same procedure of the MFTSM given in Section 3, we have

$$v(\zeta) = \sum_{i=0}^{\infty} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta + 1)},$$

and

$$\begin{aligned}\Psi_0 &= \frac{1}{\sqrt{2}},\\ \Psi_1 &= \frac{1}{\sqrt{8}},\\ \Psi_2 &= -\frac{1}{\sqrt{32}},\\ \Psi_3 &= -\frac{5}{\sqrt{128}},\\ &\vdots\end{aligned}$$

Therefore, the solution of equations (4.1)-(4.2), is given by

$$v(\zeta) = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2}\frac{\zeta^{\eta}}{\Gamma(\eta+1)} - \frac{1}{4}\frac{\zeta^{2\eta}}{\Gamma(2\eta+1)} - \frac{5}{8}\frac{\zeta^{3\eta}}{\Gamma(3\eta+1)} + ...\right). \tag{4.3}$$

If we take $\eta = 1$ in equation (4.3), the solution becomes

$$\begin{aligned} v(\zeta) &= \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2}\zeta - \frac{1}{8}\zeta^2 - \frac{5}{48}\zeta^3 + ...\right)\\ &= \sqrt{\frac{1+\tanh(\zeta)}{2}},\end{aligned}$$

which is the exact solution for equations (4.1)-(4.2), when $\eta = 1$ (See.[4]).

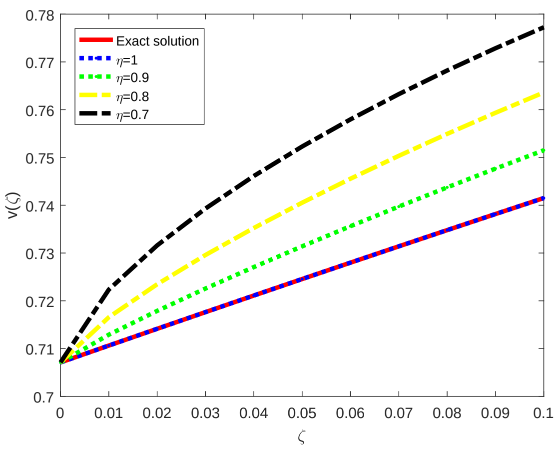

Figure 1: Graph of the exact solution and MFTSM-solution for Example 4.1

		$\eta = 2$	$\eta = 1$	*Absolute error*
ζ	v_{exact}	v_{FHATM}	v_{MFTSM}	$\lvert v_{exact} - v_{MFTSM} \rvert$
0.00	0.70711	0.70711	0.70711	0
0.02	0.71414	0.71414	0.71414	5.0793×10^{-9}
0.04	0.72110	0.72110	0.72110	8.2374×10^{-8}
0.06	0.72799	0.72799	0.72799	4.2249×10^{-7}
0.08	0.73479	0.73479	0.73479	1.3522×10^{-6}
0.1	0.74151	0.74151	0.74151	3.3415×10^{-6}

Table 1: Comparison between the exact solution, FHATM solution and MFTSM solution for Example 4.1

Example 4.2. Let us take the following nonlinear Caputo fractional Lienard equation

$$D^{2\eta}v(\zeta) - v(\zeta) + 4v^3(\zeta) + 3v^5(\zeta) = 0, 1/2 < \eta \leq 1, \zeta > 0, \tag{4.4}$$

with

$$v(0) = \frac{1}{\sqrt{1+\sqrt{2}}}, D^{\eta}v(0) = 0. \tag{4.5}$$

Using the same procedure of the MFTSM given in Section 3, we have

$$v(\zeta) = \sum_{i=0}^{\infty} \Psi_i \frac{\zeta^{i\eta}}{\Gamma(i\eta + 1)},$$

and

$$\begin{aligned}
\Psi_0 &= \frac{1}{\sqrt{1+\sqrt{2}}}, \\
\Psi_1 &= 0, \\
\Psi_2 &= -\left(\frac{4+2\sqrt{2}}{\left(1+\sqrt{2}\right)^2\sqrt{1+\sqrt{2}}}\right), \\
\Psi_3 &= 0, \\
&\vdots
\end{aligned}$$

Therefore, the solution of equations (4.4)-(4.5), is given by

$$v(\zeta) = \frac{1}{\sqrt{1+\sqrt{2}}}\left(1 - \frac{4+2\sqrt{2}}{\left(1+\sqrt{2}\right)^2}\frac{\zeta^{2\eta}}{\Gamma(2\eta+1)} + ...\right). \tag{4.6}$$

If we take $\eta = 1$ in equation (4.6), the solution becomes

$$\begin{aligned} v(\zeta) &= \frac{1}{\sqrt{1+\sqrt{2}}}\left(1 - \frac{2+\sqrt{2}}{\left(1+\sqrt{2}\right)^2}\zeta^2 + ...\right) \\ &= \sqrt{\frac{\sec h^2(\zeta)}{2\sqrt{2} + \left(1-\sqrt{2}\right)\sec h^2(\zeta)}}. \end{aligned}$$

which is the exact solution for equations (4.4)-(4.5), when $\eta = 1$ (See.[4]).

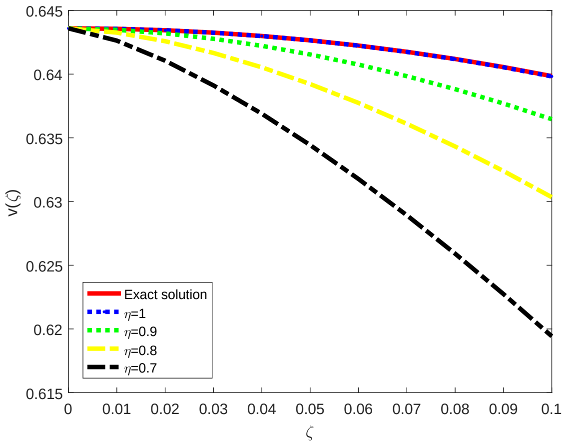

Figure 2: Graph of the exact solution and MFTSM-solution for Example 4.2

		$\eta = 2$	$\eta = 1$	*Absolute error*
ζ	v_{exact}	v_{FHATM}	v_{MFTSM}	$\lvert v_{exact} - v_{MFTSM}\rvert$
0.00	0.64359	0.64359	0.64359	0.0
0.02	0.64344	0.64344	0.64344	3.2888×10^{-8}
0.04	0.64299	0.64299	0.64299	5.2585×10^{-7}
0.06	0.64224	0.64224	0.64224	2.6590×10^{-6}
0.08	0.64119	0.64118	0.64118	8.3902×10^{-6}
0.1	0.63984	0.63982	0.63982	2.0441×10^{-5}

Table 2: Comparison between the exact solution, FHATM solution and MFTSM solution for Example 4.2

Figures 1 and 2 show the graphs of the exact solutions and the 3^{rd} order approximate solutions using the MFTSM at $\eta = 0.7, 0.8, 0.9, 1$ for equations (4.1) and (4.4), respectively. The figures show that for various fractional-order values, the proposed method is reliable, accurate and efficient. Tables 1 and 2 shows the comparison between the exact solutions, approximate soluions using FHATM at $\eta = 2$ (See.[10]) and approximate solutions using MFTSM at $\eta = 1$. The tables show that there is a very good agreement between the solutions obtained and those available in the literature.

5. Conclusion

In this article, we used a new approach known as the modified fractional Taylor series method (MFTSM) to obtain an analytical series solution of the nonlinear fractional Lienard equation with the Caputo fractional derivative. Numerical results have been presented to demonstrate the accuracy and efficiency of the MFTSM. From the obtained results, it is clear that the MFTSM provided highly accurate series solutions, which converge very rapidly to the exact solution. In addition, it has been observed that there exists a very good agreement between the solutions obtained and those available in the literature. Finally, we can conclude that the proposed method is extremely methodical, more effective and very accurate, and which can be applied to solve various classes of nonlinear fractional differential equations.

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