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FURTHER ON PETROVIĆ'S TYPES INEQUALITIES†

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Abstract. In this article, authors derived Petrović's type inequalities for a class of functions, namely, called exponentially $h-$convex functions. Also, the associated results for coordinates has been derived by defining exponentially $h-$convex functions on coordinates.

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1. Introduction and preliminaries

Convex functions are utilized to investigate a diverse range of problems that emerge in the pure and applied sciences. This theory offers us with a natural and broad framework for investigating a wide range of unconnected issues. See [2, 3, 4, 5, 8, 12, 13, 15] for further information on convex functions and their variant forms, including contemporary applications, generalizations, and other topics.

S. Varošanec [26] gave the definition of $h-$convex function and derived several results by imposing the conditions on h, which seemed like a nice generalization of the convex functions. Bernstein [5] introduced exponentially convex functions, which have applications in covariance analysis. By imposing the requirement of $r-$convex functions, Avriel [4] studied this topic. Noor and Noor [13] were motivated and inspired by these applications to analyse exponentially convex functions and investigate their basic characterizations. In information theory, optimization theory, and statistical theory, Pal and Wong [15] offered fruitful ground for exponentially convex functions.

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The well-known Petrović's inequality [16] is one of the most significant inequalities. Several authors have discovered Petrović's type inequality, see [11, 16, 17, 18, 19, 20] and references therein.

In this paper, we will denote the class of exponentially $h-$convex functions by $ESX(h, I)$, where I is an interval in $\mathbb{R}$ and the class of exponentially $h-$convex functions on coordinates by $\mathbf{ESX}(h, \Delta)$.

Rashid et al. [24] introduced exponentially $h-$convex function as follows:

Definition 1.1. Let $h : J \to \mathbb{R}$ be a non-negative function such that $(0, 1) \subseteq J$. A function $\phi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ belongs to $ESX(h, I)$, if

$$e^{\phi(\tau\varsigma+(1-\tau)\xi)} \leq h(\tau)e^{\phi(\varsigma)} + h(1-\tau)e^{\phi(\xi)}, \quad \forall \varsigma, \xi \in \Omega, \quad \tau \in (0, 1). \tag{1}$$

Remark 1.1.

Particular value of h in inequality (1) gives us the following results:

1. Take $h(\alpha) = \alpha$ gives the definition of exponentially convex functions.
2. Take $h(\alpha) = \alpha^s$ and $\alpha \in (0, 1)$ gives the definition of exponentially $s-$convex functions in the second sense.
3. Take $h(\alpha) = \frac{1}{\alpha}$ and $\alpha \in (0, 1)$ gives the definition of exponentially Godunova Levin functions.
4. Take $h(\alpha) = \frac{1}{\alpha^s}$ and $\alpha \in (0, 1)$ gives the definition of exponentially $s-$Godunova Levin functions of second sense.

The concept of convex functions on coordinates was given by Dragomir [8]. Following the idea of Dragommir, Alomari et al. [2] introduced $h-$convex on coordinates as follows:

Definition 1.2. Let $A = [x, y]$, with $x < y$ and $B = [\varsigma, \xi]$ with $\varsigma < \xi$ be intervals in $\mathbb{R}$. Also, let $\phi : A \times B \to \mathbb{R}$ be a mapping. Define partial mappings as

$$\phi_\xi : A \to \mathbb{R} \text{ defined by } \phi_\xi(x) = \phi(x, \xi) \tag{2}$$

and

$$\phi_\varsigma : B \to \mathbb{R} \text{ defined by } \phi_\varsigma(y) = \phi(\varsigma, y). \tag{3}$$

If the mappings defined in (2) and (3) are $h-$convex on A and B respectively, for all $\xi \in B$ and $\varsigma \in A$. Then ϕ is $h-$convex on coordinates.

Let us consider the bidimensional interval $\Delta = A \times B$ in $\mathbb{R}^2$. We will keep the notation Δ in the whole paper.

Definition 1.3. Let $h : J \to \mathbb{R}$ be a positive function such that $(0, 1) \subseteq J$. A mapping $\phi : \Delta \to \mathbb{R}$ is $h-$convex in Δ, if

$$\phi(\tau\varsigma + (1-\tau)\eta, \tau\xi + (1-\tau)\zeta) \leq h(\tau)\phi(\varsigma, \xi) + h(1-\tau)\phi(\eta, \zeta), \tag{4}$$
$$\forall (\varsigma, \xi), (\eta, \zeta) \in \Delta, \tau \in (0, 1).$$

Petrović [16] derived inequality for convex functions.

Theorem 1.4. *Let* $[0,d] \subseteq \mathbb{R}$ *be an interval,* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$ *and* $(w_1, w_2, ..., w_n) \in \mathbb{R}_+{}^n$ *such that*

$$\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa \in [0,d] \text{ and } \sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa \geq \varsigma_\kappa \text{ for each } \kappa = 1, ..., n. \tag{5}$$

Let ϕ *be the convex function on* $[0,d]$*, then we have*

$$\sum_{\kappa=1}^{n} w_\kappa \phi(\varsigma_\kappa) \leq \phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right) + \left(\sum_{\kappa=1}^{n} w_\kappa - 1\right)\phi(0). \tag{6}$$

The next two results has been proved by W. Iqbal et al. [11].

Theorem 1.5. *Let* $[0,d] \subseteq \mathbb{R}$ *be an interval,* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$ *and* $(w_1, w_2, ..., w_n) \in \mathbb{R}_+{}^n$ *such that* (5) *hold.*
Let a function $\phi \in ESX\left(h, [0, \infty)\right)$, *then*

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} \leq e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - 1\right) e^{\phi(0)}. \tag{7}$$

Theorem 1.6. *Let* $[0,b], [0,d] \subseteq \mathbb{R}$ *be intervals,* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$, $(\xi_1, \xi_2, ..., \xi_n) \in [0,b]^n$ *and* $(s_1, s_2, ..., s_n), (w_1, w_2, ..., w_n) \in \mathbb{R}_+{}^n$, *such that*

$$\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa \in [0,d), \quad 0 \neq \sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa \geq \varsigma_\kappa, \text{ for each } \kappa = 1, 2, ..., n \tag{8}$$

and

$$\sum_{r=1}^{n} q_r \xi_r \in [0,b), \quad 0 \neq \sum_{r=1}^{n} q_r \xi_r \geq \xi_r, \text{ for each } r = 1, 2, ..., n. \tag{9}$$

Let a function $\phi \in \mathbf{ESX}\left(h, [0,\infty)^2\right)$, *then*

$$\begin{aligned} \sum_{\kappa=1}^{n}\sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa, \xi_r)} &\leq \left\{ e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \sum\limits_{r=1}^{n} q_r \xi_r\right)} + \left(\sum_{r=1}^{n} q_r - 1\right) e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, 0\right)} \right\} \\ &+ \left(\sum_{\kappa=1}^{n} w_\kappa - 1\right)\left\{ e^{f\left(0, \sum\limits_{r=1}^{n} q_r \xi_r\right)} + \left(\sum_{r=1}^{n} q_r - 1\right) e^{\phi(0,0)} \right\}. \end{aligned} \tag{10}$$

The main purpose of this paper is to introduce a new concept of exponentially $h-$convex functions on coordinates. We derive Petrović's type inequalities for exponentially $h-$convex and exponentially $h-$convex functions on coordinates.

2. Main results

Here we give a lemma, which has important role in proving our results.

Lemma 2.1. *Let $[0,d] \subseteq \mathbb{R}$ be an interval, $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$ and $(w_1, w_2, ..., w_n) \in {\mathbb{R}_+}^n$ such that (5) hold.*

Also, let $\phi : [0,\infty) \to \mathbb{R}$ be a function and $h : J \to \mathbb{R}$ be a positive function. Then

$$e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} \geq \frac{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)}{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)} \sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)}, \tag{11}$$

if $\frac{e^{\phi(\varsigma)}}{h(\varsigma - c)}$ is increasing for $\varsigma > c$ on $[0,d]$.

Proof. Since $\frac{e^{\phi(\varsigma)}}{h(\varsigma-c)}$ is increasing on $[0,d]$ and $\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa \geq \varsigma_\kappa > c$ for all $\kappa = 1, ..., n$, we have

$$\frac{e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)}}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)} \geq \frac{e^{\phi(\varsigma_\kappa)}}{h(\varsigma_\kappa - c)}.$$

This gives

$$h(\varsigma_\kappa - c) e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} \geq h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right) e^{\phi(\varsigma_\kappa)}.$$

Multiplying above inequality by $\sum\limits_{\kappa=1}^{n} w_\kappa$ on both sides, we have

$$\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c) e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} \geq h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right) \sum_{r=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)},$$

from which one can deduce (11). □

Next two theorems are the generalization of Petrović's type inequality for exponentially $h-$convex functions.

Theorem 2.2. *Let $[0,d] \subseteq \mathbb{R}$ be an interval, $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$ and $(w_1, w_2, ..., w_n) \in {\mathbb{R}_+}^n$ such that (5) hold. Also, let $h : J \to \mathbb{R}^+$ be a supermultiplicative function such that*

$$h(\tau) + h(1-\tau) \leq 1, \forall \tau \in (0,1). \tag{12}$$

Also, let $\phi \in ESX(h, [0,\infty))$, then

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} \leq A e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - A\right) e^{\phi(c)} \quad , \tag{13}$$

where

$$A=\frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa-c)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa-c\right)}.$$

Proof. Let $\phi\in ESX\,(h,[0,\infty))$ and

$$\Psi_{(\varsigma)}=\frac{e^{\phi(\varsigma)}-e^{\phi(c)}}{h(\varsigma-c)}.$$

Let $\xi>\varsigma>c$ and $\varsigma=\tau\xi+(1-\tau)c$, where $\tau\in(0,1)$. Then

$$\begin{aligned}\Psi_{(\varsigma)}&=\frac{e^{\phi(\tau\xi+(1-\tau)c)}-e^{\phi(c)}}{h(\tau\xi+(1-\tau)c-c)}\\&\le\frac{h(\tau)e^{\phi(\xi)}+h(1-\tau)e^{\phi(c)}-e^{\phi(c)}}{h(\tau(\xi-c))}.\end{aligned}$$

As h is supermultiplicative, so we have

$$\Psi_{(\varsigma)}\le\frac{h(\tau)e^{\phi(\xi)}+h(1-\tau)e^{\phi(c)}-e^{\phi(c)}}{h(\tau)h(\xi-c)}.$$

Since $h(1-\tau)-1\le -h(\tau)$, we have

$$\Psi_{(\varsigma)}\le\frac{h(\tau)e^{\phi(\xi)}-h(\tau)e^{\phi(c)}}{h(\tau)h(\xi-c)}.$$

This gives

$$\Psi_{(\varsigma)}\le\frac{e^{\phi(\xi)}-e^{\phi(c)}}{h(\xi-c)}=\Psi_{(\xi)},$$

which shows that $\Psi_{(\varsigma)}$ is increasing on $[0,d]$.
As we have shown that, $\frac{e^{\phi(\xi)}-e^{\phi(c)}}{h(\varsigma-c)}$ is increasing for $\varsigma>c$, when $\phi\in ESX\,(h,[0,\infty))$.
Substituting $e^{\phi(\varsigma)}$ by $e^{\phi(\varsigma)}-e^{\phi(c)}$ in Lemma 2.1, one has

$$e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa\right)}-e^{\phi(c)}\ge\frac{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa-c\right)}{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa-c)}\sum_{\kappa=1}^{n} w_\kappa\left(e^{\phi(\varsigma_\kappa)}-e^{\phi(c)}\right).$$

This gives

$$\begin{aligned}&e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa\right)}\\&\ge\frac{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa-c\right)}{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa-c)}\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)}-\left(\frac{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa-c\right)}{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa-c)}\sum_{\kappa=1}^{n} w_\kappa-1\right)e^{\phi(c)}.\end{aligned}$$

This gives

$$\frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} \geq \sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} - \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)}\right) e^{\phi(c)}.$$

From above inequality one can deduce (13). □

Theorem 2.3. *Assume that the conditions given in Theorem 2.2 are valid. Then*

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} \leq \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)}\right) e^{\phi(0)} \quad . \tag{14}$$

Proof. By taking $c = 0$ in (13), we get the required result. □

The next two results has been proved by W. Iqbal et al. [11].

Corollary 2.4. *Assume that the conditions given in Theorem 1.5 are valid. If $\phi : [0, \infty) \to \mathbb{R}$ is an exponentially convex function, then*

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} \leq \frac{\sum_{\kappa=1}^{n} w_\kappa (\varsigma_\kappa - c)}{\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa (\varsigma_\kappa - c)}{\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c}\right) e^{\phi(c)}. \tag{15}$$

Proof. Substituting h with an identity function in expression (13) gives us the required result. □

Corollary 2.5. *Assume that the conditions given in Theorem 1.5 are valid. If $\phi : [0, \infty) to \mathbb{R}$ is an exponentially convex function, then*

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa)} \leq e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - 1\right) e^{\phi(0)}. \tag{16}$$

Proof. Substituting h with an identity function and $c = 0$, in (13) completes the proof. □

Here, we define the exponentially $h-$convex functions on coordinates, which is mainly due to Dragomir [8], Alomari [2] and W. Iqbal et al. [11].

Definition 2.6. Let $h : J \to \mathbb{R}$ be an arbitrary positive function such that $(0,1) \subseteq J$. A positive mapping $\phi \in \mathbf{ESX}(h,\Delta)$, if the mappings defined in (2) and (3) belongs to $ESX(h,A)$ and $ESX(h,B)$ respectively, for all $\xi \in A$ and $\varsigma \in B$.

Definition 2.7. Let $h : J \to \mathbb{R}$ be an arbitrary positive function with $(0,1) \subseteq J$. A positive mapping $\phi : \Delta \to \mathbb{R}$ belongs to $ESX(h,\Delta)$, if

$$e^{\phi(\tau\varsigma+(1-\tau)\eta,\tau\xi+(1-\tau)\zeta)} \leq h(\tau)e^{\phi(\varsigma,\xi)} + h(1-\tau)e^{\phi(\eta,\zeta)}, \tag{17}$$
$$\forall(\varsigma,\xi),(\eta,\zeta) \in \Delta, \tau \in (0,1).$$

Remark 2.1. Particular value of h in Definition 2.6 gives us the following results:

1. Take $h(\alpha) = \alpha$ gives the definition of exponentially convex functions on coordinates.
2. Take $h(\alpha) = \alpha^s$ and $\alpha \in (0,1)$ gives the definition of exponentially $s-$convex functions on coordinates in the second sense.
3. Take $h(\alpha) = \frac{1}{\alpha}$ and $\alpha \in (0,1)$ gives the definition of exponentially Godunova Levin functions on coordinates.
4. Take $h(\alpha) = \frac{1}{\alpha^s}$ and $\alpha \in (0,1)$ gives the definition of exponentially $s-$Godunova Levin functions on coordinates of second sense.

Lemma 2.8. *If $\phi \in ESX(h,\Delta)$, then $\phi \in \mathbf{ESX}(h,\Delta)$ but the converse is not true in general.*

Proof. Let $\phi \in ESX(h,\Delta)$. Also, let $\phi_\varsigma : [0,d] \to \mathbb{R}$ be a partial mapping defined as $\phi_\varsigma(\xi) := \phi(\varsigma,\xi)$. Then

$$\begin{aligned} e^{\phi_\varsigma(\tau\xi+(1-\tau)\zeta)} &= e^{\phi(\varsigma,\tau\xi+(1-\tau)\zeta)} \\ &= e^{\phi(\tau\varsigma+(1-\tau)\eta,\tau\xi+(1-\tau)\zeta)} \\ &\leq h(\tau)e^{\phi(\varsigma,\xi)} + h(1-\tau)e^{\phi(\eta,\zeta)} \\ &= h(\tau)e^{\phi_\varsigma(\xi)} + h(1-\tau)e^{\phi_\eta(\zeta)}, \forall\tau \in [0,1], \xi,\zeta \in [0,d], \end{aligned}$$

which shows that the partial mapping ϕ_ς is exponentially $h-$convex.
Similarly, one can show that the partial mapping ϕ_ξ is exponentially $h-$convex.

Now, consider the positive mapping $\phi : [0,1]^2 \to [0,\infty)$ given by $e^{\phi(\varsigma,\xi)} = \varsigma\xi$. Definitely $\phi \in \mathbf{ESX}(h,\Delta)$. But it is not exponentially $h-$convex on $[0,1]^2$.

Certainly, if $(\varsigma,0),(0,\zeta) \in [0,1]^2$ and $\tau \in (0,1)$, then

$$e^{\phi(\tau(\varsigma,0)+(1-\tau)(0,\zeta))} = e^{\phi(\tau\varsigma,(1-\tau)\zeta)} = \tau(1-\tau)\varsigma\zeta$$

and

$$h(\tau)e^{\phi(\varsigma,0)} + h(1-\tau)e^{\phi(0,\zeta)} = 0.$$

Thus, $\forall\ \tau \in (0,1)$, $\varsigma, \zeta \in (0,1)$, we have

$$e^{\phi(\tau(\varsigma,0)+(1-\tau)(0,\zeta))} > h(\tau)e^{\phi(\varsigma,0)} + h(1-\tau)e^{\phi(0,\zeta)}.$$

Hence ϕ is not exponentially $h-$convex for $\tau(1-\tau)\varsigma\zeta \neq 0$. □

In the next two theorems, we give the generalized Petrović's type inequality for exponentially $h-$convex functions on coordinates.

Theorem 2.9. *Let $[0,b],[0,d] \subseteq \mathbb{R}$ be intervals, $(\varsigma_1, \varsigma_2, ..., \varsigma_n) \in [0,d]^n$, $(\xi_1, \xi_2, ..., \xi_n) \in [0,b]^n$ and $(s_1, s_2, ..., s_n), (w_1, w_2, ..., w_n) \in {\mathbb{R}_+}^n$, such that (8) and (9) holds.*

If $h : J \to \mathbb{R}^+$ be a supermultiplicative function such that (12) hold and $\phi \in \mathbf{ESX}(h,[0,\infty)^2)$. Then

$$\begin{aligned}\sum_{\kappa=1}^{n}\sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa,\xi_r)} \leq A &\Bigg\{ B e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa, \sum\limits_{r=1}^{n} q_r\xi_r\right)} \\ &+ \left(\sum_{r=1}^{n} q_r - B\right) e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa, c\right)} \Bigg\} \\ &+ \left(\sum_{\kappa=1}^{n} w_\kappa - A\right)\left\{ B e^{f\left(c, \sum\limits_{r=1}^{n} q_r\xi_r\right)} + \left(\sum_{r=1}^{n} q_r - B\right) e^{\phi(c,c)} \right\},\end{aligned} \tag{18}$$

where

$$A = \frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa - c\right)} \text{ and } B = \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r - c)}{h\left(\sum\limits_{r=1}^{n} q_r\xi_r - c\right)}. \tag{19}$$

Proof. Since $\phi \in \mathbf{ESX}(h,[0,\infty)^2)$. Therefore, the partial mapping ϕ_ξ defined in (2) belongs to $\phi \in ESX\,(h,[0,\infty))$. Using Theorem 2.2, we have

$$\begin{aligned}&\sum_{\kappa=1}^{n} w_\kappa e^{\phi_\xi(\varsigma_\kappa)} \\ &\leq \frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa - c\right)} e^{\phi_\xi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa - c\right)}\right) e^{\phi_\xi(c)}.\end{aligned}$$

This is equivalent to

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa,\xi)}$$

$$\leq \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \xi\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)}\right) e^{\phi(c,\xi)}.$$

Replacing $\xi = \xi_r$, we have

$$\sum_{\kappa=1}^{n} w_\kappa e^{\phi(\varsigma_\kappa, \xi_r)}$$
$$\leq \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \xi_r\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)}\right) e^{\phi(c,\xi_r)}.$$

Multiplying above inequality by $\sum_{r=1}^{n} q_r$ on both sides, we have

$$\sum_{\kappa=1}^{n} \sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa, \xi_r)} \leq \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)} \sum_{r=1}^{n} q_r e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \xi_r\right)} + \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa - c)}{h\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa - c\right)}\right) \sum_{r=1}^{n} q_r e^{\phi(c,\xi_r)}. \tag{20}$$

Again by Theoram 2.2, we have

$$\sum_{r=1}^{n} q_r e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \xi_r\right)} \leq \frac{\sum_{r=1}^{n} q_r h(\xi_r - c)}{h\left(\sum_{r=1}^{n} q_r \xi_r - c\right)} e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \sum_{r=1}^{n} q_r \xi_r\right)} + \left(\sum_{r=1}^{n} q_r - \frac{\sum_{r=1}^{n} q_r h(\xi_r - c)}{h\left(\sum_{r=1}^{n} q_r \xi_r - c\right)}\right) e^{\phi\left(\sum_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, c\right)} \tag{21}$$

and

$$\sum_{r=1}^{n} q_r e^{f(c,\xi_r)} \leq \frac{\sum_{r=1}^{n} q_r h(\xi_r - c)}{h\left(\sum_{r=1}^{n} q_r \xi_r - c\right)} e^{f\left(c, \sum_{r=1}^{n} q_r \xi_r\right)} \tag{22}$$

$$+\left(\sum_{r=1}^{n} q_r - \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r - c)}{h\left(\sum\limits_{r=1}^{n} q_r \xi_r - c\right)}\right) e^{f(c,c)}.$$

Using (21) and (22) in the inequality (20). Then using the notations given in (19), one get the inequality (18). ☐

Theorem 2.10. *Let the conditions given in Theorem 2.9 are valid. Then*

$$\begin{aligned}
&\sum_{\kappa=1}^{n}\sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa, \xi_r)} \\
&\leq \frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)} \left\{ \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r)}{h\left(\sum\limits_{r=1}^{n} q_r \xi_r\right)} e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, \sum\limits_{r=1}^{n} q_r \xi_r\right)} \right. \\
&\left. + \left(\sum_{r=1}^{n} q_r - \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r)}{h\left(\sum\limits_{r=1}^{n} q_r \xi_r\right)}\right) e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa, 0\right)} \right\} \\
&+ \left(\sum_{\kappa=1}^{n} w_\kappa - \frac{\sum\limits_{\kappa=1}^{n} w_\kappa h(\varsigma_\kappa)}{h\left(\sum\limits_{\kappa=1}^{n} w_\kappa \varsigma_\kappa\right)}\right) \left\{ \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r)}{h\left(\sum\limits_{r=1}^{n} q_r \xi_r\right)} e^{f\left(0, \sum\limits_{r=1}^{n} q_r \xi_r\right)} \right. \\
&\left. + \left(\sum_{r=1}^{n} q_r - \frac{\sum\limits_{r=1}^{n} q_r h(\xi_r)}{h\left(\sum\limits_{r=1}^{n} q_r \xi_r\right)}\right) e^{\phi(0,0)} \right\}.
\end{aligned} \tag{23}$$

Proof. If we take $c = 0$ in (18), we get the required result. ☐

The next two results has been proved by W. Iqbal et al. [11].

Corollary 2.11. *Assume that the conditions given in Theorem 2.9 are valid.*

If $\phi : [0,\infty)^2 \to \mathbb{R}$ is an exponentially convex function on coordinates, then

$$\begin{aligned}
&\sum_{\kappa=1}^{n}\sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa,\xi_r)} \\
&\leq C\left\{De^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa,\sum\limits_{r=1}^{n} q_r\xi_r\right)} + \left(\sum_{r=1}^{n} q_r - D\right)e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa, c\right)}\right\} \\
&+\left(\sum_{\kappa=1}^{n} w_\kappa - C\right)\left\{De^{f\left(c,\sum\limits_{r=1}^{n} q_r\xi_r\right)} + \left(\sum_{r=1}^{n} q_r - D\right)e^{\phi(c,c)}\right\},
\end{aligned}$$

where

$$C = \left(\frac{\sum\limits_{\kappa=1}^{n} w_\kappa(\varsigma_\kappa - c)}{\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa - c}\right) \text{ and } D = \left(\frac{\sum\limits_{r=1}^{n} q_r(\xi_r - c)}{\sum\limits_{r=1}^{n} q_r\xi_r - c}\right).$$

Proof. Substituting h with an identity function in (18) completes the proof. □

Corollary 2.12. *Assume that the conditions given in Theorem 1.6 are valid. If a function $\phi : \omega to \mathbb{R}$ is exponentially convex function on coordinates, then*

$$\begin{aligned}
&\sum_{\kappa=1}^{n}\sum_{r=1}^{n} w_\kappa q_r e^{\phi(\varsigma_\kappa,\xi_r)} \\
&\leq \left\{e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa,\sum\limits_{r=1}^{n} q_r\xi_r\right)} + \left(\sum_{r=1}^{n} q_r - 1\right)e^{\phi\left(\sum\limits_{\kappa=1}^{n} w_\kappa\varsigma_\kappa, 0\right)}\right\} \qquad (24)\\
&+\left(\sum_{\kappa=1}^{n} w_\kappa - 1\right)\left\{e^{f\left(0,\sum\limits_{r=1}^{n} q_r\xi_r\right)} + \left(\sum_{r=1}^{n} q_r - 1\right)e^{\phi(0,0)}\right\}.
\end{aligned}$$

Proof. Substituting h with an identity function and $c = 0$ in (18) completes the proof. □

3. Conclusion

In this paper, we defined exponentially $h-$convex functions on coordinates and derived the Petrović's type inequalities for those functions. Also, we defined the Petrović's type inequalities for exponentially $h-$convex functions. It is demonstrated that our results can be used to get previously known results as special cases. The ideas and techniques presented in this study are designed to inspire scholars working in functional analysis and statistical theory to identify applications. This is an exciting new direction for future research.

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