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SOME PROPERTIES FOR COMPREHENSIVE SUBCLASS OF BI-UNIVALENT FUNCTIONS

TARIQ AL-HAWARY

Abstract. Herein, we construct a qualitative subclass $\mathcal{B}^{\gamma}(\kappa, \mu, \sigma)$ of the class of analytic bi-univalent functions. Next, we explore estimates of the coefficients $|a_j|$ $(j = 1, 2)$ and the Fekete-Szegö functional $\left|a_3 - \varsigma a_2^2\right|$ for functions in this subclass.

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1. Introduction and preliminaries

Let's denote by $\mathcal{A}$ the class that contain all analytic functions f, which can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

where $z \in \mathbb{U} = \{z : |z| < 1\}$, the usual open unit disk.

From Koebe one-quarter theorem [2], every univalent function $f \in \mathcal{A}$ has an inverse f^{-1}, which can be written as

$$f^{-1}(\omega) = \omega - a_2^2\omega + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{2}$$

If $\omega \in \mathbb{U}$, then the function f^{-1} is univalent. Hence, $f \in \Sigma$, the class of all analytic bi-univalent functions (see [1, 6, 10]).

Definition 1.1. [5] Given the parameters $v \geq 2$ and $0 \leq \gamma < 1$. Let $\mathcal{L}_v(\gamma)$ denote the class of all analytic univalent functions l, satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} l(z) - \gamma}{1 - \gamma} \right| d\theta \leq \pi v, \tag{3}$$

and normalized with $l(0) = 1$, where $z = re^{i\theta} \in \mathbb{U}$.

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For $v=2$ and $\gamma=0$ we get the class $\mathcal{L}:=\mathcal{L}_2(0)$ of all functions with positive real part. Note that (3) can be written as

$$l(z)=\frac{1}{2}\int_0^{2\pi}\frac{1+(1-2\gamma)\,ze^{-it}}{1-ze^{-it}}d\epsilon(t).$$

Also, we one can write

$$l(z)=\left(\frac{v}{4}+\frac{1}{2}\right)l_1(z)-\left(\frac{v}{4}-\frac{1}{2}\right)l_2(z),\qquad l_1(z),l_2(z)\in\mathcal{L}_2(\gamma),\quad z\in\mathbb{U}$$

where $\epsilon(t)$ is a real-valued function on $[0,2\pi]$, which ensures

$$\int_0^{2\pi}d\epsilon(t)=2\pi\ \ \text{and}\ \ \int_0^{2\pi}|d\epsilon(t)|\leq v,\quad v\geq 2.$$

In the following definition and in the sequel, the analytic bi-univalent function f is given by (1), its inverse function $f^{-1}:=g$ is given by (2), and $z,\omega\in\mathbb{U}$.

Definition 1.2. A function $f(z)$ belongs to the class $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$, if the following inequalities are fulfilled:

$$\operatorname{Re}\left((1-\kappa)\left(\frac{f(z)}{z}\right)^{\mu}+\kappa\left(f'(z)\right)^{1-\mu}+\sigma zf''(z)\right)\in\mathcal{L}_v(\gamma)\tag{4}$$

and

$$\operatorname{Re}\left((1-\kappa)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\kappa\left(g'(\omega)\right)^{1-\mu}+\sigma zg''(\omega)\right)\in\mathcal{L}_v(\gamma),\tag{5}$$

where $\kappa\geq 1$, σ, $\mu\geq 0, v\geq 2$ and $0\leq\gamma<1$.

The particular values of the parameters κ,μ,v,σ, and γ direct the class $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$ to different subclasses, for example:

(1) If $\kappa=1$, we obtain the subclass $\mathcal{B}^{\gamma}(1,\mu,\sigma)$.

A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(1,\mu,\sigma)$, if

$$\operatorname{Re}\left(\left(f'(z)\right)^{1-\mu}+\sigma zf''(z)\right)\in\mathcal{L}_v(\gamma)$$

and

$$\operatorname{Re}\left(\left(g'(\omega)\right)^{1-\mu}+\sigma zg''(\omega)\right)\in\mathcal{L}_v(\gamma).$$

(2) If $\kappa=1$ and $\mu=0$, we obtain the subclass $\mathcal{B}^{\gamma}(1,0,\sigma)$.

A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(1,0,\sigma)$, if

$$\operatorname{Re}\left(f'(z)+\sigma zf''(z)\right)\in\mathcal{L}_v(\gamma)$$

and

$$\operatorname{Re}\left(g'(\omega)+\sigma zg''(\omega)\right)\in\mathcal{L}_v(\gamma).$$

(3) If $\kappa=1$ and $\mu=\sigma=0$, we obtain the subclass $\mathcal{B}^{\gamma}(1,0,0)$.

A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(1,0,0)$, if

$$\operatorname{Re}\left(f'(z)\right)\in\mathcal{L}_v(\gamma)$$

and

$$\mathrm{Re}\left(g'(\omega)\right) \in \mathcal{L}_v(\gamma).$$

Remark 1.1. For $v=2$, the class $\mathcal{B}^{\gamma}(1,0,0)$ was introduced by Srivastava et al. [6].

(4) If $\kappa=\sigma=1$ and $\mu=0$, we obtain the subclass $\mathcal{B}^{\gamma}(1,0,1)$.
A function $f(z)\in\Sigma$ given by (1) belongs to the subclass $\mathcal{B}^{\gamma}(1,0,1)$, if

$$\mathrm{Re}\left(f'(z)+zf''(z)\right) \in \mathcal{L}_v(\gamma)$$

and

$$\mathrm{Re}\left(g'(\omega)+zg''(\omega)\right) \in \mathcal{L}_v(\gamma).$$

(5) If $\mu=1$, we obtain the subclass $\mathcal{B}^{\gamma}(\kappa,1,\sigma)$.
A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(\kappa,1,\sigma)$, if

$$\mathrm{Re}\left((1-\kappa)\left(\frac{f(z)}{z}\right)+\sigma zf''(z)+\kappa\right) \in \mathcal{L}_v(\gamma)$$

and

$$\mathrm{Re}\left((1-\kappa)\left(\frac{g(\omega)}{\omega}\right)+\sigma zg''(\omega)+\kappa\right) \in \mathcal{L}_v(\gamma).$$

(6) If $\mu=1$ and $\sigma=0$, we obtain the subclass $\mathcal{B}^{\gamma}(\kappa,1,0)$.
A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(\kappa,1,0)$, if

$$\mathrm{Re}\left((1-\kappa)\left(\frac{f(z)}{z}\right)+\kappa\right) \in \mathcal{L}_v(\gamma)$$

and

$$\mathrm{Re}\left((1-\kappa)\left(\frac{g(\omega)}{\omega}\right)+\kappa\right) \in \mathcal{L}_v(\gamma).$$

(7) If $\mu=\sigma=0$, we obtain the subclass $\mathcal{B}^{\gamma}(\kappa,0,0)$.
A function $f(z)$ belongs to the subclass $\mathcal{B}^{\gamma}(\kappa,0,0)$, if

$$\mathrm{Re}\left(\kappa f'(z)-\kappa+1\right) \in \mathcal{L}_v(\gamma)$$

and

$$\mathrm{Re}\left(\kappa g'(z)-\kappa+1\right) \in \mathcal{L}_v(\gamma).$$

The next lemma is significantly important to proof our main result.

Lemma 1.3. *([4]) Let the function* $\Upsilon(z)=\sum_{j=1}^{\infty}\lambda_j z^j$, $z\in\mathbb{U}$ *such that* $\Upsilon\in\mathcal{L}_v(\gamma)$, *then*

$$|\lambda_j|\le v(1-\gamma), \quad j\ge 1.$$

Motivated by the work of Frasin et al. [3], Padmanabhan and Parvatham [5], Tang et al. [4], and Yousef et al. [7, 8, 9], we provide estimates on the coefficients $|a_j|$ $(j=1,2)$ and the Fekete-Szegö functional $\left|a_3-\varsigma a_2^2\right|$ for functions belonging to the class $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$.

2. Coefficient bounds and the Fekete-Szegö inequality for the subclass $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$

In this section, we state and prove the main result for the subclass $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$, introduced by Definition 1.2.

Theorem 2.1. *Let $f(z)$ be in the class $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$, where $\kappa \geq 1$, σ, $\mu \geq 0$, and $0 \leq \gamma < 1$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{2v(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|}}, \frac{v(1-\gamma)}{|\mu(1-3\kappa)+2(\kappa+\sigma)|} \right\},$$

$$|a_3| \leq \min \left\{ \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|} + \frac{2v(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|}, \right.$$
$$\left. \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|} + \frac{v^2(1-\gamma)^2}{|\mu(1-3\kappa)+2(\kappa+\sigma)|^2} \right\},$$

and

$$\left|a_3 - \varsigma a_2^2\right| \leq \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|},$$

where

$$v \geq 2 \ \text{and} \ \varsigma = \frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2\mu(1-4\kappa)+6\kappa+12\sigma}.$$

Proof. First, we write the equivalent forms of inequalities (4) and (5) as follows:

$$(1-\kappa)\left(\frac{f(z)}{z}\right)^{\mu} + \kappa\left(f'(z)\right)^{1-\mu} + \sigma z f''(z) = r(z), \tag{6}$$

and

$$(1-\kappa)\left(\frac{g(\omega)}{\omega}\right)^{\mu} + \kappa\left(g'(\omega)\right)^{1-\mu} + \sigma z g''(\omega) = s(w), \tag{7}$$

where $g = f^{-1}$ and $r, s \in \mathcal{L}_v(\gamma)$ have the following Taylor expansions:

$$r(z) = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \cdots \ \text{and} \ s(w) = 1 + s_1 w + s_2 w^2 + s_3 w^3 + \cdots.$$

Equalizing the corresponding coefficients in (6) and (7), yields

$$[\mu(1-3\kappa)+2(\kappa+\sigma)]\, a_2 = r_1, \tag{8}$$

$$\left[\frac{\mu(\mu-1)(3\kappa+1)}{2}\right] a_2^2 + [\mu(1-4\kappa)+3\kappa+6\sigma]\, a_3 = r_2, \tag{9}$$

$$-[\mu(1-3\kappa)+2(\kappa+\sigma)]\, a_2 = s_1, \tag{10}$$

and

$$\left[\frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2}\right] a_2^2 - [\mu(1-4\kappa)+3\kappa+6\sigma]\, a_3 = s_2. \tag{11}$$

In view of $r, s \in \mathcal{L}_v(\gamma)$ and Lemma 1.3, the following inequalities hold:

$$|r_v| \leq v(1-\gamma), \quad |s_v| \leq v(1-\gamma), \quad v \geq 1. \tag{12}$$

From (9), (11), and (12) we find that

$$|a_2| \leq \sqrt{\frac{2v(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|}}. \tag{13}$$

From (8) and (10), we have

$$r_1 = -s_1, \tag{14}$$

and

$$a_2^2 \leq \frac{r_1^2}{(\mu(1-3\kappa)+2(\kappa+\sigma))^2}, \tag{15}$$

which, by (12), shows

$$|a_2| \leq \frac{v(1-\gamma)}{|\mu(1-3\kappa)+2(\kappa+\sigma)|}.$$

Next, by subtracting (11) from (9), we get

$$a_3 = \frac{r_2 - s_2}{2\mu(1-4\kappa)+6\kappa+12\sigma} + a_2^2. \tag{16}$$

By using (13) and (15) in (16), additionally, by means of the inequalities (12), we have

$$|a_3| \leq \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|} + \frac{2v(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|},$$

and

$$|a_3| \leq \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|} + \frac{v^2(1-\gamma)^2}{|\mu(1-3\kappa)+2(\kappa+\sigma)|^2},$$

respectively.

Also, from (11), we find that

$$\frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2\mu(1-4\kappa)+6\kappa+12\sigma}a_2^2 - a_3 = \frac{s_2}{\mu(1-4\kappa)+3\kappa+6\sigma}.$$

Consequently, we have

$$\left|a_3 - \varsigma a_2^2\right| \leq \frac{|s_2|}{|\mu(1-4\kappa)+3\kappa+6\sigma|} \leq \frac{v(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|},$$

where

$$\varsigma = \frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2\mu(1-4\kappa)+6\kappa+12\sigma}.$$

Which completes the proof. □

Choosing $v = 2$ in Theorem 2.1, we get the following consequence:

Corollary 2.2. *Let $f(z)$ be in the class $\mathcal{B}^{\gamma}(\kappa,\mu,\sigma)$, where $\kappa \geq 1$, σ, $\mu \geq 0$ and $0 \leq \gamma < 1$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|}}, \frac{2(1-\gamma)}{|\mu(1-3\kappa)+2(\kappa+\sigma)|}\right\},$$

$$|a_3| \leq \min\left\{\frac{2(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|}+\frac{4(1-\gamma)}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|},\right.$$
$$\left.\frac{2(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|}+\frac{4(1-\gamma)^2}{|\mu(1-3\kappa)+2(\kappa+\sigma)|^2}\right\},$$

and

$$\left|a_3-\varsigma a_2^2\right| \leq \frac{2(1-\gamma)}{|\mu(1-4\kappa)+3\kappa+6\sigma|},$$

where

$$\varsigma = \frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2\mu(1-4\kappa)+6\kappa+12\sigma}.$$

Putting $\gamma = 0$ in Corollary 2.2, we immediately obtain the following consequence:

Corollary 2.3. *Let $f(z)$ given by (1) be in the class $\mathcal{B}^{0}(\kappa,\mu,\sigma)$, where $\kappa \geq 1$ and σ, $\mu \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{2}{\sqrt{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|}}, \frac{2}{|\mu(1-3\kappa)+2(\kappa+\sigma)|}\right\},$$

$$|a_3| \leq \min\left\{\frac{2}{|\mu(1-4\kappa)+3\kappa+6\sigma|}+\frac{4}{|\mu^2(3\kappa+1)+\mu(1-11\kappa)+6\kappa+12\sigma|},\right.$$
$$\left.\frac{2}{|\mu(1-4\kappa)+3\kappa+6\sigma|}+\frac{4}{|\mu(1-3\kappa)+2(\kappa+\sigma)|^2}\right\},$$

and

$$\left|a_3-\varsigma a_2^2\right| \leq \frac{2}{|\mu(1-4\kappa)+3\kappa+6\sigma|},$$

where

$$\varsigma = \frac{\mu^2(3\kappa+1)+\mu(3-19\kappa)+12\kappa+24\sigma}{2\mu(1-4\kappa)+6\kappa+12\sigma}.$$

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TARIQ AL-HAWARY received M.Sc. from Al al-Bayt University and Ph.D. at National University of Malaysia. Since 2013 he has been at Al-Balqa' Applied University. His research interests include Geometric Function Theory.

Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816. Jordan.
e-mail: tariq_amh@bau.edu.jo