J. Appl. Math. & Informatics Vol. **40***(2022), No. 5 - 6, pp. 949 - 958*
https://doi.org/10.14317/jami.2022.949

DISCRETE COMPACTNESS PROPERTY FOR GENERAL QUADRILATERAL MESHES

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ABSTRACT. The aim of this papaer is to prove the discrete compactness property for modified Raviart-Thomas element(MRT) of lowest order on quadrilateral meshes. Then MRT space can be used for eigenvalue problems, and is more efficient than the lowest order ABF space since it has less degrees of freedom.

AMS Mathematics Subject Classification : 65N30, 65N25.
Key words and phrases : Discrete compactness property, finite element methods, eigenvalue problem.

1. Introduction

The Raviart-Thomas(RT) finite elements have been frequently employed for numerical computations of various practical poblems in electromagnetics in $\mathbb{R}^2$ [1, 2, 3, 9]. Because this element is more suitable to approximate vector functions together with their rotations. As for the meshes, quadrilateral meshes have the advantages of two standard methods, triangular meshes and rectangular meshes. That is, they not only can fit complex geometry well, but also maintain the data structures of rectangular meshes. However, when we use the RT elements to arbitrary quadrilaterals, the velocity vector does not converge in the divergence norm. D. A. Arnold, D. Boffi, and R. S. Falk introduced a new finite element spaces(ABF), which provides an improved error estimate over original RT element on quadrilateral meshes[10]. However, ABF element has additional degrees of freedom than RT element. On the other hand, Do Y. Kwak, and H. C. Pyo also introduced a modified Raviart-Thomas element of lowest order (MRT), which has the same degrees of freedom as the RT element[12]. The existing theory cannot be applied because of the violation of the condition $\operatorname{div}\mathbf{V}_h \subset W_h$. But the modified part of MRT element maintain the condition

Received March 30, 2022. Revised May 10, 2022. Accepted June 13, 2022.

$\operatorname{div}\mathbf{V}_h = W_h$. Hence this approach provides an optimal order approximation in $H(\operatorname{div})$ on quadrilateral meshes.

When we apply finite elements to some electromagnetic problems, it is convenient to show some compactness properties related to the original problems and then to establish their discrete analogs[4, 5, 6, 8]. In this paper, we will show that MRT element space satisfy the discrete compactness property, which is effectively employed to assure the convergence of the finite element solutions.

The organization of this paper is as follows: In the next section, we introduce the model problem and associated variational form. Section 3 contains the discretization of the problem using modified RT space. In Section 4, we prove the main result of this paper concerning the discrete compactness property and state the error estimates. Finally, we present the conclusion.

2. Model problem

Let Ω be a convex in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following eigenvalue problem: find $\lambda \in \mathbb{R}$ such that for a nonvanishing $\mathbf{u}$, it holds

$$\left\{\begin{array}{rcll} -\nabla\operatorname{div}\mathbf{u} &=& \lambda\mathbf{u}, & \text{in } \Omega, \\ \operatorname{rot}\mathbf{u} &=& 0, & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &=& 0, & \text{on}\partial\Omega, \end{array}\right. \tag{1}$$

where $\mathbf{n}$ is an unit outward normal vector, and the rotation operator rot is understood as

$$\operatorname{rot}\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

for a two-dimensional vector function $\mathbf{v} = (v_1, v_2)^T$.

We need to describe some function spaces related to this problem. First, $L^2(\Omega)$ and $H_0^1(\Omega)$ are the usual real Hilbert spaces, and the norm and inner product of $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. For a positive parameter s, we denote by $H^s(\Omega)$ the Sobolev space of order s. Also, let $\mathbf{H}^s(\Omega)$ be the space of vectors each of whose component lies in $H^s(\Omega)$. For both of the spaces $H^s(\Omega)$ and $\mathbf{H}^s(\Omega)$, we shall denote their norms(semi-norms) by $\|\cdot\|_s$ ($|\cdot|_s$). Let

$$H(\operatorname{div},\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$$

with norm $\|\mathbf{v}\|_{\operatorname{div}}^2 = \|\mathbf{v}\|_0^2 + \|\operatorname{div}\mathbf{v}\|_0^2$ and

$$H_0(\operatorname{div},\Omega) = \{\mathbf{v} \in H(\operatorname{div},\Omega) \mid (\mathbf{v}, \mathrm{grad\ q}) = -(\operatorname{div}\mathbf{v}, \mathrm{q}),\ \forall \mathrm{q} \in \mathrm{H}^1(\Omega)\}.$$

And also we let

$$\mathbb{K} = \{\mathbf{v} \in H_0(\operatorname{div},\Omega) \mid (\mathbf{v}, \mathbf{rot}\ q) = 0,\ \forall q \in H_0^1(\Omega)\}.$$

Then problem (1) can be written in the following mixed formulation: find $\lambda \in \mathbb{R}$ such that there exist $(\mathbf{u}, p) \in H_0(\operatorname{div},\Omega) \times H_0^1(\Omega)$ with $\mathbf{u} \neq 0$

$$\left\{\begin{array}{rcll} (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\mathbf{v}, \mathbf{rot}\ p) &=& \lambda(\mathbf{u},\mathbf{v}), & \forall \mathbf{v} \in H_0(\operatorname{div},\Omega), \\ (\mathbf{u}, \mathbf{rot}\ q) &=& 0, & \forall q \in H_0^1(\Omega), \end{array}\right. \tag{2}$$

where the rotation operator **rot** is defined by

$$\mathbf{rot}\ \phi = \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right)^T$$

for a scalar function ϕ . It is well-known that the problem (2) holds the inf-sup condition and the ellipticity. That is, (2) is well-posed[11].

3. Discretization using modified RaviartThomas space

In this section, we briefly recall the definition of a modified Raviart-Thomas space of the lowest order(MRT) and its approximation properties[12]. Let $\tau_h = \{K\}$ be a triangulation of the domain Ω into quadrilaterals whose diameters are bounded by $h > 0$. Let $\widehat{\mathbf{x}} = (\hat{x}, \hat{y})$ and $\mathbf{x} = (x, y)$. We use the unit square $\widehat{K} = [0,1] \times [0,1]$ as the reference element in the $\hat{x}\hat{y}-$plane with the vertices

$$\widehat{\mathbf{x}}_1 = (0,0),\ \widehat{\mathbf{x}}_2 = (1,0),\ \widehat{\mathbf{x}}_3 = (1,1),\ \widehat{\mathbf{x}}_4 = (0,1).$$

Let K be a convex quadrilateral with vertices $\mathbf{x}_i$. Then there exists a unique bilinear map $F_K : \widehat{K} \to K$ satisfying

$$F_K(\widehat{\mathbf{x}}_i) = \mathbf{x}_i,\ \ i = 1, 2, 3, 4.$$

From simple calculation, we know that the determinant J_K of the Jacobian matrix DF_K is a linear function of $\hat{x}$ and $\hat{y}$,

$$J_K(\hat{x}, \hat{y}) = \alpha + \beta\hat{x} + \gamma\hat{y} \tag{3}$$

for some constants $\alpha,\ \beta$ and γ, where

$$DF_K = \begin{pmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{pmatrix}.$$

And we note that

$$\mid K \mid = \int_K 1\ d\mathbf{x} = \int_{\hat{K}} J_K\ d\hat{\mathbf{x}} = \alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma.$$

The vector valued functions on $\widehat{K}$ are transformed into vector valued functions on K by the Piola transformation P_{F_K} associated with F_K. That is,

$$\mathbf{v} = P_{F_K}\widehat{\mathbf{v}} = \frac{DF_K}{J_K}\widehat{\mathbf{v}} \circ F_K^{-1}.$$

Also, this transformation maps $H(\mathrm{div}\,, \widehat{K})$ space on reference element onto $H(\mathrm{div}\,, K)$.

Let $\mathbf{V}_h(\widehat{K})$ be the local space on the reference element $\widehat{K}$ consisting of all functions of the form

$$\widehat{\mathbf{v}} = \begin{pmatrix} a + b\hat{x} + \frac{\beta(b+d)}{2|K|}\hat{x}(\hat{x}-1) \\ c + d\hat{y} + \frac{\gamma(b+d)}{2|K|}\hat{y}(\hat{y}-1) \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$ and β, γ are from (3). Then the local space $\mathbf{V}_h(K)$ on each quadrilateral K is defined by

$$\mathbf{V}_h(K) = \{\mathbf{v} = P_{F_K}\widehat{\mathbf{v}} \mid \widehat{\mathbf{v}} \in \mathbf{V}_h(\widehat{K})\}.$$

The modified Raviart-Thomas space of the lowest order(MRT) is defined as follows :

$$\mathrm{MRT} := \{\mathbf{v} \in \mathrm{H}(\mathrm{div}\,,\Omega) \mid \mathbf{v}|_{\mathrm{K}} \in \mathbf{V}_{\mathrm{h}}(\mathrm{K}),\ \forall \mathrm{K} \in \tau_{\mathrm{h}}\}.$$

The degrees of freedom for MRT on the reference element $\widehat{K}$ are

$$\int_{\hat{e}} \widehat{\mathbf{u}} \cdot \widehat{\mathbf{n}}\hat{q}\ d\hat{s},\ \forall \hat{q} \in P_0(\hat{e}),$$

where $\widehat{\mathbf{n}}$ and $\hat{e}$ denote the unit outward normal on $\partial\widehat{K}$ and a side of $\widehat{K}$, respectively. So it has the same degrees of freedom as the RT element. Since the divergence of an arbitrary vector on each element K is constant, it gives the following optimal order approximation for velocity and its divergence:

$$\| \mathbf{u} - \mathbf{u}_h \|_0 \leq Ch \mid \mathbf{u} \mid_1, \tag{4}$$

$$\| \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \|_0 \leq Ch \mid \operatorname{div} \mathbf{u} \mid_1 . \tag{5}$$

Then we can have the following finite dimensional problem corresponding (2) by means of MRT: find $\lambda_h \in \mathbb{R}$ such that there exist $(\mathbf{u}_h, p_h) \in \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega) \times \mathrm{Q}_{1,1} \cap \mathrm{H}_0^1(\Omega)$ with $\mathbf{u}_h \neq 0$

$$\left\{ \begin{array}{rcll} (\operatorname{div}\mathbf{u}_h, \operatorname{div}\mathbf{v}_h) + (\mathbf{v}_h, \mathbf{rot}\ p_h) & = & \lambda_h(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega), \\ (\mathbf{u}_h, \mathbf{rot}\ q_h) & = & 0, & \forall q_h \in Q_{1,1} \cap H_0^1(\Omega), \end{array} \right. \tag{6}$$

where $Q_{1,1}$ is the space of polynomials of degree at most 1 separately in each variable.

4. Discrete compactness property and error estimates

First, let us introduce some spaces for vector functions. The discrete kernel $\mathbb{K}_h$ is defined as

$$\mathbb{K}_h = \{\mathbf{v}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega) \mid (\mathbf{v}_{\mathrm{h}}, \mathbf{rot}\ \mathrm{q}_{\mathrm{h}}) = 0,\ \forall \mathrm{q}_{\mathrm{h}} \in \mathrm{Q}_{1,1} \cap \mathrm{H}_0^1(\Omega)\}.$$

Also, we define

$$H(\mathrm{rot}\,,\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \mathrm{rot}\,\mathbf{v} \in L^2(\Omega)\}$$

and

$$H_0(\mathrm{rot}\,,\Omega) = \{\mathbf{v} \in H(\mathrm{rot}\,,\Omega) \mid \mathrm{rot}\,\mathbf{v} = 0\}.$$

In order to analyze the convergence of the discrete eigensolutions to the continuous one, it is necessary to show that the following assumptions, called the discrete compactness property, are satisfied on the finite element space[5, 6, 7]:
(**Assumption 1**) There exists constant $c > 0$, independent of h, such that

$$(\operatorname{div}\mathbf{w}_h, \operatorname{div}\mathbf{w}_h) \geq c \| \mathbf{w}_h \|_{\mathrm{div}}^2,\ \forall \mathbf{w}_h \in \mathbb{K}_h.$$

(**Assumption 2**) For every $p \in H_0^1(\Omega)$,

$$\sup_{\mathbf{w}_h \in \mathbb{K}_h} \frac{(\mathbf{w}_h, \mathbf{rot}\ p)}{\| \mathbf{w}_h \|_{\mathrm{div}}} \leq Ch \| p \|_1 .$$

(**Assumption 3**) For every $\mathbf{u} \in \mathbb{K}$, there exists $\mathbf{u}_h \in \mathbb{K}_h$ such that

$$\| \mathbf{u} - \mathbf{u}_h \|_{\mathrm{div}} \leq Ch(\| \mathbf{u} \|_1 + \| \operatorname{div} \mathbf{u} \|_1).$$

To prove the Assumption 1, we will first show that the following commutativity holds:

Lemma 4.1. *The space $\widehat{Q}_{1,1}$ and $\widehat{\mathrm{MRT}}$ and their corresponding spaces defined on the element K satisfy $\mathbf{rot}(q) = P_{F_K}(\widehat{\mathbf{rot}}\ \hat{q})$.*

$$\begin{array}{ccc} \widehat{Q}_{1,1} & \xrightarrow{\widehat{\mathbf{rot}}} & \widehat{\mathrm{MRT}} \\ \downarrow & & \downarrow {\scriptstyle P_{F_K}} \\ Q_{1,1} & \xrightarrow[\mathbf{rot}]{} & \mathrm{MRT} \end{array}$$

Proof. Let $\hat{q} = a + b\hat{x} + c\hat{y} + d\hat{x}\hat{y}$ be any element in $\widehat{Q}_{1,1}$. Then

$$\widehat{\mathbf{rot}}\ \hat{q} = \begin{pmatrix} c + d\hat{x} \\ -b - d\hat{y} \end{pmatrix} \in \widehat{\mathrm{RT}} \subseteq \widehat{\mathrm{MRT}}.$$

Set $\widehat{\mathbf{u}} = \widehat{\mathbf{rot}}\ \hat{q}$. We will show that $\mathbf{rot}\ q = P_{F_K}(\widehat{\mathbf{u}})$. By definition of the bilinear map F_K, we have $\hat{q}(F_K^{-1}(x, y)) = q(x, y)$. Then

$$\mathbf{rot}\ q = \begin{pmatrix} \hat{q}_{\hat{x}} \frac{\partial \hat{x}}{\partial y} + \hat{q}_{\hat{y}} \frac{\partial \hat{y}}{\partial y} \\ -\hat{q}_{\hat{x}} \frac{\partial \hat{x}}{\partial x} - \hat{q}_{\hat{y}} \frac{\partial \hat{y}}{\partial x} \end{pmatrix}.$$

Since $\mathbf{u} = P_{F_K} \widehat{\mathbf{u}} = \frac{DF_K}{J_K} \widehat{\mathbf{u}} \circ F_K^{-1}$ by the Piola transformation,

$$\begin{aligned} P_{F_K} \widehat{\mathbf{u}} &= \begin{pmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{pmatrix} \begin{pmatrix} \hat{q}_{\hat{y}} \\ -\hat{q}_{\hat{x}} \end{pmatrix} J_K^{-1} = \begin{pmatrix} \frac{\partial \hat{y}}{\partial y} & -\frac{\partial \hat{x}}{\partial y} \\ -\frac{\partial \hat{y}}{\partial x} & \frac{\partial \hat{x}}{\partial x} \end{pmatrix} \begin{pmatrix} \hat{q}_{\hat{y}} \\ -\hat{q}_{\hat{x}} \end{pmatrix} = \begin{pmatrix} \hat{q}_{\hat{x}} \frac{\partial \hat{x}}{\partial y} + \hat{q}_{\hat{y}} \frac{\partial \hat{y}}{\partial y} \\ -\hat{q}_{\hat{x}} \frac{\partial \hat{x}}{\partial x} - \hat{q}_{\hat{y}} \frac{\partial \hat{y}}{\partial x} \end{pmatrix} \\ &= \mathbf{rot}\ q. \end{aligned}$$

□

Lemma 4.2. *Let $\mathbf{v}_h \in \mathrm{MRT}$ such that $\operatorname{div} \mathbf{v}_h = 0$. Then there exists $q_h \in (Q_{1,1} \cap H_0^1)/\mathbb{R}$ such that $\mathbf{v}_h = \mathbf{rot}\ q_h$.*

Proof. Since $\operatorname{div} \mathbf{v}_h = \frac{b+d}{|K|} = 0$, we have $\widehat{\operatorname{div}}\, \widehat{\mathbf{v}}_h = \frac{b+d}{|K|} J_K = 0$. Then $\widehat{\mathbf{v}}_h = \widehat{\mathbf{rot}}\ \hat{q}_h$ for $\hat{q}_h \in H_0^1(\widehat{K})/\mathbb{R}$ by Helmholtz decomposition. Let

$$\widehat{\mathbf{rot}}\ \hat{q}_h = \begin{pmatrix} (\hat{q}_h)_{\hat{y}} \\ -(\hat{q}_h)_{\hat{x}} \end{pmatrix} = \begin{pmatrix} \hat{v}_h^1 \\ \hat{v}_h^2 \end{pmatrix} = \widehat{\mathbf{v}}_h.$$

Since $\widehat{\mathbf{v}}_h \in \widehat{MRT}_0$,

$$\hat{v}_h^1 = a + b\hat{x} + (b+d)\frac{\beta}{2|K|}\hat{x}(\hat{x}-1) = (\hat{q}_h)_{\hat{y}} \tag{7}$$

and

$$\hat{v}_h^2 = c + d\hat{y} + (b+d)\frac{\gamma}{2|K|}\hat{y}(\hat{y}-1) = -(\hat{q}_h)_{\hat{x}} \tag{8}$$

Integrate both side of (7) with respect to $\hat{y}$, then

$$\hat{q}_h = \phi(\hat{x}) + a\hat{y} + b\hat{x}\hat{y} + (b+d)\frac{\beta}{2|K|}\hat{x}(\hat{x}-1)\hat{y}. \tag{9}$$

Differentiate both side of (9) with respect to $\hat{x}$, then

$$(\hat{q}_h)_{\hat{x}} = \phi'(\hat{x}) + b\hat{y} + (b+d)\frac{\beta}{2|K|}(2\hat{x}-1)\hat{y} \tag{10}$$

By making equal the corresponding coefficients in the two expressions (8) and (10), we know that $b+d=0$ and $\phi(\hat{x}) \in P_1(\widehat{K})$. Therefore, $\hat{q}_h = s + t\hat{x} + a\hat{y} + b\hat{x}\hat{y} \in Q_{1,1}(\widehat{K})$ and $\hat{q}_h \in (Q_{1,1}(\widehat{K}) \cap H_0^1(\widehat{K}))/\mathbb{R}$. By Lemma 4.1, we have $\mathbf{v}_h = \mathbf{rot}\ q_h$ for $q_h \in (Q_{1,1} \cap H_0^1)/\mathbb{R}$. □

We consider the following Laplace mixed problem with datum $-\mathrm{div}\,\mathbf{w}_h$: find $(\mathbf{u}, p) \in H_0(\mathrm{div}\,,\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\mathbf{u},\mathbf{v}) + (\mathrm{div}\,\mathbf{v}, p) &= 0, & \forall \mathbf{v} \in H_0(\mathrm{div}\,,\Omega), \\ (\mathrm{div}\,\mathbf{u}, q) &= (\mathrm{div}\,\mathbf{w}_h, q), & \forall q \in L^2(\Omega). \end{cases} \tag{11}$$

Let $\mathbf{V}_h = \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega)$ and $W_h = \{q_h \mid q_h|_K = \hat{q}_h,\ \forall \hat{q}_h \in \widehat{\mathrm{div}}\,(\widehat{\mathrm{MRT}})\}$. Then we have the following finite dimensional problem corresponding (11) : find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{cases} (\mathbf{u}_h,\mathbf{v}_h) + (\mathrm{div}\,\mathbf{v}_h, p_h) &= 0, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathrm{div}\,\mathbf{u}_h, q_h) &= (\mathrm{div}\,\mathbf{w}_h, q_h), & \forall q_h \in W_h. \end{cases} \tag{12}$$

It is well-known that the problem (11) and (12) are well-posed and stable[12]. Using the problem (12) and Lemma 4.2, we will show the following discrete Helmholtz decomposition.

Lemma 4.3. *Let* $\mathbf{w}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega)$. *Then there exist unique* $\mathbf{u}_h \in \mathbb{K}_h$ *and* $r_h \in (Q_{1,1} \cap H_0^1(\Omega))/\mathbb{R}$ *such that* $\mathbf{w}_h = \mathbf{u}_h + \mathbf{rot}\ r_h$.

Proof. Let $\mathbf{u}_h$ be the first component of the solution of the problem (12). From second equation of (12), we have

$$(\mathrm{div}\,(\mathbf{w}_h - \mathbf{u}_h), q_h) = 0,\ \forall q_h \in W_h.$$

Let

$$\mathbb{K}_{h,0}^{\mathrm{div}} = \{\mathbf{v}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\mathrm{div}\,,\Omega) \mid (\mathrm{div}\,\mathbf{v}_\mathrm{h}, \mathrm{q}_\mathrm{h}) = 0,\ \forall \mathrm{q}_\mathrm{h} \in \mathrm{W}_\mathrm{h}\}$$

and

$$\mathbb{K}_0^{\mathrm{div}} = \{\mathbf{v}_h \in H_0(\mathrm{div}\,,\Omega) \mid \mathrm{div}\,\mathbf{v} = 0\}.$$

Then $\mathbf{w}_h - \mathbf{u}_h \in \mathbb{K}_{h,0}^{\mathrm{div}} \subseteq \mathbb{K}_0^{\mathrm{div}}$ and hence $\mathrm{div}\,(\mathbf{w}_h - \mathbf{u}_h) = 0$. From Lemma 4.2, we know that there exists $r_h \in (Q_{1,1} \cap H_0^1)/\mathbb{R}$ such that $\mathbf{w}_h - \mathbf{u}_h = \mathbf{rot}\ r_h$. Set $\mathbf{v}_h = \mathbf{rot}\ r_h$ in the first equation of (12). Then $(\mathbf{u}_h, \mathbf{rot}\ r_h) = 0$, since $\mathrm{div}\,(\mathbf{rot}\ r_h) = 0$. Therefore, $\mathbf{u}_h \in \mathbb{K}_h$. Since the decomposition of $\mathbf{w}_h$ is an orthonormal decomposition, we obtain the desired result. □

Theorem 4.4. *There exists $c > 0$, independent of h, such that*

$$(\mathrm{div}\,\mathbf{w}_h, \mathrm{div}\,\mathbf{w}_h) \geq c \parallel \mathbf{w}_h \parallel_{\mathrm{div}}^2, \ \forall \mathbf{w}_h \in \mathbb{K}_h.$$

Proof. Since $\mathbf{w}_h \in \mathbb{K}_h$, the decomposition of $\mathbf{w}_h = \mathbf{u}_h$ from Lemma 4.3. Hence by the stability of solution of (12), we have

$$\parallel \mathbf{u}_h \parallel_0 \leq C \parallel \mathrm{div}\,\mathbf{u}_h \parallel_0 .$$

So we have the proof. □

To show the Assumption 2, weak approximation, we will use the following result.

Lemma 4.5. *For $\mathbf{w}_h \in \mathbb{K}_h$, there exist $\mathbf{w} \in \mathbb{K}$ such that*

$$\parallel \mathbf{w}_h - \mathbf{w} \parallel_0 \leq Ch \parallel \mathbf{w}_h \parallel_{\mathrm{div}} .$$

Proof. Let $\mathbf{u}_h$ be the first component of the solution of problem (12). Then $\mathbf{u}_h \in \mathbb{K}_h$, from the proof of Lemma 4.3. And we also have $\mathrm{div}\,(\mathbf{u}_h - \mathbf{w}_h) = 0$ for $\mathbf{w}_h \in \mathbb{K}_h$. From Lemma 4.2, there exists $q_h \in (Q_{1,1} \cap H_0^1)/\mathbb{R}$ such that $\mathbf{u}_h - \mathbf{w}_h = \mathbf{rot}\ q_h$. Since $\mathbf{u}_h, \mathbf{w}_h \in \mathbb{K}_h$,

$$(\mathbf{u}_h - \mathbf{w}_h, \mathbf{u}_h - \mathbf{w}_h) = (\mathbf{u}_h - \mathbf{w}_h, \mathbf{rot}\ q_h) = 0.$$

Therefore, by the stability of problem (12), $\mathbf{w}_h$ solves the problem (12). Now, we define $\mathbf{w}$ as the first component of the solution of the corresponding continuous problem (11). Set $\mathbf{v} = \mathbf{rot}\ r$ for $r \in H_0^1(\Omega)$ in the first equation of (11). Then $(\mathbf{w}, \mathbf{rot}\ r) = 0$. Hence, $\mathbf{w} \in \mathbb{K}$. From the estimate for the regularity of the solution, we already know that $\parallel \mathbf{w} - \mathbf{w}_h \parallel_0 \leq Ch \mid \mathbf{w} \mid_1$. Therefore,

$$\parallel \mathbf{w} - \mathbf{w}_h \parallel_0 \leq Ch \parallel \mathrm{div}\,\mathbf{w}_h \parallel_0 \leq Ch \parallel \mathbf{w}_h \parallel_{\mathrm{div}} .$$

□

Theorem 4.6. *For all $p \in H_0^1(\Omega)$, we have*

$$\sup_{\mathbf{w}_h \in \mathbb{K}_h} \frac{(\mathbf{w}_h, \mathbf{rot}\ p)}{\parallel \mathbf{w}_h \parallel_{\mathrm{div}}} \leq Ch \parallel p \parallel_1 .$$

Proof. Let $\mathbf{w}_h \in \mathbb{K}_h$. From Lemma 4.5, we know that there exists $\mathbf{w} \in \mathbb{K}$ such that $\parallel \mathbf{w}_h - \mathbf{w} \parallel_0 \leq Ch \parallel \mathbf{w}_h \parallel_{\mathrm{div}}$. Since $\mathbf{w} \in \mathbb{K}$, we have

$$(\mathbf{w}_h, \mathbf{rot}\ p) = (\mathbf{w}_h - \mathbf{w}, \mathbf{rot}\ p).$$

Therefore,

$$\sup_{\mathbf{w}_h \in \mathbb{K}_h} \frac{(\mathbf{w}_h, \mathbf{rot}\ p)}{\parallel \mathbf{w}_h \parallel_{\mathrm{div}}} = \sup_{\mathbf{w}_h \in \mathbb{K}_h} \frac{(\mathbf{w}_h - \mathbf{w}, \mathbf{rot}\ p)}{\parallel \mathbf{w}_h \parallel_{\mathrm{div}}}$$

$$\begin{aligned} &\leq \sup_{\mathbf{w}_h \in \mathbb{K}_h} \frac{\| \mathbf{w}_h - \mathbf{w} \|_0 \| \mathbf{rot}\ p \|_0}{\| \mathbf{w}_h \|_{\mathrm{div}}} \\ &\leq Ch \| \mathbf{rot}\ p \|_0 \\ &\leq Ch \| p \|_1 . \end{aligned}$$

□

Finally, we will show the Assumption 3, the strong approximation.

Theorem 4.7. *For every $\mathbf{u} \in \mathbb{K}$, there exists $\mathbf{u}_h \in \mathbb{K}_h$ such that*

$$\| \mathbf{u} - \mathbf{u}_h \|_{\mathrm{div}} \leq Ch(\| \mathbf{u} \|_1 + \| \operatorname{div} \mathbf{u} \|_1).$$

Proof. First, we consider the following source problem associate with (2) : find $(\mathbf{u}, p) \in H_0(\operatorname{div}, \Omega) \times H_0^1(\Omega)$ such that

$$\left\{ \begin{array}{rcll} (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) + (\mathbf{v}, \mathbf{rot}\ p) & = & (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}), & \forall \mathbf{v} \in H_0(\operatorname{div}, \Omega), \\ (\mathbf{u}, \mathbf{rot}\ q) & = & 0, & \forall q \in H_0^1(\Omega). \end{array} \right. \tag{13}$$

For $\mathbf{u} \in \mathbb{K}$, the second equation of (13) is satisfied. If we take $\mathbf{v} = \mathbf{rot}\ q$, then $p = 0$ and the first equation of (13) is also satisfied. So $(\mathbf{u}, 0)$ be the solution of (13). Then we can define $\mathbf{u}_h$ as the solution of the corresponding discrete problem : find $(\mathbf{u}_h, p_h) \in \mathrm{MRT} \cap \mathrm{H}_0(\operatorname{div}, \Omega) \times \mathrm{Q}_{1,1} \cap \mathrm{H}_0^1(\Omega)$ such that

$$\left\{ \begin{array}{rcll} (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v}_h) + (\mathbf{v}_h, \mathbf{rot}\ p_h) & = & (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\operatorname{div}, \Omega), \\ (\mathbf{u}_h, \mathbf{rot}\ q_h) & = & 0, & \forall q_h \in Q_{1,1} \cap H_0^1(\Omega). \end{array} \right. \tag{14}$$

Then $\mathbf{u}_h \in \mathbb{K}_h$ trivially. From Theorem 4.4 and Theorem 4.6, we know that problem (14) is well-posed. Using the error estimates for MRT, we obtain

$$\begin{aligned} \| \mathbf{u} - \mathbf{u}_h \|_{\mathrm{div}} &\leq C \inf_{\mathbf{v}_h \in \mathrm{MRT} \cap \mathrm{H}_0(\operatorname{div}, \Omega)} \| \mathbf{u} - \mathbf{v}_h \|_{\mathrm{div}} \\ &\leq Ch(| \mathbf{u} |_1 + | \operatorname{div} \mathbf{u} |_1) \\ &\leq Ch(\| \mathbf{u} \|_1 + \| \operatorname{div} \mathbf{u} \|_1 . \end{aligned}$$

□

Let T and T_h be the resolvent operator associated with problem (2) and (6), respectively. Since Assumption 1 - Assumption 3 are verified, we know that the sequences T_h converges uniformly to T. And we have that the eigensolutions of the discrete problem (6) converge to those of the continuous problem (2), from the following theorem [11, 7] .

Theorem 4.8. *Let λ_i be an eigenvalue of problem (2) with multiplicity m_i and E_i be the corresponding eigenspace. Then, exactly m_i discrete eigenvalues $\lambda_{h,1}, \cdots, \lambda_{h,m_i}$ converge to λ_i and*

$$| \lambda_i - \frac{1}{m_i} \sum_{j=1}^{m_i} \lambda_{h,j} | \leq Ch^2.$$

Let $\overline{E_{h,j}}$ be the direct sum of the eigenspaces corresponding to $\lambda_{h,1}, \cdots, \lambda_{h,m_i}$. Then

$$| E_i - \overline{E_{h,j}} | \leq Ch.$$

5. Conclusion

We proved the discrete compactness property related to the Modified Raviart-Thomas space of the lowest order(MRT). The stiffness matrix resulting from MRT has a similar data structure as standard RT space. Also, it has 2 fewer degrees of freedom on each elements than Arnold-Boffi-Falk space of the lowest order(ABF), provides optimal order approximation on quadrilateral meshes. Hence MRT space can be more efficiently applied to electromagnetic eigenvalue problems and Maxwell's eigenproblem for cavity resonator on general quadrilateral meshes than ABF space.

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