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GENERALIZED ABSOLUTE CESÀRO SUMMABILITY OF FACTORED INFINITE SERIES

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Abstract. In this paper, we have proved a general theorem dealing with $\varphi-\mid C,\alpha,\beta\mid_k$ summability factors of infinite series. Also, we have obtained some new and known results related to the different special summability methods.

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1. Introduction

Let $\sum a_n$ be an infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n), that is (see [6])

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \qquad (1)$$

where

$$A_n^{\alpha+\beta}\simeq\frac{n^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)},\quad A_0^{\alpha+\beta}=1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta}=0 \quad \texttt{for} \quad n>0. \qquad (2)$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi-\mid C,\alpha,\beta\mid_k$, $k\geq 1$, if (see [4])

$$\sum_{n=1}^{\infty}\frac{1}{n^k}\mid\varphi_n t_n^{\alpha,\beta}\mid^k<\infty. \qquad (3)$$

In the special case when $\varphi_n=n^{1-\frac{1}{k}}$, $\varphi-\mid C,\alpha,\beta\mid_k$ summability is the same as $\mid C,\alpha,\beta\mid_k$ summability (see [7]). Also, if we take $\varphi_n=n^{\delta+1-\frac{1}{k}}$, then $\varphi-\mid C,\alpha,\beta\mid_k$ summability reduces to $\mid C,\alpha,\beta;\delta\mid_k$ summability (see [5]). If we take $\beta=0$, then we have $\varphi-\mid C,\alpha\mid_k$ summability (see [1]). If we take

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$\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [8]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \delta \mid_k$ summability (see [9]). The following theorem is known dealing with the $\varphi- \mid C, \alpha \mid_k$ summability factors of infinite series.

Theorem 1.1[2] Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{6}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{7}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \left|\varphi_n\right|^k)$ is non increasing and if the sequence (ω_n^α) defined by (see [11])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1\leq v\leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n^k}(|\varphi_n|\, \omega_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, where $k \geq 1$ and $(\alpha + \epsilon) > 1$.

2. Main result

The aim of this paper is to generalize Theorem 1.1 for $\varphi - \mid C, \alpha, \beta \mid_k$ summability method. Now we shall prove the following theorem.

Theorem 2.1 Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (4)-(7) of Theorem 1.1 are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \left|\varphi_n\right|^k)$ is non increasing and if the sequence $(\omega_n^{\alpha,\beta})$ be a sequence defined by (see [3])

$$\omega_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1, \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{10}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n^k}(|\varphi_n|\, \omega_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, where $k \geq 1$, $\beta > -1$, and $(\alpha + \beta)k + \epsilon > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2 [3] If $0<\alpha\leq 1$, $\beta>-1$, and $1\leq v\leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^{\beta}a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^{\beta}a_p \mid . \tag{12}$$

Lemma 2.3 [10] Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem 1.1, the following conditions hold, when (6) is satisfied;

$$n\beta_nX_n = O(1) \quad as \quad n\to\infty \tag{13}$$

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty. \tag{14}$$

3. Proof of Theorem 2.1

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v.$$

Applying Abel's transformation first and then using Lemma 2. 2, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

$$\begin{aligned}\mid T_n^{\alpha,\beta}\mid &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\mid\\ &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\omega_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid\omega_n^{\alpha,\beta}=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.\end{aligned}$$

To complete the proof of Theorem 2.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^k}\mid\varphi_nT_{n,r}^{\alpha,\beta}\mid^k<\infty, \quad \text{for} \quad r=1,2.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}\sum_{n=2}^{m+2}\frac{1}{n^k}\mid\varphi_nT_{n,1}^{\alpha,\beta}\mid^k &\leq \sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha+\beta})^{-k}\mid\varphi_n\mid^k\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}\omega_v^{\alpha,\beta}\mid\Delta\lambda_v\mid\}^k\\ &\leq \sum_{n=2}^{m+1}n^{-k}n^{-(\alpha+\beta)k}\mid\varphi_n\mid^k\sum_{v=1}^{n-1}(A_v^{\alpha+\beta})^k(\omega_v^{\alpha,\beta})^k\beta_v.\{\sum_{v=1}^{n-1}\beta_v\}^{k-1}\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v \sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}\mid\varphi_n\mid^k}{n^{(\alpha+\beta)k+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v v^{\epsilon-k}\mid\varphi_v\mid^k \sum_{n=v+1}^{m+1}\frac{1}{n^{(\alpha+\beta)k+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k v^{\epsilon-k}\mid\varphi_v\mid^k \beta_v\int_v^{\infty}\frac{dx}{x^{(\alpha+\beta)k+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v\beta_v v^{-k}(\omega_v^{\alpha,\beta}\mid\varphi_v\mid)^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v} r^{-k}(\omega_r^{\alpha,\beta}\mid\varphi_r\mid)^k\\
&\quad +O(1)m\beta_m\sum_{v=1}^{m} v^{-k}(\omega_v^{\alpha,\beta}\mid\varphi_v\mid)^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by the hypotheses of Theorem 2.1 and Lemma 2.3. Since, $\mid\lambda_n\mid = O(1)$ by (7), finally we have that

$$
\begin{aligned}
&\sum_{n=1}^{m}\frac{1}{n^k}\mid\varphi_n T_{n,2}^{\alpha,\beta}\mid^k\\
= \quad & O(1)\sum_{n=1}^{m}\mid\lambda_n\mid n^{-k}(\omega_n^{\alpha,\beta}\mid\varphi_n\mid)^k = O(1)\sum_{n=1}^{m-1}\Delta\mid\lambda_n\mid\sum_{v=1}^{n} v^{-k}(\omega_v^{\alpha,\beta}|\varphi_v|)^k\\
&+O(1)|\lambda_m|\sum_{n=1}^{m} n^{-k}(\omega_n^{\alpha,\beta}|\varphi_n|)^k = O(1)\sum_{n=1}^{m-1}\mid\Delta\lambda_n\mid X_n + O(1)\mid\lambda_m\mid X_m\\
= \quad & O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)\mid\lambda_m\mid X_m = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by the hypotheses of Theorem 2.1 and Lemma 2.3. This completes the proof.

4. Conclusions

If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C,\alpha,\beta\mid_k$ summability factors of infinite series. Also, if we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C,\alpha;\delta\mid_k$

summability factors of infinite series. Finally, if we set $\beta = 0$, then we obtain Theorem 1.1.

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