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SOME IDENTITIES FOR MULTIPLE (h, p, q)-HURWITZ-EULER ETA FUNCTION†

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Abstract. In this paper, we construct the multiple (h, p, q)-Hurwitz-Euler eta function by generalizing the multiple Hurwitz-Euler eta function. We get some explicit formulas and properties of the higher-order (h, p, q)-Euler numbers and polynomials.

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1. Introduction

The field of the special functions such as the gamma and beta functions, special polynomials, the hypergeometric functions, the zeta and related functions, q-series, (p, q)-series, and series representations is a ever expanding area in advanced mathematics, applied mathematics, probability, mathematical statistics, and physics. In particular, special polynomials play a fundamental role in applied mathematics, physics, science, and industry(see [1-15]). Choi and Srivastava presented a generalized Hurwitz formula and Hurwitz-Euler eta function(see [5, 6]). It is the purpose of this paper to introduce and investigate a new some generalizations of the (p, q)-Euler numbers and polynomials, (p, q)-Euler zeta function, (p, q)-Hurwiz-Euler zeta function. We call them multiple (h, p, q)-Euler numbers and polynomials, multiple (h, p, q)-Euler zeta function, and multiple (h, p, q)-Hurwitz-Euler eta function. The structure of the paper is as follows: In Sect. 2, we define higher-order (h, p, q)-Euler numbers and polynomials and derive some of their properties involving elementary properties, distribution relation, and so on. In Sect. 3, by using the higher-order (h, p, q)-Euler numbers and polynomials, multiple (h, p, q)-Euler zeta function

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and multiple (h,p,q)-Hurwitz-Euler eta function are defined. We also contains some connection formulae between the higher-order (h,p,q)-Euler polynomials and the multiple (h,p,q)-Hurwitz-Euler eta function.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0,-1,-2,-3,\dots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We use the notation

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty} = \sum_{k_1,\cdots,k_r=0}^{\infty}.$$

We would like to review definitions related to q-number and (p,q)-number used in this paper. For any $m\in\mathbb{N}$, q-number can be defined as follows

$$[m]_q = \frac{1-q^m}{1-q} = \sum_{i=0}^{m-1} q^i = 1+q+q^2+\cdots+q^{m-1}.$$

For $z\in\mathbb{C}$, the (p,q)-number is defined by

$$[z]_{p,q} = \frac{p^z-q^z}{p-q}, (p\neq q).$$

With the (p,q)-number, the necessary elements of the (p,q)-calculus, namely, (p,q)-integration, (p,q)-differentiation, (p,q)-exponential, were worked by many mathematicians. Many (p,q)-extensions of some special functions and polynomials have been studied(see [1, 2, 7, 12, 13, 14, 15]).

The binomial formulae are known as

$$(1-b)^n = \sum_{k=0}^{n}\binom{n}{k}(-b)^k, \text{ where } \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!},$$

and

$$\frac{1}{(1-b)^n} = (1-b)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-b)^k = \sum_{k=0}^{\infty}\binom{n+k-1}{i}b^k.$$

Choi and Srivastava [5] constructed and studied the multiple Hurwitz-Euler eta function $\eta_r(s,a)$ defined by following r-ple series:

$$\eta_r(s,a) = \sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}}{(k_1+\cdots+k_r+a)^s}, \quad (Re(s)>0; a>0; r\in\mathbb{N}).$$

It is known that $\eta_r(s,a)$ can be continued analytically to be whole complex s-plane. Inspired by their work, the (h,p,q)-extension of the multiple Hurwitz-Euler eta function can be defined as follows: For $s,x\in\mathbb{C}$ with $Re(x)>0$ and $r\in\mathbb{N}$, the multiple (h,p,q)-Hurwitz-Euler eta function $\eta_{p,q}^{(r,h)}(s,x)$ is define by

$$\eta_{p,q}^{(r,h)}(s,x) = [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}p^{hk_1+\cdots+hk_r}q^{k_1+\cdots+k_r}}{[k_1+\cdots+k_r+x]_{p,q}^s}.$$

Note that if $p=1, q\to 1$, then $\eta_{p,q}^{(r,h)}(s,a)=2^r\eta_r(s,a)$. Another type of multiple (h,p,q)-Euler zeta function $\zeta_{p,q}^{(r,h)}(s)$ can be defined as follows. For $s\in\mathbb{C}$, we define

$$\zeta_{p,q}^{(r,h)}(s)=[2]_q^r\sum_{m=1}^{\infty}\binom{m+r-1}{m}\frac{(-1)^m p^{hm}q^m}{[m]_{p,q}^s}.$$

Observe that if $r=1$, then $\zeta_{p,q}^{(r,h)}(s)=\zeta_{p,q}^{(h)}(s)$(see [13]). By using the symmetric properties about the multiple (h,p,q)-Hurwitz-Euler eta function, we obtain symmetric identities about the higher-order (h,p,q)-Euler numbers and polynomials. Firstly, we introduce the basic definitions related to higher-order (h,p,q)-Euler numbers and polynomials.

Definition 1.1. The classical Euler polynomials $E_n(x)$ are defined by the following generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (|t|<\pi).$$

When $x=0, E_n=E_n(0)$ are called the Euler numbers E_n.

Definition 1.2. For $r\in\mathbb{N}$, the classical higher-order Euler polynomials $E_n^{(r)}(x)$ are defined by the following generating function:

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},\quad (|t|<\pi).$$

As usual, the numbers $E_n^{(r)}=E_n^{(r)}(0)$ are called higher-order Euler numbers.

Much research has been done in the area of special functions by using (p,q)-number(see [1, 2, 7, 12, 13, 15]). Some interesting properties of the (h,p,q)-Euler numbers $E_{n,p,q}^{(h)}$ polynomials $E_{n,p,q}^{(h)}(x)$ were first investigated by Ryoo [13].

Definition 1.3. For $0<q<p\le 1$ and $h\in\mathbb{Z}$, (h,p,q)-Euler numbers $E_{n,p,q}^{(h)}$ and (h,p,q)-Euler polynomials $E_{n,p,q}^{(h)}(x)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty}E_{n,p,q}^{(h)}\frac{t^n}{n!}=[2]_q\sum_{l=0}^{\infty}(-1)^l p^{hl}q^l e^{[l]_{p,q}t}$$

and

$$\sum_{n=0}^{\infty}E_{n,p,q}^{(h)}(x)\frac{t^n}{n!}=[2]_q\sum_{l=0}^{\infty}(-1)^l p^{hl}q^l e^{[l+x]_{p,q}t}$$

respectively.

2. Higher-order (h, p, q)-Euler numbers and polynomials

In this section, we consider the higher-order (h, p, q)-Euler numbers and polynomials as follows:

Definition 2.1. For $0 < q < p \leq 1$, $h \in \mathbb{Z}$, and $r \in \mathbb{N}$, higher-order (h, p, q)-Euler numbers $E_{n,p,q}^{(r,h)}$ and higher-order (h, p, q)-Euler polynomials $E_{n,p,q}^{(r,h)}(x)$ are defined by the following generating functions

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)}(x)\frac{t^n}{n!} = [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{h(k_1+\cdots+k_r)} e^{[k_1+\cdots+k_r+x]_{p,q}t}, \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)}\frac{t^n}{n!} = [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{h(k_1+\cdots+k_r)} e^{[k_1+\cdots+k_r]_{p,q}t}, \tag{2}$$

respectively.

Note that if $r = 1$, then $E_{n,p,q}^{(r,h)} = E_{n,p,q}^{(h)}$ and $E_{n,p,q}^{(r,h)}(x) = E_{n,p,q}^{(h)}(x)$. Observe that if $p = 1, q \to 1$, then $E_{n,p,q}^{(r,h)} \to E_n^{(r)}$ and $E_{n,p,q}^{(r,h)}(x) \to E_n^{(r)}(x)$.

From (1) and (2), we note that

Theorem 2.2. *For $0 < q < p \leq 1$, $h \in \mathbb{Z}$, and $r \in \mathbb{N}$, we have*

$$\begin{aligned} E_{n,p,q}^{(r,h)}(x+y) &= \sum_{l=0}^{n} \binom{n}{l} p^{lx} q^{y(n-l)} [y]_{p,q}^{l} E_{n-l,p,q}^{(r,h+l)}(x), \\ E_{n,p,q}^{(r)}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{xl} [x]_{p,q}^{l} E_{n-l,p,q}^{(r,h+l)}. \end{aligned} \tag{3}$$

Theorem 2.3. *For $r \in \mathbb{N}$ and $h \in \mathbb{Z}$, we have*

$$\begin{aligned} E_{n,p,q}^{(r,h)}(x) &= [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} [k_1+\cdots+k_r+x]_{p,q}^n \\ &= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \left(\frac{1}{1+q^{l+1}p^{h+n-l}} \right)^r. \end{aligned}$$

Proof. By the Taylor series expansion of $e^{[x]_{p,q}t}$, we have

$$\begin{aligned} &\sum_{l=0}^{\infty} E_{l,p,q}^{(r)}(x)\frac{t^l}{l!} \\ &= [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r} e^{[k_1+\cdots+k_r+x]_{p,q}t} \\ &= \sum_{l=0}^{\infty} \left([2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{h(k_1+\cdots+k_r)} [k_1+\cdots+k_r+x]_{p,q}^l \right) \frac{t^l}{l!}. \end{aligned}$$

The first part of the theorem follows when we compare the coefficients of $\frac{t^l}{l!}$ in the above equation. By (p,q)-numbers and binomial expansion, we also note that

$$\begin{aligned}
E_{n,p,q}^{(r,h)}(x) &= [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{h(k_1+\cdots+k_r)} [k_1+\cdots+k_r+x]_{p,q}^n \\
&= [2]_q^r \sum_{k_1,\cdots,k_r=0}^{\infty} (-q)^{k_1+\cdots+k_r} p^{h(k_1+\cdots+k_r)} \left(\frac{p^{k_1+\cdots+k_r+x} - q^{k_1+\cdots+k_r+x}}{p-q} \right)^n \\
&= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \\
&\quad \times \sum_{k_1,\cdots,k_r=0}^{\infty} (-1)^{k_1+\cdots+k_r} q^{(l+1)(k_1+\cdots+k_r)} p^{(h+n-l)(k_1+\cdots+k_r)} \\
&= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \left(\frac{1}{1+q^{l+1}p^{h+n-l}} \right)^r .
\end{aligned} \tag{4}$$

This completes the proof of Theorem 2.3. □

Theorem 2.4. *For $r \in \mathbb{N}$, we have*

$$E_{n,p,q}^{(r,h)}(x) = [2]_q^r \sum_{m=0}^{\infty} \binom{r+m-1}{m} (-1)^m q^m p^{hm} [m+x]_{p,q}^n .$$

Proof. By Taylor-Maclaurin series expansion of $(1-a)^{-n}$, we have

$$\left(\frac{1}{1+q^{l+1}p^{n-l+h}} \right)^r = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m (q^{l+1}p^{n-l+h})^m .$$

Also, by (4) and binomial expansion, one can obtain the desired result immediately.□

For $d \in \mathbb{N}$ with $d \equiv 1(\mod 2)$, by Theorem 2.3, we can show

$$\begin{aligned}
E_{n,p,q}^{(r,h)}(x) &= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \\
&\quad \times \sum_{a_1,\cdots,a_r=0}^{d-1} \sum_{k_1,\cdots,k_r=0}^{\infty} (-1)^{a_1+\cdots+a_r} (-1)^{k_1+\cdots+k_r} \\
&\quad \times q^{(l+1)(a_1+dk_1+\cdots+a_r+dk_r)} p^{(n-l+h)(a_1+dk_1+\cdots+a_r+dk_r)} .
\end{aligned}$$

Theorem 2.5. *(Distribution relation of higher-order (h,p,q)-Euler polynomials). For $d \in \mathbb{N}$ with $d \equiv 1(\ mod\ 2)$, we have*

$$E^{(r,h)}_{n,p,q}(x) = \frac{[2]^r_q}{[2]^r_{q^d}}[d]^n_{p,q} \sum_{a_1,\cdots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E^{(r,h)}_{n,p^d,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right).$$

Proof. Since

$$E^{(r,h)}_{n,p^d,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right) = \frac{[2]^r_{q^d}}{(p^d-q^d)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(a_1+\cdots+a_r+x)} p^{(n-l)(a_1+\cdots+a_r+x)} \left(\frac{1}{1+q^{d(l+1)}p^{d(n-l+h)}}\right)^r,$$

we have

$$\begin{aligned}&\sum_{a_1,\cdots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E^{(r,h)}_{n,p^d,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\\ &= \frac{[2]^r_{q^d}}{(p^d-q^d)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} p^{(n-l)x}\\ &\times \sum_{a_1,\cdots,a_r=0}^{d-1} (-1)^{a_1+\cdots+a_r} q^{(l+1)(a_1+\cdots+a_r)} p^{(n-l+h)(a_1+\cdots+a_r)} \left(\frac{1}{1+q^{d(l+1)}p^{d(n-l+h)}}\right)^r.\end{aligned} \tag{5}$$

Hence, by (5) and Theorem 2.3, we have

$$\begin{aligned}&\frac{[2]^r_q}{[2]^r_{q^d}}[d]^n_{p,q} \sum_{a_1,\cdots,a_r=0}^{d-1} (-q)^{a_1+\cdots+a_r} p^{ha_1+\cdots+ha_r} E^{(r,h)}_{n,p^d,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\\ &= \frac{[2]^r_q}{(p-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \left(\frac{1}{1+q^{l+1}p^{n-l+h}}\right)^r.\end{aligned}$$

This completes the proof of Theorem 2.5. □

3. Multiple (h,p,q)-Hurwitz-Euler eta function

In this section, we define multiple (h,p,q)-Hurwitz-Euler eta function. This function interpolates the higher-order (h,p,q)-Euler polynomials at negative integers.

Choi and Srivastava [5] defined the multiple Hurwitz-Euler eta function $\eta_r(s,a)$ by means of

$$\eta_r(s,a) = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{(k_1+\cdots+k_r+a)^s}, \quad (Re(s)>0; a>0; r\in\mathbb{N}).$$

It is known that $\eta_r(s,a)$ can be continued analytically to be whole complex s-plane(see [5]). The (h,p,q)-extension of the multiple Hurwitz-Euler eta function can be defined as follows:

Definition 3.1. For $s,x\in\mathbb{C}$ with $Re(x)>0$, the multiple (h,p,q)-Hurwitz-Euler eta function $\eta_{p,q}^{(r,h)}(s,x)$ is define by

$$\eta_{p,q}^{(r,h)}(s,x)=[2]_q^r\sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}p^{h(k_1+\cdots+k_r)}q^{k_1+\cdots+k_r}}{[k_1+\cdots+k_r+x]_{p,q}^s}. \tag{6}$$

Observe that if $p=1,q\to 1$, then $2^r\eta_{p,q}^{(r,h)}(s,a)=\eta_r(s,a)$. Let

$$\begin{aligned}F_{p,q}^{(r,h)}(t,x)&=\sum_{n=0}^{\infty}E_{n,p,q}^{(r,h)}(x)\frac{t^n}{n!}\\&=[2]_q^r\sum_{k_1,\cdots,k_r=0}^{\infty}(-1)^{k_1+\cdots+k_r}p^{h(k_1+\cdots+k_r)}q^{k_1+\cdots+k_r}e^{[k_1+\cdots+k_r+x]_{p,q}t}.\end{aligned} \tag{7}$$

Theorem 3.2. *For $r\in\mathbb{N}$, we have*

$$\eta_{p,q}^{(r,h)}(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty F_{p,q}^{(r,h)}(x,-t)t^{s-1}dt, \tag{8}$$

where $\Gamma(s)=\int_0^\infty z^{s-1}e^{-z}dz$.

Proof. From (7) and Definition 3.1, we get

$$\begin{aligned}&\eta_{p,q}^{(r,h)}(s,x)\\&=[2]_q^r\sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}p^{h(k_1+\cdots+k_r)}q^{k_1+\cdots+k_r}}{[k_1+\cdots+k_r+x]_{p,q}^s}\\&=[2]_q^r\frac{1}{\Gamma(s)}\sum_{k_1,\cdots,k_r=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_r}p^{h(k_1+\cdots+k_r)}q^{k_1+\cdots+k_r}}{[k_1+\cdots+k_r+x]_{p,q}^s}\int_0^\infty z^{s-1}e^{-z}dz\\&=\frac{[2]_q^r}{\Gamma(s)}\sum_{k_1,\cdots,k_r=0}^{\infty}(-q)^{k_1+\cdots+k_r}p^{h(k_1+\cdots+k_r)}\int_0^\infty e^{[k_1+\cdots+k_r+x]_{p,q}t}t^{s-1}dt\\&=\frac{1}{\Gamma(s)}\int_0^\infty F_{p,q}^{(r,h)}(x,-t)t^{s-1}dt.\end{aligned}$$

This completes the proof of Theorem 3.2. □

The value of multiple (h,p,q)-Hurwitz-Euler eta function $\eta_{p,q}^{(r,h)}(s,x)$ at negative integers is given explicitly by the following theorem:

Theorem 3.3. *Let $n\in\mathbb{N}$. Then we obtain*

$$\eta_{p,q}^{(r,h)}(-n,x)=E_{n,p,q}^{(r,h)}(x).$$

Proof. Again, by (7) and (8), we have

$$\begin{aligned}
&\eta_{p,q}^{(r,h)}(s,x)\\
&=\frac{1}{\Gamma(s)}\int_0^\infty F_{p,q}^{(r,h)}(x,-t)t^{s-1}dt\\
&=\frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}E_{m,p,q}^{(r,h)}(x)\frac{(-1)^m}{m!}\int_0^\infty t^{m+s-1}dt.
\end{aligned}\tag{9}$$

We note that

$$\begin{aligned}
\Gamma(-n)&=\int_0^\infty e^{-z}z^{-n-1}dz\\
&=\lim_{z\to 0}2\pi i\frac{1}{n!}\left(\frac{d}{dz}\right)^n\left(z^{n+1}e^{-z}z^{-n-1}\right)\\
&=2\pi i\frac{(-1)^n}{n!}.
\end{aligned}\tag{10}$$

For $n\in\mathbb{N}$, let us take $s=-n$ in (8). Then, by (9), (10), and Cauchy residue theorem, we have

$$\begin{aligned}
\eta_{p,q}^{(r,h)}(-n,x)&=\lim_{s\to -n}\frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}E_{m,p,q}^{(r,h)}(x)\frac{(-1)^m}{m!}\int_0^\infty t^{m-n-1}dt\\
&=2\pi i\left(\lim_{s\to -n}\frac{1}{\Gamma(s)}\right)\left(E_{n,p,q}^{(r,h)}(x)\frac{(-1)^n}{n!}\right)\\
&=2\pi i\left(\frac{1}{2\pi i\frac{(-1)^n}{n!}}\right)\left(E_{n,p,q}^{(r,h)}(x)\frac{(-1)^n}{n!}\right)\\
&=E_{n,p,q}^{(r,h)}(x).
\end{aligned}$$

This completes the proof of Theorem 3.3. □

Let

$$\begin{aligned}
F_{p,q}^{(r,h)}(t)&=\sum_{l=0}^{\infty}E_{l,p,q}^{(r,h)}\frac{t^l}{l!}\\
&=[2]_q^r\sum_{k_1,\cdots,k_r=0}^{\infty}(-1)^{k_1+\cdots+k_r}p^{hk_1+\cdots+hk_r}q^{k_1+\cdots+k_r}e^{[k_1+\cdots+k_r]_{p,q}t}.
\end{aligned}\tag{11}$$

By the l-th differentiation on both side of (11) at $t=0$, we obtain the following

$$\begin{aligned}
&\frac{d^l}{dt^l}F_{p,q}^{(r,h)}(t)\Big|_{t=0}\\
&=[2]_q^r\sum_{k_1,\cdots,k_r=0}^{\infty}(-1)^{k_1+\cdots+k_r}p^{hk_1+\cdots+hk_r}q^{k_1+\cdots+k_r}[k_1+\cdots+k_r]_{p,q}^l\\
&=E_{l,p,q}^{(r,h)},\ (l\in\mathbb{N}).
\end{aligned}\tag{12}$$

By using the above equation, we are now ready to define multiple (h,p,q)-Euler eta function. We define multiple (h,p,q)-Euler eta function as follows:

Definition 3.4. For $s \in \mathbb{C}$, we define

$$\eta_{p,q}^{(r,h)}(s) = [2]_q^r \sum_{k_1,\cdots,k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r}}{[k_1+\cdots+k_r]_{p,q}^s}.$$

Relation between $\zeta_{p,q}^{(r,h)}(s)$ and $= E_{n,p,q}^{(r,h)}$ is given by the following theorem.

Theorem 3.5. *Let $n \in \mathbb{N}$, We have*

$$\zeta_{p,q}^{(r,h)}(-n) = E_{n,p,q}^{(r,h)}.$$

By (4), we have

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)} \frac{t^n}{n!} = [2]_q^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^m p^{hm} e^{[m]_{p,q}t}.$$

By using Taylor series of $e^{[m]_{p,q}t}$ in the above, we have

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(r,h)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left([2]_q^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^m q^{hm} [m]_{p,q}^n \right) \frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have

$$E_{n,p,q}^{(r,h)} = [2]_q^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^m q^{hm} [m]_{p,q}^n. \tag{13}$$

By using (13), another type of multiple (h,p,q)-Euler zeta function can be defined as follows.

Definition 3.6. For $s \in \mathbb{C}$, we define

$$\zeta_{p,q}^{(r,h)}(s) = [2]_q^r \sum_{m=1}^{\infty} \binom{m+r-1}{m} \frac{(-1)^m p^{hm} q^m}{[m]_{p,q}^s}. \tag{14}$$

The function $\zeta_{p,q}^{(h)}(s)$ interpolates the number $E_{n,p,q}^{(r,h)}$ at negative integers. Substituting $s = -n$ with $n \in \mathbb{N}$ into (14), and using (13), we obtain the following theorem and corollary:

Theorem 3.7. *Let $l \in \mathbb{N}$. We have*

$$\eta_{p,q}^{(r,h)}(-l) = \zeta_{p,q}^{(r,h)}(-l) = E_{l,p,q}^{(r,h)}.$$

Corollary 3.8. *For $0 < q < p \leq 1$, $h \in \mathbb{Z}$, $r \in \mathbb{N}$, and $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{\infty} \binom{m+r-1}{m} (-1)^m p^{hm} q^m [m]_{p,q}^n$$
$$= \sum_{k_1,\cdots,k_r=1}^{\infty} (-1)^{k_1+\cdots+k_r} p^{hk_1+\cdots+hk_r} q^{k_1+\cdots+k_r} [k_1+\cdots+k_r]_{p,q}^n.$$

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