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THE FORMS AND PROPERTIES OF DIFFERENTIAL EQUATIONS OF HIGHER ORDER FOR q-TANGENT POLYNOMIALS

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ABSTRACT. We find several q-differential equations of higher order that has q-tangent polynomials as the solution and obtain its associated symmetric properties.

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1. Introduction

One of the differential equations that can convert nonlinear equations into linear equations is the Bernoulli differential equation. A Bernoulli differential equation is an equation of the form

$$\frac{dy}{dx}+p(x)y-g(x)y^{m}=0, \tag{1.1}$$

where m is any real number, $p(x)$ and $g(x)$ are continuous functions on the interval. If $m=0$ or $m=1$, the above equation is linear, and if not, the equation is nonlinear. The Bernoulli differential equation can be reduced to a linear differential equation with substitution $u=y^{1-m}$. Then for u we obtain a linear equation $\frac{du}{dx}+(1-m)p(x)u=(1-m)g(x)$. This Bernoulli differential equation has many application to problems modeled by nonlinear differential equations, equations about the population expressed in logistic equations or Verhulst equations, physics and so on.

If $m=0$ in (1.1), then the Bernoulli differential equation has the solution which is a generating function of the tangent polynomials. The equation is as follows.

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$$\frac{d}{dx}T_n(x)+\frac{1}{2}T_n(x)+\frac{1}{2}T_0(x)-x^n=0 \tag{1.2}$$

where $T_n(x)$ is the tangent polynomials, see [8].

The tangent numbers and polynomials can be expressed as

$$\sum_{n=0}^{\infty}T_n\frac{t^n}{n!}=\frac{2}{e^{2t}+1},\quad \sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}=\frac{2}{e^{2t}+1}e^{tx},\quad \text{respectively.}$$

Based on the concept above, we can consider the q-Bernoulli differential equation of the first order $D_qy+p(x)y-g(x)y^m=0$ in q-calculus. When $m=0$ in (1.1), the q-tangent polynomials is a solution of the following q-differential equation of the first order.

$$D_{q,x}^{(1)}T_{n,q}(x)+2^{-1}(T_{0,q}(x)+T_{n,q}(x))-x^n=0, \tag{1.3}$$

where D_q is the derivative in q-calculus and $T_{n,q}(x)$ is the q-tangent polynomials. For $e_q(2t)\neq -1$, the q-tangent numbers and polynomials can be expressed as

$$\sum_{n=0}^{\infty}T_{n,q}\frac{t^n}{[n]_q!}=\frac{2}{e_q(2t)+1},\quad \sum_{n=0}^{\infty}T_{n,q}(x)\frac{t^n}{[n]_q!}=\frac{2}{e_q(2t)+1}e_q(tx),\quad \text{respectively.}$$

We note that (1.3) becomes (1.2) when $q\to 1$.

The aim of this paper is to find out the form of differential equations of higher order for q-tangent polynomials through the equation in (1.3). To obtain the above aim, we briefly review several concepts of q-calculus which we need for this paper.

Let $n,q\in\mathbb{R}$ with $q\neq 1$. The number

$$[n]_q=\frac{1-q^n}{1-q}$$

is called q-number, see [1], [2]. We note that $\lim_{q\to1}[n]_q=n$. In particular, for $k\in\mathbb{Z}$, $[k]_q$ is called q-integer.

The q-Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \\ r\end{bmatrix}_q=\frac{[m]_q!}{[m-r]_q![r]_q!},$$

where m and r are non-negative integers, see [5]. For $r=0$, the value is 1 since the numerator and the denominator are both empty products. One notes $[n]_q!=[n]_q[n-1]_q\cdots[2]_q[1]_q$ and $[0]_q!=1$.

Definition 1.1. Let z be any complex numbers with $|z|<1$. Two forms of q-exponential functions can be expressed as

$$e_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!},\qquad E_q(z)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{z^n}{[n]_q!}.$$

We note that $\lim_{q\to1}e_q(z)=e^z$, see [1], [4].

Theorem 1.2. *From Definition 1.1, we note that*

$$\text{(i)}\quad e_q(x)e_q(y) = e_q(x+y), \quad \text{if} \quad yx = qxy.$$
$$\text{(ii)}\quad e_q(x)E_q(-x) = 1.$$
$$\text{(iii)}\quad e_{q^{-1}}(x) = E_q(x).$$

From the result of using the two concepts of q-exponential functions, new types of Bernoulli, Euler, and Genocchi polynomials are appeared and many mathematicians have studied their properties and identities, see [3], [6]-[9]. By using computer, this topic is studied in various research way. The generating functions of q-Euler polynomials used in this paper can be confirmed in definitions 1.3.

Definition 1.3. The generating function for the q-Euler numbers and polynomials are

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}, \quad \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}e_q(tx), \quad \text{respectively.}$$

Let $q \to 1$ in Definition 1.3. Then, we can find the Euler numbers and polynomials as

$$\sum_{n=0}^{\infty}\mathcal{E}_{n}\frac{t^n}{n!} = \frac{2}{e^t+1}, \quad \sum_{n=0}^{\infty}\mathcal{E}_{n}(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{tx}, \qquad |t| < \pi.$$

Definition 1.4. The q-derivative of a function f with respect to x is defined by

$$D_{q,x}f(x) := D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}, \qquad \text{for} \quad x \neq 0,$$

and $D_qf(0) = f'(0)$.

We can prove that f is differentiable at zero, and it is clear that $D_qx^n = [n]_qx^{n-1}$. From Definition 1.4, we can see some formulae for q-derivative.

Theorem 1.5. *From Definition 1.4, we note that*

$$\begin{aligned}\text{(i)}\quad D_q(f(x)g(x)) &= q(x)D_qf(x) + f(qx)D_qg(x)\\ &= f(x)D_qg(x) + g(qx)D_qf(x),\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad D_q\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)}\\ &= \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)},\end{aligned}$$

(iii) for any constants a and b,

$$D_q(af(x)+bg(x)) = aD_qf(x) + bD_qg(x).$$

Based on the previous content, our purpose is to find various q-differential equations of higher order that contain q-tangent polynomials as solution of the q-differential equation of higher order. In Section 2, we find q-differential equations of higher order that has q-tangent polynomials as the solution and check its associated symmetric properties.

2. Main results

In this section, we find some basic q-differential equations of higher order of q-tangent polynomials using q-tangent numbers and polynomials. Moreover, we introduce a special q-differential equation of higher order which is related to a symmetric property for q-tangent polynomials.

Lemma 2.1. *For $0 < q < 1$, we have*

$$\text{(i)} \quad T_{n-k,q}(x) = \frac{[n-k]_q!}{[n]_q!} D_{q,x}^{(k)} T_{n,q}(x),$$

$$\text{(ii)} \quad T_{n-k,q}(q^{-1}x) = \frac{q^k [n-k]_q!}{[n]_q!} D_{q,x}^{(k)} T_{n,q}(q^{-1}x).$$

Proof. (i) We will show the proof using mathematical induction. Applying q-derivative in q-tangent polynomials, we find

$$D_{q,x}^{(1)} \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(2t)+1} D_{q,x}^{(1)} e_q(tx) = \sum_{n=0}^{\infty} [n]_q T_{n-1,q}(x) \frac{t^n}{[n]_q!}. \tag{2.1}$$

From the Equation (2.1), we obtain a relation such as

$$D_{q,x}^{(1)} T_{n,q}(x) = [n]_q T_{n-1,q}(x).$$

In a similar method, we have

$$D_{q,x}^{(2)} T_{n,q}(x) = [n]_q [n-1]_q T_{n-2,q}(x).$$

Therefore, we can find a relation as

$$D_{q,x}^{(k)} T_{n,q}(x) = [n]_q [n-1]_q \cdots [n-(k-1)]_q T_{n-k,q}(x),$$

which is the desired result.
(ii) We omit the proof of Lemma 2.1.(ii) because we can derive the required result if we use a similar method as the proof in Lemma 2.1.(i).

□

Theorem 2.2. *The q-tangent polynomials $T_{n,q}(x)$ is a solution of the following q-differential equation of higher order.*

$$\begin{aligned}
&\frac{2^{n-1}}{[n]_q!} D_{q,x}^{(n)} T_{n,q}(x) + \frac{2^{n-2}}{[n-1]_q!} D_{q,x}^{(n-1)} T_{n,q}(x) + \frac{2^{n-3}}{[n-2]_q!} D_{q,x}^{(n-2)} T_{n,q}(x) + \cdots \\
&+ \frac{2^3}{[4]_q!} D_{q,x}^{(4)} T_{n,q}(x) + \frac{2^2}{[3]_q!} D_{q,x}^{(3)} T_{n,q}(x) + \frac{2}{[2]_q!} D_{q,x}^{(2)} T_{n,q}(x) + D_{q,x}^{(1)} T_{n,q}(x) \\
&+ 2^{-1}(T_{0,q}(x) + T_{n,q}(x)) - x^n = 0.
\end{aligned}$$

Proof. Using q-derivative in q-tangent polynomials, we have

$$\sum_{n=0}^{\infty} T_{n,q}(x)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty} 2^n \frac{t^n}{[n]_q!}+1\right) = 2\sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!}. \tag{2.2}$$

From Equation (2.2), we have

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^k T_{n-k,q}(x) = 2x^n - T_{n,q}(x). \tag{2.3}$$

Using Lemma 2.1.(i) in the left-hand side of (2.3), we obtain

$$\sum_{k=0}^{n} \frac{2^{k-1}}{[k]_q!} D_{q,x}^{(k)} T_{n,q}(x) = x^n - 2^{-1}T_{n,q}(x). \tag{2.4}$$

From Equation (2.4), we complete the required result. □

Corollary 2.3. *When $q \to 1$ in Theorem 2.2, the tangent polynomials $T_n(x)$ is a solution of the following difference equation of higher order.*

$$\begin{aligned}&\frac{2^{n-1}}{n!}\frac{d^n}{dx^n}T_n(x) + \frac{2^{n-2}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}T_n(x) + \frac{2^{n-3}}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}T_n(x) + \cdots \\ &+ \frac{2^2}{3!}\frac{d^3}{dx^3}T_n(x) + \frac{2}{2!}\frac{d^2}{dx^2}T_n(x) + \frac{d}{dx}T_n(x) + 2^{-1}(T_0(x) + T_n(x)) - x^n = 0,\end{aligned}$$

where $T_n(x)$ is the tangent polynomials.

Theorem 2.4. *The q-tangent polynomials $T_{n,q}(x)$ is a solution of the following q-differential equation of higher order.*

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{2^{n-k-1}T_{k,q}}{[n-k-1]_q![k]_q!} D_{q,x}^{(n-1)} T_{n-1,q}(x) + \sum_{k=0}^{n-2} \frac{2^{n-k-2}qT_{k,q}}{[n-k-2]_q![k]_q!} D_{q,x}^{(n-2)} T_{n-1,q}(x) + \cdots \\ &+ \sum_{k=0}^{2} \frac{2^{2-k}q^{n-3}T_{k,q}}{[2-k]_q![k]_q!} D_{q,x}^{(2)} T_{n-1,q}(x) + \sum_{k=0}^{1} \frac{2^{1-k}q^{n-2}T_{k,q}}{[1-k]_q![k]_q!} D_{q,x}^{(1)} T_{n-1,q}(x) \\ &+ (q^{n-1}T_{0,q} - q^n x)T_{n-1,q}(x) + T_{n,q}(qx) = 0.\end{aligned}$$

Proof. We consider q-derivative after substituting qx instead of x in the generating function of q-tangent polynomials. Then, we have

$$\begin{aligned}
&D_{q,t}\sum_{n=0}^{\infty}T_{n,q}(qx)\frac{t^n}{[n]_q!}\\
&=e_q(qtx)D_{q,t}\left(\frac{2}{e_q(2t)+1}\right)+\frac{2}{e_q(2qt)+1}D_{q,t}e_q(qtx)\\
&=\sum_{n=0}^{\infty}q^nT_{n,q}(x)\frac{t^n}{[n]_q!}\left(qx-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q2^{n-k}T_{k,q}\right)\frac{t^n}{[n]_q!}\right)\\
&=\sum_{n=0}^{\infty}\left(q^{n+1}xT_{n,q}(x)-\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q2^{l-k}q^{n-l}T_{k,q}T_{n-l,q}(x)\right)\frac{t^n}{[n]_q!}.
\end{aligned}\tag{2.5}$$

To make calculations easier, we multiply t in Equation (2.5). Then, we obtain

$$\begin{aligned}
&tD_{q,t}\sum_{n=0}^{\infty}T_{n,q}(qx)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}[n]_qq^nxT_{n-1,q}(x)\frac{t^n}{[n]_q!}\\
&-\sum_{n=0}^{\infty}[n]_q\left(\sum_{l=0}^{n-1}\sum_{k=0}^{l}\begin{bmatrix}n-1\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q2^{l-k}q^{n-l-1}T_{k,q}T_{n-l-1,q}(x)\right)\frac{t^n}{[n]_q!}.
\end{aligned}\tag{2.6}$$

On the other hand, we can obtain the following equation from the generating function of q-tangent polynomials such as

$$tD_{q,t}\sum_{n=0}^{\infty}T_{n,q}(qx)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}[n]_qT_{n,q}(qx)\frac{t^n}{[n]_q!}.\tag{2.7}$$

By comparing the coefficients of Equations (2.6) and (2.7), we have

$$\begin{aligned}
&\sum_{l=0}^{n-1}\sum_{k=0}^{l}\begin{bmatrix}n-1\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q2^{l-k}q^{n-l-1}T_{k,q}T_{n-l-1,q}(x)\\
&=q^nxT_{n-1,q}(x)-T_{n,q}(qx).
\end{aligned}\tag{2.8}$$

In Lemma 2.2.(i), we consider the following equation.

$$T_{n-k-1,q}(x)=\frac{[n-k-1]_q!}{[n-1]_q!}D_{q,x}^{(k)}T_{n-1,q}(x).\tag{2.9}$$

Substituting the right hand side of (2.9) to the left hand side of (2.8), we find

$$\begin{aligned}
&\sum_{l=0}^{n-1}\sum_{k=0}^{l}\begin{bmatrix}n-1\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q2^{l-k}q^{n-l-1}T_{k,q}T_{n-l-1,q}(x)\\
&=\sum_{l=0}^{n-1}\sum_{k=0}^{l}\frac{2^{l-k}q^{n-l-1}T_{k,q}}{[l-k]_q![k]_q!}D_{q,x}^{(l)}T_{n-1,q}(x).
\end{aligned}\tag{2.10}$$

Combining Equations (2.8) and (2.10), we find the required result. □

Corollary 2.5. *When $q \to 1$ in Theorem 2.4, a solution of the following difference equation of higher order*

$$\sum_{k=0}^{n-1} \frac{2^{n-k-1}T_k}{(n-k-1)!k!} \frac{d^{n-1}}{dx^{n-1}} T_{n-1}(x) + \sum_{k=0}^{n-2} \frac{2^{n-k-1}T_k}{(n-k-2)!k!} \frac{d^{n-2}}{dx^{n-2}} T_{n-1}(x) + \cdots$$
$$+ \sum_{k=0}^{2} \frac{2^{2-k}T_k}{(2-k)!k!} \frac{d^2}{dx^2} T_{n-1}(x) + \sum_{k=0}^{1} \frac{2^{1-k}T_k}{(1-k)!k!} \frac{d}{dx} T_{n-1}(x)$$
$$+ (T_0 - x)T_{n-1}(x) + T_n(x) = 0,$$

where $T_n(x)$ is the tangent polynomials .

Theorem 2.6. *The q-tangent polynomials $T_{n,q}(x)$ satisfies a following q-differential equation of higher order.*

$$\sum_{k=0}^{n-1} \frac{2^{n-1}\mathcal{E}_{k,q}}{[n-k-1]_q![k]_q!} D_{q,x}^{(n-1)} T_{n-1,q}(x) + \sum_{k=0}^{n-2} \frac{2^{n-2}q\mathcal{E}_{k,q}}{[n-k-2]_q![k]_q!} D_{q,x}^{(n-2)} T_{n-1,q}(x) + \cdots$$
$$+ \sum_{k=0}^{2} \frac{2^2 q^{n-3}\mathcal{E}_{k,q}}{[2-k]_q![k]_q!} D_{q,x}^{(2)} T_{n-1,q}(x) + \sum_{k=0}^{1} \frac{2q^{n-2}\mathcal{E}_{k,q}}{[1-k]_q![k]_q!} D_{q,x}^{(1)} T_{n-1,q}(x)$$
$$+ (q^{-1}\mathcal{E}_{0,q} - x)q^n T_{n-1,q}(x) + T_{n,q}(qx) = 0,$$

where $\mathcal{E}_{n,q}$ is the q-Euler numbers.

Proof. To find a q-differential equation of higher order which contained the q-Euler numbers, we can transform the Equation (2.5) as

$$\begin{aligned}
&D_{q,t} \sum_{n=0}^{\infty} T_{n,q}(qx) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} q^n T_{n,q}(x) \frac{t^n}{[n]_q!} \left(qx - \sum_{n=0}^{\infty} 2^n \mathcal{E}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} 2^n \frac{t^n}{[n]_q!} \right) \\
&= \sum_{n=0}^{\infty} \left(q^{n+1} x T_{n,q}(x) - \sum_{l=0}^{n} \sum_{k=0}^{l} \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q 2^l q^{n-l} \mathcal{E}_{k,q} T_{n-l,q}(x) \right) \frac{t^n}{[n]_q!}.
\end{aligned}$$

Therefore, we have

$$\sum_{l=0}^{n-1} \sum_{k=0}^{l} \frac{2^l q^{n-l-1}\mathcal{E}_{k,q}}{[l-k]_q![k]_q!} D_{q,x}^{(l)} T_{n-1,q}(x) - q^n x T_{n-1,q}(x) + T_{n,q}(qx) = 0,$$

which is the desired result. □

Corollary 2.7. *Let $q \to 1$ in Theorem 2.6. Then, the tangent polynomials $T_n(x)$ satisfies a following difference equation of higher order.*

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{2^{n-1}\mathcal{E}_k}{(n-k-1)!k!}\frac{d^{n-1}}{dx^{n-1}}T_{n-1}(x)+\sum_{k=0}^{n-2}\frac{2^{n-2}q\mathcal{E}_k}{(n-k-2)!k!}\frac{d^{n-2}}{dx^{n-2}}T_{n-1}(x)+\cdots\\&+\sum_{k=0}^{2}\frac{2^2q^{n-3}\mathcal{E}_k}{(2-k)!k!}\frac{d^2}{dx^2}T_{n-1}(x)+\sum_{k=0}^{1}\frac{2q^{n-2}\mathcal{E}_k}{(1-k)!k!}\frac{d}{dx}T_{n-1}(x)\\&+(\mathcal{E}_0-x)T_{n-1}(x)+T_n(x)=0,\end{aligned}$$

where $\mathcal{E}_n$ is the Euler numbers.

Theorem 2.8. *The q-tangent polynomials $T_{n,q}(x)$ is a solution of following q-differential equation of higher order.*

$$\begin{aligned}&\frac{T_{n-1,q}(2)}{[n-1]_q!}D_{q,x}^{(n-1)}T_{n-1,q}(x)+\frac{qT_{n-2,q}(2)}{[n-2]_q!}D_{q,x}^{(n-2)}T_{n-1,q}(x)+\cdots\\&+\frac{q^{n-4}T_{3,q}(2)}{[3]_q!}D_{q,x}^{(3)}T_{n-1,q}(x)+\frac{q^{n-3}T_{2,q}(2)}{[2]_q!}D_{q,x}^{(2)}T_{n-1,q}(x)\\&+q^{n-2}T_{1,q}(2)D_{q,x}^{(1)}T_{n-1,q}(x)+(q^{-1}T_{0,q}(2)-x)q^nT_{n-1,q}(x)+T_{n,q}(qx)=0.\end{aligned}$$

Proof. To use q-tangent polynomials as coefficients in q-differential equation of higher order, we can find the other form from equation (2.5):

$$\begin{aligned}&D_{q,t}\sum_{n=0}^{\infty}T_{n,q}(qx)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\left(q^{n+1}xT_{n,q}(x)-\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n-k}T_{k,q}(2)T_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.11}$$

Multiplying t in Equation (2.11), we have

$$\begin{aligned}tD_{q,t}\sum_{n=0}^{\infty}T_{n,q}(qx)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}[n]_q q^n xT_{n-1,q}(q^{-1}x)\frac{t^n}{[n]_q!}\\-\sum_{n=0}^{\infty}[n]_q\sum_{k=0}^{n-1}&\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{n-k-1}T_{k,q}(2)T_{n-k-1,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.12}$$

Comparing the coefficients of Equations (2.5) and (2.12), we obtain

$$\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{n-k-1}T_{k,q}(2)T_{n-k-1,q}(x)=q^nxT_{n-1,q}(x)-T_{n,q}(qx).\tag{2.13}$$

Applying a relation between $D^n_{q,x}T_{n,q}(x)$ and $T_{n,q}(x)$ in the left-hand side of (2.13), we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{n-k-1}T_{k,q}(2)T_{n-k-1,q}(x)\\&=\sum_{k=0}^{n-1}\frac{q^{n-k-1}T_{k,q}(2)}{[k]_q!}D^{(k)}_{q,x}T_{n-1,q}(x).\end{aligned}\tag{2.14}$$

We can find a equation combining the right hand side of (2.13) and (2.14), which shows the required result. □

Corollary 2.9. *The tangent polynomials $T_n(x)$ when $q \to 1$ in Theorem 2.8 is a solution of following differential equation of higher order.*

$$\begin{aligned}&\frac{T_{n-1}(2)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}T_{n-1}(x)+\frac{T_{n-2}(2)}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}T_{n-1}(x)+\cdots+\frac{T_3(2)}{3!}\frac{d^3}{dx^3}T_{n-1}(x)\\&+\frac{T_2(2)}{2!}\frac{d^2}{dx^2}T_{n-1}(x)+T_1(2)\frac{d}{dx}T_{n-1}(x)+(T_0(2)-x)T_{n-1}(x)+T_n(x)=0.\end{aligned}$$

Theorem 2.10. *Let $a, b \neq 0$ and $0 < q < 1$. Then, we find a general symmetric property of q-differential equation of higher order:*

$$\begin{aligned}&\frac{T_{n,q}(b^{-1}y)}{[n]_q!}D^{(n)}_{q,x}T_{n,q}(a^{-1}x)+\frac{b^{-1}T_{n-1,q}(b^{-1}y)}{[n-1]_q!}D^{(n-1)}_{q,x}T_{n,q}(a^{-1}x)+\cdots\\&+b^{1-n}T_{1,q}(b^{-1}y)D^{(1)}_{q,x}T_{n,q}(a^{-1}x)+b^{-n}T_{0,q}(b^{-1}y)T_{n,q}(a^{-1}x)\\&=\frac{T_{n,q}(a^{-1}y)}{[n]_q!}D^{(n)}_{q,x}T_{n,q}(b^{-1}x)+\frac{a^{-1}T_{n-1,q}(a^{-1}y)}{[n-1]_q!}D^{(n-1)}_{q,x}T_{n,q}(b^{-1}x)+\cdots\\&+a^{1-n}T_{1,q}(a^{-1}y)D^{(1)}_{q,x}T_{n,q}(b^{-1}x)+a^{-n}T_{0,q}(a^{-1}y)T_{n,q}(b^{-1}x).\end{aligned}$$

Proof. To find q-differential equation of higher order using a symmetric property of q-tangent polynomials, we can construct form A such as

$$A:=\frac{4e_q(tx)e_q(ty)}{(e_q(2at)+1)(e_q(2bt)+1)}.$$

Using the generating function of q-tangent polynomials and Cauchy products, form A is transformed as

$$A=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^{n-k}b^kT_{k,q}(b^{-1}y)T_{n-k,q}(a^{-1}x)\right)\frac{t^n}{[n]_q!},\tag{2.15}$$

and

$$A=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^kb^{n-k}T_{k,q}(a^{-1}y)T_{n-k,q}(b^{-1}x)\right)\frac{t^n}{[n]_q!}.\tag{2.16}$$

From (2.15) and (2.16), we find a symmetric property such as

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^{n-k}b^{k}T_{k,q}(b^{-1}y)T_{n-k,q}(a^{-1}x)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^{k}b^{n-k}T_{k,q}(a^{-1}y)T_{n-k,q}(b^{-1}x).\end{aligned}\tag{2.17}$$

Applying a relation between $D_{q,x}^{(n)}T_{n,q}(x)$ and $T_{n,q}(x)$ in Equation (2.17), we have

$$\sum_{k=0}^{n}\frac{b^{k-n}T_{k,q}(b^{-1}y)}{[k]_q!}D_{q,x}^{(k)}T_{n,q}(a^{-1}x)=\sum_{k=0}^{n}\frac{a^{k-n}T_{k,q}(a^{-1}y)}{[k]_q!}D_{q,x}^{(k)}T_{n,q}(b^{-1}x).$$

From the above equation, we express the required result and complete the proof of Theorem 2.10. □

Corollary 2.11. *Setting $a=1$ in Theorem 2.10, one holds*

$$\begin{aligned}&\frac{T_{n,q}(b^{-1}y)}{[n]_q!}D_{q,x}^{(n)}T_{n,q}(x)+\frac{b^{-1}T_{n-1,q}(b^{-1}y)}{[n-1]_q!}D_{q,x}^{(n-1)}T_{n,q}(x)+\cdots\\&+b^{1-n}T_{1,q}(b^{-1}y)D_{q,x}^{(1)}T_{n,q}(x)+b^{-n}T_{0,q}(b^{-1}y)T_{n,q}(x)\\&=\frac{T_{n,q}(y)}{[n]_q!}D_{q,x}^{(n)}T_{n,q}(b^{-1}x)+\frac{T_{n-1,q}(y)}{[n-1]_q!}D_{q,x}^{(n-1)}T_{n,q}(b^{-1}x)+\cdots\\&+T_{1,q}(y)D_{q,x}^{(1)}T_{n,q}(b^{-1}x)+T_{0,q}(y)T_{n,q}(b^{-1}x).\end{aligned}$$

Corollary 2.12. *Let $a,b\neq 0$, $0<q<1$ and $q\to 1$ in Theorem 2.10. Then, the following holds*

$$\begin{aligned}&\frac{T_n(b^{-1}y)}{n!}\frac{d^n}{dx^n}T_n(a^{-1}x)+\frac{b^{-1}T_{n-1}(b^{-1}y)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}T_n(a^{-1}x)+\cdots\\&+b^{1-n}T_1(b^{-1}y)\frac{d}{dx}T_n(a^{-1}x)+b^{-n}T_0(b^{-1}y)T_{n,q}(a^{-1}x)\\&=\frac{T_n(a^{-1}y)}{n!}\frac{d^n}{dx^n}T_n(b^{-1}x)+\frac{a^{-1}T_{n-1}(a^{-1}y)}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}T_n(b^{-1}x)+\cdots\\&+a^{1-n}T_1(a^{-1}y)\frac{d}{dx}T_n(b^{-1}x)+a^{-n}T_0(a^{-1}y)T_n(b^{-1}x),\end{aligned}$$

where $T_n(x)$ is the tangent polynomials.

Theorem 2.13. *Let $a,b\neq 0$ and $0<q<1$. Then, we derive*

$$\begin{aligned}&\frac{2^n\mathcal{E}_{n,q}(2^{-1}a^{-1}x)}{[n]_q!}D_{q,y}^{(n)}T_{n,q}(b^{-1}y)+\frac{2^{n-1}a^{-1}\mathcal{E}_{n-1,q}(2^{-1}a^{-1}x)}{[n-1]_q!}D_{q,y}^{(n-1)}T_{n,q}(b^{-1}y)\\&+\cdots+2a^{1-n}\mathcal{E}_{1,q}(2^{-1}a^{-1}x)D_{q,y}^{(1)}T_{n,q}(b^{-1}y)+a^{-n}\mathcal{E}_{0,q}(2^{-1}a^{-1}x)T_{n,q}(b^{-1}y)\\&=\frac{2^n\mathcal{E}_{n,q}(2^{-1}b^{-1}x)}{[n]_q!}D_{q,y}^{(n)}T_{n,q}(a^{-1}y)+\frac{2^{n-1}b^{-1}\mathcal{E}_{n-1,q}(2^{-1}b^{-1}x)}{[n-1]_q!}D_{q,y}^{(n-1)}T_{n,q}(a^{-1}y)\\&+\cdots+2b^{1-n}\mathcal{E}_{1,q}(2^{-1}b^{-1}x)D_{q,y}^{(1)}T_{n,q}(a^{-1}y)+b^{-n}\mathcal{E}_{0,q}(2^{-1}b^{-1}x)T_{n,q}(a^{-1}y).\end{aligned}$$

Proof. To find the other symmetric property of q-differential equation of higher order containing q-Euler polynomials, we can consider

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(2^{-1}x)\frac{t^n}{[n]_q!}=\frac{2}{e_q(2t)+1}e_q(tx). \tag{2.18}$$

Using the generating function of q-tangent polynomials, Equation (2.18), and Cauchy products, form A is transformed as

$$A=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(2a)^kb^{n-k}\mathcal{E}_{k,q}(2^{-1}a^{-1}x)T_{n-k,q}(b^{-1}y)\right)\frac{t^n}{[n]_q!}, \tag{2.19}$$

and

$$A=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(2a)^kb^{n-k}\mathcal{E}_{k,q}(2^{-1}b^{-1}x)T_{n-k,q}(a^{-1}y)\right)\frac{t^n}{[n]_q!}. \tag{2.20}$$

Applying the coefficient comparison method on Equations (2.19) and (2.20), we find a symmetric property which is related to q-Euler polynomials and q-tangent polynomials.

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(2a)^kb^{n-k}\mathcal{E}_{k,q}(2^{-1}a^{-1}x)T_{n-k,q}(b^{-1}y)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(2a)^kb^{n-k}\mathcal{E}_{k,q}(2^{-1}b^{-1}x)T_{n-k,q}(a^{-1}y).\end{aligned} \tag{2.21}$$

Applying a relation between $D_{q,x}^{(n)}T_{n,q}(x)$ and $T_{n,q}(x)$ in Equation (2.21), we obtain

$$\begin{aligned}&\sum_{k=0}^{n}\frac{2^ka^{k-n}\mathcal{E}_{k,q}(2^{-1}a^{-1}x)}{[k]_q!}D_{q,y}^{(k)}T_{n,q}(b^{-1}y)\\&=\sum_{k=0}^{n}\frac{2^kb^{k-n}\mathcal{E}_{k,q}(2^{-1}b^{-1}x)}{[k]_q!}D_{q,y}^{(k)}T_{n,q}(a^{-1}y).\end{aligned}$$

From the above equation, we complete the proof of Theorem 2.13. □

Corollary 2.14. *Putting $a=1$ in Theorem 2.13, the following holds*

$$\begin{aligned}&\frac{2^n\mathcal{E}_{n,q}(2^{-1}x)}{[n]_q!}D_{q,y}^{(n)}T_{n,q}(b^{-1}y)+\frac{2^{n-1}\mathcal{E}_{n-1,q}(2^{-1}x)}{[n-1]_q!}D_{q,y}^{(n-1)}T_{n,q}(b^{-1}y)\\&+\cdots+2\mathcal{E}_{1,q}(2^{-1}x)D_{q,y}^{(1)}T_{n,q}(b^{-1}y)+\mathcal{E}_{0,q}(2^{-1}x)T_{n,q}(b^{-1}y)\\&=\frac{2^n\mathcal{E}_{n,q}(2^{-1}b^{-1}x)}{[n]_q!}D_{q,y}^{(n)}T_{n,q}(y)+\frac{2^{n-1}b^{-1}\mathcal{E}_{n-1,q}(2^{-1}b^{-1}x)}{[n-1]_q!}D_{q,y}^{(n-1)}T_{n,q}(y)\\&+\cdots+2b^{1-n}\mathcal{E}_{1,q}(2^{-1}b^{-1}x)D_{q,y}^{(1)}T_{n,q}(y)+b^{-n}\mathcal{E}_{0,q}(2^{-1}b^{-1}x)T_{n,q}(y).\end{aligned}$$

Corollary 2.15. *Let $a, b \neq 0$, $0 < q < 1$ and $q \to 1$ in Theorem 2.13. Then, one holds*

$$\begin{aligned}
&\frac{2^n\mathcal{E}_n(2^{-1}a^{-1}x)}{n!}\frac{d^n}{dy^n}T_n(b^{-1}y) + \frac{2^{n-1}a^{-1}\mathcal{E}_{n-1}(2^{-1}a^{-1}x)}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}T_n(b^{-1}y)\\
&+\cdots + 2a^{1-n}\mathcal{E}_1(2^{-1}a^{-1}x)\frac{d}{dy}T_n(b^{-1}y) + a^{-n}\mathcal{E}_0(2^{-1}a^{-1}x)T_n(b^{-1}y)\\
&= \frac{2^n\mathcal{E}_n(2^{-1}b^{-1}x)}{n!}\frac{d^n}{dy^n}T_n(a^{-1}y) + \frac{2^{n-1}b^{-1}\mathcal{E}_{n-1}(2^{-1}b^{-1}x)}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}T_n(a^{-1}y)\\
&+\cdots + 2b^{1-n}\mathcal{E}_1(2^{-1}b^{-1}x)\frac{d}{dy}T_n(a^{-1}y) + b^{-n}\mathcal{E}_0(2^{-1}b^{-1}x)T_n(a^{-1}y),
\end{aligned}$$

where $\mathcal{E}_n(x)$ is the Euler polynomials and $T_n(x)$ is the tangent polynomials.

3. Conclusion

We study the q-differential equations of higher order related to the q-tangent polynomials and confirm the properties. Moreover, the relationship between q-Euler number and q-differential equations of higher order for q-tangent polynomials was confirmed.

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