J. Appl. Math. & Informatics Vol. **40**(2022), No. 5 - 6, pp. 883 - 896
https://doi.org/10.14317/jami.2022.883

GOODSTEIN'S GENERALIZED THEOREM: FROM ROOTED TREE REPRESENTATIONS TO THE HYDRA GAME†

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Abstract. A hereditary base-b representation, used in the celebrated Goodstein's theorem, can easily be converted into a labeled rooted tree. In this way it is possible to give a more elementary geometric proof of the aforementioned theorem and to establish a more general version, geometrically proved. This view is very useful for better understanding the underlying logical problems and the need to use transfinite induction in the proof. Similar problems will then be considered, such as the so-called "hydra game".

AMS Mathematics Subject Classification : 03D20, 11U99, 05C05.
Key words and phrases : Goodstein's theorem, rooted trees, unimaginable numbers, Knuth's up-arrow notation, number representation.

1. Introduction

A positive integer is said to be *unimaginable* if it exceeds the threshold of one *googol*, that is 10^{100}. We know very little about unimaginable numbers, because it is not only difficult to study them, but also simply to write them down. Several notations have been devised for this purpose, two of the most well known are the *Knuth's up-arrow* and the *Steinhaus-Moser notation.* In Section 2 of the present work, we will begin by recalling their generalized definitions, as well as a comparison result between them, obtained in [26].

A well-known case in which numbers immediately become unimaginable is that of Goodstein sequences. Recall that the *hereditary base-b notation* expresses a positive integer as the sum of powers with base b, and does the same thing with exponents, so that only coefficients between 1 and $b-1$ appear (in addition to the base b itself). This representation is the subject of the celebrated

Received June 29, 2021. Revised April 14, 2022. Accepted April 20, 2022. *Corresponding author.

†This work has been partially supported by POR Calabria FESR-FSE 2014-2020, with the grant for research project "IoT&B", CUP J48C17000230006.

Goodstein theorem (see [21] and Corollary 3.8): replacing the base b with $b+1$ and subtracting 1 from the number thus obtained, we obtain a sequence whose elements immediately become enormous, unimaginable, but after a finite number of iterations (unimaginably large) the sequence decreases to zero. Drawing inspiration from graph theory, it is possible to convert a base-b hereditary representation into a labeled rooted tree (see [26]): they will be studied in Section 3 and, in particular, a more general version will be given in Subsection 3.3. Such representations by generalized rooted trees allow to generalize Goodstein's theorem in a similar way to what was done in [30], and to give new simpler geometric proofs. This will be discussed in detail in Subsection 3.4 of this paper, where we will provide three different sketches of proof of Theorem 3.7. Some explicit bounds for the length of Goodstein's sequence in particular cases have also been obtained in [26], and will be considered at the end of the same section.

Finally, in the last section, we will describe the "hydra game" (so named by L. Kirby and J. Paris from the mythological *Hydra of Lerna*, see [23]) using unlabeled redundant rooted trees and, at the other hand, we will give new geometric points of view of the problem through unredundant labeled rooted trees introduced in Subsection 3.3.

2. Two notations for "unimaginable numbers"

The original source for Knuth's up-arrow notation is [24], and for Steinhaus-Moser notation [33]. In addition to the basic definitions, we recall without proof, a fundamental result obtained in [26]. For more on Knuth's powers and unimaginable numbers see [10, 12] and the references therein.

2.1. Knuth's up-arrow notation. In this paper we limit ourselves to providing the following recursive definition for $\uparrow(A,B,k)$, while in [26] it has been generalized to $\uparrow(A,B,k,C)$. We set

(1) $\uparrow(A,B,0) := AB$;
(2) $\uparrow(A,0,k) := 1$ for $k \geq 1$;
(3) $\uparrow(A,B+1,k) := \uparrow(A,\uparrow(A,B,k),k-1)$.

The original Knuth's notation is $A \uparrow^k B$ to mean $\uparrow(A,B,k)$ (see [24]). In the first few cases we have, for instance,

$$\begin{aligned} A \uparrow B &:= \uparrow(A,B,1) && [\textbf{Normal exponentiation}];\\ A \uparrow\uparrow B &:= \uparrow(A,B,2) && [\textbf{Tetration}];\\ A \uparrow\uparrow\uparrow B &:= \uparrow(A,B,3) && [\textbf{Pentation}];\\ A \uparrow^k B &:= \uparrow(A,B,k) && [k+2\ \ \textbf{hyper-operation}]. \end{aligned}$$

2.2. The generalized Steinhaus-Moser notation. As usual, we denote by f^k the functional power, i.e., the compositum $f \circ f \circ \ldots \circ f$ of the function f with itself k-times.

Definition 2.1 (Generalized Steinhaus-Moser notation)**.** For any $n, k \in \mathbb{N},\ k \geq 3$, we define the *generalized Steinhaus-Moser notation* $SM_k(n)$ recursively as follows:

(1) $SM_3(n) := n^n$;
(2) $SM_{k+1}(n) := SM_k^n(n)$.

Remark 2.1. For the first three values of k, i.e. $k = 3, 4, 5$, the number $SM_k(n)$ is originally denoted in [33] by n inscribed in a regular k-gon, where a circle takes the place of the pentagon. In this way the values of *mega* ② and *megiston* ⑩ are defined (see [26, 33]).

Unless otherwise specified, we will use only positive integers in the following.

Theorem 2.2. *The Steinhaus-Moser generalized function is comparable to Knuth's up-arrow notation as follows:*

$$n \uparrow^m (n+1) \leq SM_{m+2}(n) \leq n \uparrow^{m-1} (n+m-1) \uparrow^m n < (n+m-1) \uparrow^m (n+1).$$

More precisely we have

$$n \uparrow^m (k+1) \leq SM_{m+1}^k(n) \leq n \uparrow^{m-1} (n+m-1) \uparrow^m k.$$

For the proof of the previous theorem, see [26, Theorem 4.9].

Unimaginably large numbers are obviously not to be confused with infinitely large, or infinite numbers. The latter pervade every field of mathematics, just think of the cardinal and ordinal numbers, or the infinites and infinitesimals of non-standard analysis, etc. Recently, a new type of numerical-computational infinity called *grossone* has also been introduced (see [2, 3, 7, 8, 9, 11, 13, 14, 17, 20, 31, 32] and the references inside), whose symbol ①, a circled 1, is very similar to the notation for mega and megiston seen above. It also has properties very similar to those of the smallest infinite ordinal ω, which plays an essential role in the classical proof of Goodstein's theorem (see Section 4).

3. The rooted tree representation

3.1. Base equal to 2. Let $\mathcal{T}$ be the set characterized by:

- $\emptyset \in \mathcal{T}$
- A set containing a finite number of elements of $\mathcal{T}$ is itself an element of $\mathcal{T}$ (that is, $A = \{a_i \in \mathcal{T}\}_{i \in I}$ then $A \in \mathcal{T}$), and vice versa any element belonging to $\mathcal{T}$ contains only finitely many elements of $\mathcal{T}$, without infinite descending chains.

The set $\mathcal{T}$ has also the following properties:

- For any element $t \in \mathcal{T}$, a rooted tree can be associated to it: the tree is built recursively by constructing the branches (subtrees) associated with the elements that make up t, and then connecting their roots to a new root which will be that of the tree corresponding to the entire set t. The tree thus obtained is also said to be *unredundant*, since there are no two identical branches originating from the same node: this is a consequence of the fact that the elements of a set are all distinct. A branchless root is associated with the empty set.

- A *height* function $H : \mathcal{T} \to \mathbb{N}$ is defined by

$$H(\emptyset) := 0, \qquad H(A) := 1 + \max_{t \in A} H(t).$$

  Note that $H(A)$ is always well defined because there are no infinite descending chains in $\mathcal{T}$.
- There is also a bijective function $f : \mathcal{T} \overset{\cong}{\to} \mathbb{N}$ defined recursively by
  - $f(\emptyset) = 0$;
  - $f(A) = \sum_{t \in A} 2^{f(t)}$.

For the sake of clarity, we write the bijectivity of f as a separate statement, and prove it briefly.

Proposition 3.1. *The function f just defined is a bijection from $\mathcal{T}$ to the set of natural numbers $\mathbb{N}$, i.e.:*

$$f(A) = f(B) \quad \to \quad A = B.$$

Proof. We use a complete induction argument on $\max(H(A), H(B))$. After checking that in the case $\max(H(A), H(B)) = 0$ we must have $A = B = \emptyset$, we suppose that the thesis $f(a) = f(b) \to a = b$ is true whenever $\max(H(a), H(b)) < \max(H(A), H(B))$. Consider now the unique base-2 representation of $f(A)$ and $f(B)$, and note that the 1s correspond to elements $a \in A$, $b \in B$, respectively, such that $f(a)$ and $f(b)$ return their exact position. Hence, by inductive hypothesis, from $f(a) = f(b)$ we can deduce $a = b$, and this means that A and B have the same elements, so they are equal. $\square$

Using the above bijection, we will not distinguish between A and $f(A)$ in the following. We also set

$$M_k := \max\{A \in \mathcal{T} | H(A) = k\},$$

$$m_k := \min\{A \in \mathcal{T} | H(A) = k\}.$$

Note that M_k corresponds to a set $A \in \mathcal{T}$ which contains all elements t with height less than k. For instance, we have $M_0 = 0,\ M_1 = 1,\ M_2 = 2^0 + 2^1 = 2^2 - 1 = 3,\ M_3 = 2^0 + 2^1 + 2^2 + 2^3 = 2^4 - 1 = 15,\ M_4 = 2^0 + \ldots + 2^{15} = 2^{16} - 1.$

On the other hand, m_k is easily given by the following recursion: $m_0 = 0$ and $m_k = \{m_{k-1}\}$.

Let now $\{a_i\}_i$ be the sequence of integers recursively obtained by

(i) $a_0 = 0$ and
(ii) $a_{i+1} = 2^{a_i}$.

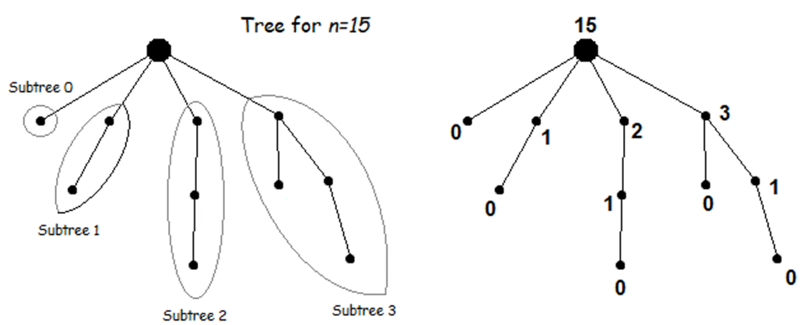
By a trivial induction on k the reader can prove that $M_k = a_{k+1} - 1$ and $m_k = a_k$; this implies that the height function H is non-decreasing. We can also place each element $A \in \mathcal{T}$ in a suitable interval, using its height $H(A)$ and Knuth's up-arrow notation:

$$2 \uparrow\uparrow (H(A) - 1) \leq A < 2 \uparrow\uparrow H(A).$$

Example 3.2. For $k = 3$ we have $M_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, i.e. M_3 consists of all elements with height less or equal than 2. We can equivalently say that M_3 is the largest set/element having height 3, and we find:

$$f(M_3) = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15.$$

The rooted tree associated to M_3 is drawn below. In the rightmost picture we have indicated the integer corresponding to each node/subtree.



Remark 3.1. Adopting the usual notation $\mathcal{P}(A)$ for the power set of A (i.e., $\mathcal{P}(A) = \{X \subset A\}$), note that we get for all $k \geq 1$

(i) $M_k = \mathcal{P}(M_{k-1})$;
(ii) $|M_k| = m_k$.

The equality in (ii) holds because M_k contains all sets whose image through f belong to $\{0, 1, \ldots, m_k - 1\}$.

3.2. Algorithms with unredundant rooted trees. Comparison. To compare two elements $A, B \in \mathcal{T}$ we can simply consider the symmetric difference $A \Delta B$ with the natural order induced by f on its elements: if the greatest element of $A \Delta B$ belong to A then A will be greater than B, and vice versa.

Remark 3.2. For the purposes of this paper the order of the elements in the sets does not matter, that is, the sets are not ordered. If, on the other hand, we considered the sets in $\mathcal{T}$ already ordered according to the natural order, then comparing two elements and deciding which is the biggest would be trivial.

Successor operation. Let $A \in \mathcal{T}$ and we want to compute its successor $s(A)$. If A is equal to some M_k, $k \geq 0$, then we have immediately $s(A) = m_{k+1}$. On the other hand, if $A \neq M_k \ \forall k \geq 0$, consider the smallest non-negative integer n_A which does not belong to A. Then $n_A \neq A$ because $A \neq M_k \ \forall k \geq 0$, and to get the successor $s(A)$ of A we have to add n_A to A if n_A is zero, or replace the subset $\{0, \ldots, n_A - 1\} \subset A$ with n_A if $n_A \neq 0$.

Predecessor operation. The computation of the predecessor $s^{-1}(A)$ of some $A \in \mathcal{T}$ doesn't require much strategy either: one simply takes the smallest digit of the number and replaces it with all lesser digits available. For example, the tree $\{7, 5, 4\}$ becomes $\{7, 5, 3, 2, 1, 0\}$ substituting the digit $4 = \{\{\{\emptyset\}\}\}$ with the set of all smaller natural numbers $\{0, 1, 2, 3\}$.

Addition. To add A and B we just need to join their elements together. In the event that some elements appear twice, successors and carries are used in a natural way.

Multiplication. To multiply A and B one uses the following identity:

$$A \cdot B = \sum_{(a,b) \in A \times B} (c := \{a + b\})$$

which means, in usual terms,

$$\left(\sum_{a \in A} 2^a\right) \cdot \left(\sum_{b \in B} 2^b\right) = \sum_{a \in A,\, b \in B} 2^{a+b}.$$

3.3. Generalized rooted tree representation. Generalization to bases greater than 2. The set $\mathcal{T}$ seen above is no longer useful if we want to consider a base greater than 2. For such bases we introduce a more complex notation where a representation is a pair (b, s) where b is an integer greater or equal than 2 and s is a string in the language $\{1, 2, \ldots, b-1, \text{“}+\text{”}, \text{“(”}, \text{“)”}\}$. We must interpret a digit followed by a parenthesis by inserting "$\cdot b \uparrow$" each time, to get the complete expression. For example,

$$b = 3: \quad 2(1() + 2(1()) + 1(2())) = 2 \cdot 3 \uparrow (1 + 2 \cdot 3 \uparrow (1) + 1 \cdot 3 \uparrow (2)) = 2 \cdot 3^{16}.$$

Remark 3.3. We again remark that order is irrelevant for the addition, and we could interpret the "plus" symbol as a separator (indeed, it could be completely removed because the expression would be uniquely interpretable anyways). From a computational point of view it is instead convenient to order sums by exponents.

We now consider also rooted trees whose edges are **labeled** with a digit belonging to $\{1, \ldots, b-1\}$.

Definition 3.3. Given a base $b > 2$, we consider the set $\mathcal{T}_b$ of **labeled unredundant rooted trees** such that:

- the zero-tree consisting of a single node is an element of $\mathcal{T}_b$;
- the joining of any number of different trees in $\mathcal{T}_b$ from their roots to a new root with an edge labeled from 1 to $b-1$ gives a new tree in $\mathcal{T}_b$.

Example 3.4. If we use different colors, **blue** and **red**, to better distinguish the labels 1 and 2, respectively, we obtain the tree shown in Fig. 1 associated to the bracketed expression below

$$3: \quad 2() + 1(1() + 2(1())) = 2 \cdot 3^0 + 1 \cdot 3^{1 \cdot 3^0 + 2 \cdot 3^{1 \cdot 3^0}}$$
$$= 2 + 3^{1+6} = 2 + 2187 \ = 2189.$$

FIGURE 1

Also in this case it is possible to define the **height** of a tree and, consequently, to find suitable sequences of minimal or maximal elements having a fixed height:

$$b \uparrow\uparrow (H(A) - 1) \leq A < b \uparrow\uparrow H(A).$$

In fact, the minimum $m_k = b \uparrow\uparrow (H(A) - 1)$ is obtained with a single path of digits 1, instead the maximum is the sum of $(b-1) \times b^k$ where $k < b \uparrow\uparrow (H(A)-1)$, and this gives a geometric progression whose sum is $M_k := [b \uparrow\uparrow H(A)] - 1$.

Generalization to base ω (See [34] for the theory of ordinal numbers). When we don't give an upper bound to the digits of an unredundant rooted tree, we can suppose that the expression obtained is an ordinal with the assumption $b = \omega$. Thus we can consider the set $\mathcal{T}_\omega$ of unredundant rooted trees with labeled edges, where the labels are strictly positive integers which in this case are not bounded by a finite b. This means:

- the zero-tree consisting of a single node is an element of $\mathcal{T}_\omega$;
- the joining of any number of different trees in $\mathcal{T}_\omega$ from their roots to a new root with an edge labeled with a strictly positive number gives a new tree in $\mathcal{T}_\omega$.

Definition 3.5. We give a total order to the set $\mathcal{T}_\omega$:

$$T_1 < T_2 \ \leftrightarrow \ b : T_1 < b : T_2,$$

where the notation "b :" gives the partial inverse of the inclusion $\mathcal{T}_b \subseteq \mathcal{T}_\omega$ (that is, the interpretation of the tree in base b), which is defined whenever b is an integer bigger than every label in T_1 and T_2. This is a good definition because it is independent on the choice of b, because changing the base to a tree is always order-preserving.

This ordered set is known to be isomorphic to $\epsilon_0 := \omega \uparrow\uparrow \omega$, i.e. to the set of ordinals $\kappa < \epsilon_0$, with the same recursive map as before. The ordinal corresponding to a tree T is given by:

$$\sum_{S \text{ sub-tree of } T} l \cdot \omega^{val(S)}$$

where l is the label of the edge connecting the sub-tree S to the root and $val(S)$ is the numerical value of the sub-tree which can be found recursively.

3.4. Generalized Goodstein's theorem. Goodstein's theorem (see [21]) is very important in logic because it remarks clearly the limits of Peano axioms (see remark at the end of [30]), which are insufficient to prove it. Transfinite induction up to the aforementioned ordinal ϵ_0 is required, which is equivalent to the well-ordering of base-ω trees (so we will be able to use them in order to obtain new geometrical proofs). Goodstein's theorem has an interesting interpretation within the topic of rooted tree notation, and here we give a new geometrical proof of it.

Definition 3.6. Given an increasing sequence of bases $\{b_0 \leq b_1 \leq \ldots\} \subseteq \mathbb{Z}_{\geq 1}$ and a starting labeled unredundant rooted tree A_0 in base b_0, we consider the following geometrical definition of generalized Goodstein's sequence:

$$(b_{k+1}, A_{k+1}) := s^{-1}(b_{k+1}, A_k)$$

where s^{-1} is the predecessor function.

Remark 3.4. A tree in base b_k can be reinterpreted in a natural way as a base b_{k+1} tree as we did for tree ordering, because if labels are smaller than b_k they are also smaller than b_{k+1}. Thus, in the former definition it is well defined (b_{k+1}, A_k) as the interpretation of A_k in the base b_{k+1}.

Theorem 3.7. *Whatever is the sequence of bases and the starting tree A_0, $\exists k \in \mathbb{Z}_{\geq 1}$ such that A_k is the zero tree (for all greater k the pairs are undefined).*

Corollary 3.8. *Goodstein's original theorem can be obtained as a corollary of the general version we just stated by setting $b_k = b + k$ for some starting base $b_0 = b$, so that at every step the base is increased by 1.*

In the following we give three sketches of proof of Theorem 3.7.

Sketch of proof 1. We consider the set of pairs obtained by the iteration:

$$\{F^k(b_0, A_0) = (b_k, A_k) | k \in \mathbb{N}\}.$$

To each pair of this sequence we consider the corresponding base-ω tree. The sequence of trees obtained this way is strictly decreasing because the base-ω substitution is order-preserving, so by infinite descent (thus using the aforementioned well-ordering of base-ω trees) we know it must stop somewhere, and the only possibility is the zero-tree.

Sketch of proof 2. There is also a more constructive proof: the one given in [30] can be geometrically interpreted as applying complete induction to the height of the tree (we remark that the well-ordering of the trees, i.e. the possibility to apply induction, is stronger than Peano axioms). We give a sketch of the proof with our notations:

– the proof is based on the lemma that when the Goodstein sequence converges for the starting pair (b, T_1), it must converge for all smaller pairs (b, T_2) $(T_2 < T_1)$;
– one proceeds by induction on k, and Theorem 3.9 proves the case $k = 2$;
– one considers only the biggest power at each step (by the aforementioned lemma the other ones will vanish as well), and notices that their exponents form another Goodstein sequence starting from (b, M_{k-1}): this proves the assertion.

Sketch of proof 3. A very short proof can be made by proving the following simple lemma: in a finite number of steps the digit with lowest exponent will decrease by 1 without any change on the remaining tree. Complete induction again gives another proof of generalized Goodstein's theorem. More precisely, after one step the lowest digit will decrease by 1 and there will pop up a finite number of other finite lower-exponent digits, which by induction will vanish in a finite number of steps, so that the lemma is proved. The sum of values of the digits is finite, thus one by one they will vanish from any starting tree, proving the theorem. □

We also give a specific bound in the following simpler case (proof can be found in [26]):

Theorem 3.9. *We will denote by* $\check{x} := x - 1$ *the predecessor of a number* x*. Let* $b > 1$ *and* $B_k(b)$ $(k < b)$ *be the base that brings the pair* $(b, \check{b}(\check{k}) + \ldots + \check{b}(1) + \check{b}())$ *to the stopping value* -1*. Then the following equalities hold*

$$B_1(b) = 2 \cdot b,$$
$$B_k(b) = B^b_{k-1}(b),$$

where B^b_{k-1} *stays for* $B_{k-1} \circ B_{k-1} \circ \ldots \circ B_{k-1}$*, i.e. the* b*-times composition of* B_{k-1} *with itself. Let's see explicitly what this means in the case* $k = 2$ *(see* [26] *for the meta-algorithm notation):*

$$B_2(b) = EXPAND \left[\overbrace{2\cdot}^{b}\ b \right] = 2^b b.$$

Remark 3.5. The sequence in the theorem is a special case of Löb-Wainer fast growing hierarchy (see [16]).

Corollary 3.10. *Let* A *be a tree relative to the base* $b > 2$ *such that* $H(A) \leq 2$*. Then Goodstein's algorithm starting from the pair* (b, A) *arrives at the stopping* *pair* $(B,-1)$ *when* $B = B_b(b)$. *We have moreover*

$$B_b(b) < SM_{b+1}(b) \leq (2b-2) \uparrow^{b-1} (b+1)$$

where SM_k *stays for the generalized* k*-gon Steinhaus-Moser notation (see Definition 2.1) and the last inequality follows from Theorem 2.2.*

From the previous corollary we have that $B-b-1$ is an effective bound for Goodstein's algorithm to arrive at the zero value, as it needs $B-b$ steps to reach -1.

4. Redundant trees and the hydra game

We consider now a finite rooted tree where we don't require branches from the same node to differ from each other (see [23]). This will represent an "hydra", following the lore of Herculean myth.

4.1. The hydra game. The hydra, mythical creature classically defeated by Hercules, has stimulated mathematicians to analyze the battle from a schematic point of view. The monster is represented by a redundant rooted (unlabeled) tree, with the following interpretation:

- the root of the tree represents the body of the hydra;
- ending segments (leaves) of the tree are the heads of the hydra;
- points (nodes) of the tree are the ramifications of the hydra's head system;
- at each turn ($k = 1, 2, 3, \ldots$) Hercules can cut a head, but there is a function $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ so that the branch starting from the parent node of the head basis (unless the head is attached to the body directly so that this node does not exist) is copied $f(k)$ times (usually one takes $f(k) = 2$ or $f(k) = k$).

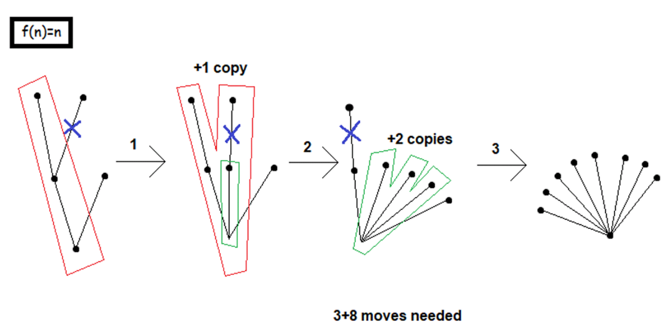


FIGURE 2. An example of hydra game: 8 more moves are needed to remove the remaining branches.

The game ends when Hercules has cut all the heads and only the body of the hydra is left in the battle (see Fig. 2). Using the ordinal theory mentioned above mathematicians have proven that:

Theorem 4.1. *Given any starting tree, the hydra game is won with any strategy in a finite number of steps (although this number may be unimaginably large).*

4.2. Connection with unredundant tree representation. An hydra tree can be represented also as an unredundant base ω tree in the following way: from each node one superposes the identical branches and then labels the unique remaining segment with the number of the original branches (see Fig. 3).

The resulting tree gives a simple strategy to solve the game which does not require more than basic induction: removing the same leaf from all identical trees to which it is attached means removing one segment from the base ω tree and increasing the label of the parent segment by a finite value (see for example Fig. 4). This means at each iteration of this process the number of segments of the trees is decreased by one, so that the tree will eventually be left with only the root.

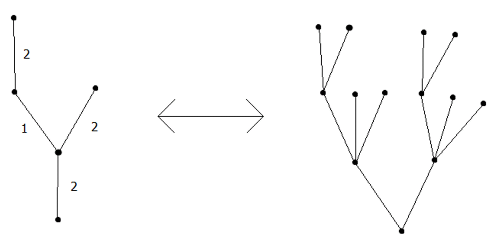


FIGURE 3. An example of hydra tree both in labeled and redundant variants.

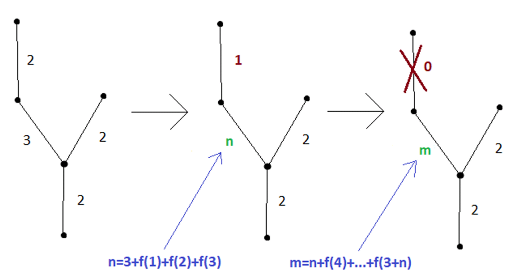


FIGURE 4. Simple strategy to solve the Hydra Game.

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