J. Appl. Math. & Informatics Vol. **40***(2022), No. 5 - 6, pp. 925 - 931*
https://doi.org/10.14317/jami.2022.925

ON THE CLASS OF TRANSMUTED-G DISTRIBUTIONS†

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Abstract. In this article, we compare the reliability and the hazard function between a baseline distribution and the corresponding transmuted-G distribution. Some examples based on existing transmuted-G distributions in literature are used. Three tests of parameter significance are utilized to test the importance of a transmuted-G distribution over the baseline distribution, and real data is used in an application of the inference about the importance of transmuted-G distributions.

AMS Mathematics Subject Classification : 62N02, 62N03, 62E10.
Key words and phrases : Transmuted-G, generalized distributions, exponentiated distributions, T-$X\{Y\}$, hazard, survival, reliability, test of significance.

1. Introduction

The transmuted-G (TG) distributions were originally proposed by Shaw and Buckley (2007)[12] using quadratic transmutation mapping as follows:
Let X be a continuous random variable with cumulative distribution function (CDF) $G(x)$ and probability density function (PDF) $g(x)$, then the CDF of the TG distribution is given by

$$F_{TG}(x) = (1+\lambda)G(x) - \lambda G^2(x), -1 \leq \lambda \leq 1. \quad (1)$$

The PDF of the TG distribution can be written as

$$f_{TG}(x) = g(x)(1 + \lambda - 2\lambda G(x)). \quad (2)$$

Many TG distributions appeared in literature since 2007, such as the transmuted-Weibull (TW) (Aryal and Tsokos, 2011)[4], the transmuted-Rayleigh (TR) (Merovci, 2013)[7], and the transmuted exponentiated additive Weibull (TEAW) (Nofal et al., 2018)[10]. Bourguignon et al. (2016)[6] discussed general properties of the

Received August 29, 2021. Revised February 10, 2022. Accepted March 19, 2022.
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†The work of the first author was supported by the Deanship of Scientific Research at the University of Jordan.

TG distributions, such as the fact that the CDF in (1) can be expressed as a linear mixture between the baseline CDF $G(x)$ and the exponentiated CDF $G(x)^2$. Bourguignon et al. (2016)[6] discussed also the asymptots and shapes, moments and generating functions, mean deviations, and multivariate extensions of the TG distributions, for any baseline CDF $G(x)$.

Let X and T be continuous random variables with CDFs $F_X(x)$ and $R_T(x)$ and PDFs $f_X(x)$ and $r_T(x)$, respectively, where $support(T) = [a, b]$. Alzaatreh et al. (2013)[1] defined the T-$X(W)$ class of distributions by using a transformation $W(F_X(x))$ of the CDF $F_X(x)$ that satisfies the following conditions:

$$\begin{cases} (i) & W\left[F_X(x)\right] \in [a, b], \\ (ii) & W\left[F_X(x)\right] \text{ is differentiable and monotonically non-decreasing, and} \\ (iii) & W\left[F_X(x)\right] \to a \text{ as } x \to -\infty \text{ and } W\left[F_X(x)\right] \to b \text{ as } x \to \infty. \end{cases} \tag{3}$$

The class of T-$X(W)$ distributions is defined by

$$G_{TX(W)}(x) = \int_a^{W(F_X(x))} r_T(t)dt. \tag{4}$$

Aljarrah et al. (2014)[2] defined the CDF of the T-$X\{Y\}$ frame as follows:

$$G_{TX\{Y\}}(x) = \int_a^{Q_Y(F_X(x))} f_T(t)dt, \tag{5}$$

where X, Y and T are random variables with CDFs $F_X(x)$, $F_Y(x)$ and $F_T(x)$, PDFs $f_X(x)$, $f_Y(x)$ and $f_T(x)$, respectively, $support(T) = support(Y) = [a, b]$, and $Q_Y()$ is the quantile function of Y. It is worth mentioning that the TG CDF in (1) can be obtained from the T-$X\{Y\}$ frame by using $f_T(x) = 1 + \lambda - 2\lambda x$ and $f_Y(x) = x$, on the support of $[0, 1]$. Therefore, we may consider any TG distribution in (1) as a member of of the T-$X\{Y\}$ frame in (5).

Furthermore, Merovci et al. (2016)[8] defined another generalized transmuted G (AGT-G) family by using $r_T(x) = 1 + \lambda - 2\lambda x$ and $W(F_X(x)) = (1 - (1 - F_X(x))^\alpha)$ in (4). The CDF for the AGTG family is given by

$$G_{AGT-G}(x) = (1 + \lambda)(1 - G(x))^\alpha - \lambda(1 - G(x))^{2\alpha}, -1 \leq \lambda \leq 1, \alpha > 0. \tag{6}$$

Merovci et al. (2016)[8] studied general properties of the AGT-G family, such as asymptotes and shapes, useful expansions, moments, mean deviations, entropy, characterization properties, and order statistics.

In section 2 of this article, we discuss a re-parametrization of the transmuted-G family (1). In section 3, we discuss inference about the parameter λ to compare a transmuted TG distribution to its baseline G distribution. In section 4, we compare two existing nested transmuted distribution through inference about a distribution parameter. In section 5, we use two real data sets in application of the inference from sections 3 and 4. Finally, in section 6, we conclude this article.

2. Re-parametrization of the transmuted-G distributions

Let X be a continouos random variable with CDF $G(x)$, reliability/survival funciton $R(x) = 1 - G(x)$, PDF $g(x)$, and hazard rate function (HRF) $h(x) = g(x)/R(x)$. By using $\alpha = -\lambda$ in (1), (Bourguignon et al., 2016)[6]), the CDF of the TG distribution in (1) can be rewritten as

$$G_\alpha(x) = (1-\alpha)G(x) + \alpha G^2(x) = G(x)(1-\alpha R(x)), -1 \leq \alpha \leq 1. \tag{7}$$

The reliability, PDF and HRF of (7) can be written, respectively, as

$$R_\alpha(x) = 1 - G_\alpha(x) = R(x)(1 + \alpha G(x)) \tag{8}$$

$$g_\alpha(x) = G'_\alpha(x) = g(x)(1 - \alpha + 2\alpha G(x)) \tag{9}$$

$$h_\alpha(x) = \frac{g_\alpha(x)}{R_\alpha(x)} = \frac{g(x)(1 + \alpha G(x) + \alpha G(x) - \alpha)}{R(x)(1 + \alpha G(x))} = h(x)(1 - \alpha\frac{R(x)}{1 + \alpha G(x)}) \tag{10}$$

3. Inference about λ

Note that when $\lambda = 0$, then (1) reduces to $F_{TG}(x) = G(x)$. Hence, we can compare a TG distribution to its baseline distribution, G, by testing

$$\begin{cases} H_0 : \lambda = 0, \\ H_1 : \lambda \neq 0. \end{cases} \tag{11}$$

In this section, the hypotheses in (11) are tested using the likelihood ratio test (Neyman and Pearson, 1928)[9], the Wald test (Wald, 1943)[13], or the score test (Rao, 1948)[11]. For testing the hypotheses in (11), the likelihood ratio test statistic $LR = -2[_G(\tilde{\Phi}) - {}_{TG}(\hat{\Theta})]$, which has asymptotic chi-square distribution with one degree of freedom, where $_G(\tilde{\Phi})$ and $_{TG}(\hat{\Theta})$ are the log-likelihood values for the G and TG distributions, respectively. The Wald test statistic for (11), $Z_{\hat{\lambda}} = \hat{\lambda}/SE_{\hat{\lambda}}$ has approximate standard normal distribution, where $\hat{\lambda}$ is the maximum likelihood estimate (MLE) of λ from the TG distribution, and $SE_{\hat{\lambda}}$ is the standard error of $\hat{\lambda}$. The score test statistic for (11) is $S = V^T I^{-1} V$, where V is the gradient vector, $I = -E[H]$ is the information matrix and H is the hessian matrix for the TG distribution. The test statistic S is calculated under the null hypothesis and is approximately chi-square with one degree of freedom.

4. Inference about TW and TR

Two examples from the transmuted-G class are the transmuted-Weibull distribution (TW) (Aryal and Tsokos, 2011)[4] with PDF

$$f_{TW}(x) = \frac{\eta}{\sigma}(\frac{x}{\sigma})^{\eta-1}exp(-(\frac{x}{\sigma})^{\eta})(1-\lambda+2\lambda exp(-(\frac{x}{\sigma})^{\eta})), -1 \leq \lambda \leq 1. \quad (12)$$

and its sub-model when $\eta = 2$, the transmuted-Raleigh distribution (TR) (Merovci, 2013)[7], which has a PDF that can be rewritten as

$$f_{TR}(x) = \frac{x}{\sigma^2}exp(-\frac{x^2}{2\sigma^2})(1-\lambda+2\lambda exp(-\frac{x^2}{2\sigma^2}), -1 \leq \lambda \leq 1, \quad (13)$$

A comparison between a TW distribution to its sub-model, TR, can be performed by testing

$$\begin{cases} H_0 : \eta = 2, \\ H_1 : \eta \neq 2. \end{cases} \quad (14)$$

In a similar way to inference about λ, the hypotheses in (14) can be tested by using the likelihood ratio test statistic $LR = -2[{}_{TR}(\tilde{\sigma}, \tilde{\lambda}) - {}_{TW}(\hat{\eta}, \hat{\sigma}, \hat{\lambda})]$, which is asymptotically chi-square with one degree of freedom, where ${}_{TR}(\tilde{\sigma}, \tilde{\lambda})$ and ${}_{TW}(\hat{\eta}, \hat{\sigma}, \hat{\lambda})$ are the log-likelihood values of TR and TW, respectively; The Wald test statistic for (14), $Z_{\hat{\eta}} = (\hat{\eta}-2)/SE_{\hat{\eta}}$ has approximate standard normal distribution, where $\hat{\eta}$ is the MLE of the TW; The score test statistic for (14), $S = V^T I^{-1} V$, is calculated for the log-likelihood function of the TW under the null hypothesis $\eta = 2$, and is approximately chi-square with one degree of freedom.

5. Application

The data set used here is on the lifetime of Kevlar 49/epoxy strands when subjected to constant sustained pressure at the 90% stress level until all units failed. The data is obtained from Andrews and Herzberg (1985)[3] and Barlow et al. (1984)[5], has 101 observations and is highly right skewed ($skewness = 3.05$ and $kurtosis = 14.47$). The data is used in application of the inference about λ for the TW distribution, and inference about η when comparing TW and TR distributions. The TW, TR and the Weibull distribution (WD) ($F_{WD} = 1 - exp(-(x/\sigma)^{\eta})$) are fitted using the NLMIXED procedure in SAS. The fitted distributions are compared based on the p-value of Kolmogorov-Smirnov (K-S) statistic, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The MLEs and goodness of fit statistics are in table 1. The histogram and the fitted PDFs as well as the empirical CDF and the fitted CDFs are in figure 1.

By looking at the goodness of fit statistics, we can see that TW provides the best fit based on the K-S statistics, but the WD is better based on AIC and BIC statictics. Further, we can observe that the TR fit is inadequate. When comparing the TW to the WD using this data, all the three tests are insignificant,

TABLE 1. MLEs (standard errors in parentheses) for the glass fibers data.

Distribution	WD	TW	TR
Parameter estimates (standard error)	$\hat{\sigma}$ =0.9899 (0.1118) $\hat{\eta}$ =0.9259 (0.0726)	$\hat{\sigma}$ =0.8227 (0.1973) $\hat{\eta}$ =0.8664 (0.1036) $\hat{\lambda}$ =-0.2940 (0.3280)	$\hat{\sigma}$ =1.2563 (0.0733) $\hat{\lambda}$ =0.8119 (0.0933)
Log Likelihood	-102.9768	-102.6701	-164.2631
AIC	210.0	211.3	332.5
BIC	215.2	219.2	337.8
K-S (p-value)	0.0906 (0.3778)	0.0832 (0.4861)	0.2461 (9.75E-6)

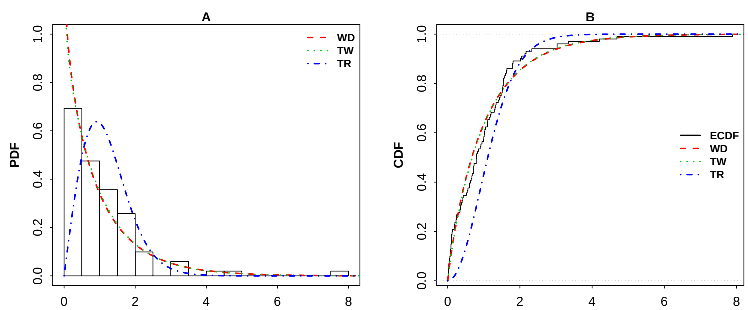

FIGURE 1. Kevlar 49 at 90 percent using K-S statistics

($LR = 0.6334, p.value = 0.4261$), ($Z_{\hat{\lambda}} = -0.8963, p.value = 0.3701$) and ($S = 2.3564, p.value = 0.1248$). Hence the TW is as good as the WD. On the contrary, when comparing the TW to its sub-model, the TR, all three tests are significant, ($LR = 123.2, p = 0$), ($Z_{\hat{\eta}} = -10.94, p = 0$) and ($S = 161.58, p = 0$). Henceforth, the TW is better than TR for fitting this data.

6. Concluding remarks

In this article, we discuss the relation between a transmuted-G distribution to its baseline G distribution. When $0 \leq \alpha \leq 1$, the transmuted-G distribution has higher reliability and smaller hazard rate than its baseline G. On the other hand, when $-1 \leq \alpha \leq 0$, then the reliability is smaller but the hazard rate is higher for the transmuted-G than for G. Three tests are used in an inference of the transmutation importance as well as in comparison between TW and its submodel TR. The data set used shows that TW is as good as Weibull, however, on the other hand, when comparing TW to TR, the TW provided better fit.

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