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# SUFFICIENT OSCILLATION CONDITIONS FOR DYNAMIC EQUATIONS WITH NONMONOTONE DELAYS

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ABSTRACT. In this article, we analyze the first order delay dynamic equations with several nonmonotone arguments. Also, we present new oscillation conditions involving lim sup and lim inf for the solutions of these equations. Finally, we give an example to demonstrate the results.

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## 1. Introduction

After Stefan Hilger published his Ph.D. thesis in 1988 about the theory of dynamic equations on time scales (or measure chain) [15]-[16], this theory has managed to attract the remarkable attention of many scientists studying differential-difference equations. Both partial and ordinary differential-difference, dynamic equations, with or without delay, are very commonly employed to model real problem coming from, physical electronics, nuclear physics, optics and astrophysics, biomathematics, and also from probability when full moment problems are involved; see [13]-[17]-[21]. Also, the oscillation of first-order delay dynamic equations have numerous applications in the study of oscillatory behavior of higher-order dynamic equations; please see for more detail, [7].
Consider the first order delay dynamic equation with several nonmonotone arguments

$$x^{\Delta}(t)+\sum_{i=1}^{m} p_i(t)x\left(\phi_i(t)\right)=0, \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$, $p_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, $\phi_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are not necessarily monotone for $1\le i\le m$ such that

$$\phi_i(t)\le t \ \text{ for all } t\in\mathbb{T}, \quad \lim_{t\to\infty}\phi_i(t)=\infty. \tag{2}$$

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First of all, we recall some information and basic concepts on time scales calculus. If a function $p:\mathbb{T}\to\mathbb{R}$ is rd-continuous and satisfies $1+\mu(t)p(t)>0$ for all $t\in\mathbb{T}$, where $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is the graininess function defined by $\mu(t):=\sigma(t)-t$ with the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ defined by $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ for $t\in\mathbb{T}$, it is called positively regressive (we write $p\in\mathcal{R}^+$). If $\sigma(t)=t$ or $\mu(t)=0$, a point $t\in\mathbb{T}$ is called right-dense, otherwise it is called right-scattered.
The $\Delta$-derivative $x^\Delta$ for a function $x$ described on $\mathbb{T}$, then we have
(i) if $\mathbb{T}=\mathbb{R}$, $x^\Delta=x'$ is the usual derivative
(ii) if $\mathbb{T}=\mathbb{Z}$, $x^\Delta=\Delta x$ is the usual forward operator.
If a function $x:\mathbb{T}\to\mathbb{R}$ is delta differentiable for $t\in\mathbb{T}^\kappa$ and satisfies equation (1) for $t\in\mathbb{T}^\kappa$, it is called a solution of the equation (1). We say that a solution $x$ of equation (1) has a generalized zero at $t$ if $x(t)=0$ or if $\mu(t)>0$ and $x(t)x(\sigma(t))<0$. Let $\sup\mathbb{T}=\infty$ and then a nontrivial solution $x$ of equation (1) is called *oscillatory* on $[t,\infty)$ if it has arbitrarily large generalized zeros in $[t,\infty)$.
For more comprehensive information, we recommend Bohner and Peterson's monographs [4], [5], which summarize and organize this topic, to the readers.
From above statements, for $\mathbb{T}=\mathbb{R}$, Eq. (1) reduces to the first order delay differential equation

$$x'(t)+\sum_{i=1}^{m}p_i(t)x(\phi_i(t))=0, \quad t\in\mathbb{R}. \tag{3}$$

In 2016, Braverman et al. [3] obtained the following conditions for the oscillation of (3). Assume that (2) is satisfied and $\phi_i(t)$ are not necessarily monotone for $1\le i\le m$. Also,

$$\psi_i(t)=\sup_{0\le s\le t}\{\phi_i(s)\},\ \psi(t)=\max_{1\le i\le m}\{\psi_i(t)\},\ t\ge 0. \tag{4}$$

If

$$\liminf_{t\to\infty}\int_{\psi(t)}^{t}\sum_{i=1}^{m}p_i(s)ds>\frac{1}{e} \tag{5}$$

or

$$\limsup_{t\to\infty}\int_{\psi(t)}^{t}\sum_{i=1}^{m}p_i(s)ds>1, \tag{6}$$

then all solutions of (3) are oscillatory.
In 2016, J. P. Dix and J. G. Dix [10] found out that if

$$\liminf_{t\to\infty}\int_{\psi(t)}^{t}\sum_{i=1}^{m}p_i(s)\exp\left\{\int_{\phi_i(s)}^{\psi(s)}\sum_{j=1}^{m}p_j(r)dr\right\}ds>\frac{1}{e} \tag{7}$$

or

$$\limsup_{t\to\infty}\int_{\psi(t)}^{t}\sum_{i=1}^{m}p_i(s)\exp\left\{\int_{\phi_i(s)}^{\psi(t)}\sum_{j=1}^{m}p_j(r)dr\right\}ds>1, \tag{8}$$

where $\phi(t)=\max_{1\le i\le m}\{\phi_i(t)\}$, $\psi(t)=\max_{s\le t}\{\phi(s)\}$, then all solutions of (3) are oscillatory.

Furthermore, you can see some results about equation (3) in [8] and [9].

If we take $\mathbb{T}=\mathbb{Z}$, Eq. (1) reduces to the first order delay difference equation

$$\Delta x(t)+\sum_{i=1}^{m}p_i(t)x(\phi_i(t))=0,\ t\in\mathbb{Z}. \tag{9}$$

In 2006, Berezansky and Braverman [1] obtained that if $(\phi_i(t))$ are not necessarily monotone for $1\le i\le m$ and

$$\limsup_{t\to\infty}\sum_{i=1}^{m}p_i(t)>0 \quad \text{and} \quad \liminf_{t\to\infty}\sum_{j=\phi(t)}^{t-1}\sum_{i=1}^{m}p_i(j)>\frac{1}{e}, \tag{10}$$

where $\phi(t)=\max_{1\le i\le m}\{\phi_i(t)\}$ for all $t\ge 0$, then all solutions of (9) are oscillatory.

In 2015, Braverman et al. [2], established the following result.

If $(\phi_i(t))$ are not necessarily monotone for $1\le i\le m$ and

$$\limsup_{t\to\infty}\sum_{j=\psi(t)}^{t}\sum_{i=1}^{m}p_i(j)\prod_{l=\phi_i(j)}^{\psi(t)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}>1, \tag{11}$$

where $\psi_i(t)=\max_{s\le t}\{\phi_i(s)\}$, $\psi(t)=\max_{1\le i\le m}\{\psi_i(t)\}$, then all solutions of (9) are oscillatory.

Clearly, the following result is obtained by (11) immediately; if $(\phi_i(t))$ are not necessarily monotone for $1\le i\le m$ and

$$\limsup_{t\to\infty}\sum_{j=\psi(t)}^{t}\sum_{i=1}^{m}p_i(j)>1,$$

where $\psi_i(t)=\max_{s\le t}\{\phi_i(s)\}$, $\psi(t)=\max_{1\le i\le m}\{\psi_i(t)\}$, then all solutions of (9) are oscillatory.

In 2020, Kılıç and Öcalan [19] established following result.

If $(\phi_i(t))$ are not necessarily monotone for $1\le i\le m$ and

$$\liminf_{t\to\infty}\sum_{j=\phi(t)}^{t-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\phi_i(j)}^{\psi(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}>\frac{1}{e}, \tag{12}$$

where $\psi_i(t) = \max_{s\leq t}\{\phi_i(s)\}$, $\psi(t) = \max_{1\leq i\leq m}\{\psi_i(t)\}$, then all solutions of (9) are oscillatory.

For $m = 1$ in Eq. (1), we have

$$x^{\Delta}(t) + p(t)x(\phi(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{13}$$

When $\phi(t)$ is not necessarily monotone, Öcalan et al. [22] studied the equation (13), then they obtained the following result.

If

$$\limsup_{t\to\infty} \int_{\psi(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{14}$$

where $\psi(t) = \sup_{s\leq t}\{\phi(s)\}$, then all solutions of (13) are oscillatory.

In 2020, Öcalan [23] established the following oscillation criteria in the general case that the delay argument $\phi(t)$ is not necessarily monotone.

If $-p \in \mathcal{R}^+$ and

$$\limsup_{t\to\infty} \int_{\psi(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}(\psi(t), \phi(s))}\Delta s > 1 \tag{15}$$

or

$$\liminf_{t\to\infty} \int_{\phi(t)}^{t} \frac{p(s)}{e_{-p}(\psi(s), \phi(s))}\Delta s > \frac{1}{e}, \tag{16}$$

where $\psi(t) = \sup_{s\leq t}\{\phi(s)\}$,

$$e_{-\lambda p}(t, \phi(t)) = \exp\left\{ \int_{\phi(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s \right\}$$

and

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & , \text{ if } h \neq 0 \\ z & , \text{ if } h = 0 \end{cases}$$

then all solutions of (13) are oscillatory.

Moreover, (16) implies that if $\phi(t)$ is not necessarily monotone and

$$\liminf_{t\to\infty} \int_{\phi(t)}^{t} p(s)\Delta s > \frac{1}{e}, \tag{17}$$

where $\psi(t) = \sup_{s\leq t}\{\phi(s)\}$, then all solutions of (13) are oscillatory.

Very recently, Kılıç and Öcalan [18] presented the following results which are the first results for equation (1) with nonmonotone arguments in the literature.

Suppose that $-\sum_{i=1}^{m} p_i \in \mathcal{R}^+$. If $\phi_i(t)$ are not necessarily monotone for $1 \le i \le m$ and

$$\limsup_{t\to\infty} \int_{\psi(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s)\Delta s > 1 \tag{18}$$

or

$$\liminf_{t\to\infty} \int_{\phi(t)}^{t} \sum_{i=1}^{m} p_i(s)\Delta s > \frac{1}{e}, \tag{19}$$

where $\psi_i(t) = \sup_{s\le t} \{\phi_i(s)\}$ and $\psi(t) = \max_{1\le i\le m} \{\psi_i(t)\},\ t \in \mathbb{T},\ t \ge 0.$
$\phi(t) = \max_{1\le i\le m} \{\phi_i(t)\}$, then all solutions of (1) oscillate.
As seen above differential and difference equations with several arguments and also, dynamic equations with one delay have been analyzed by several authors, while dynamic equations with several deviating arguments have been studied rarely. Thus, in this paper our aim is to essentially develop these results under the assumptions that $\phi_i(t)$ are not necessarily monotone for $1 \le i \le m$ and also to generalize the results (15) and (16) which are obtained for (13) to (1).

## 2. Main results

In this section, we present some sufficient conditions for the oscillation of all solutions of (1).
Set

$$\psi_i(t) = \sup_{s\le t} \{\phi_i(s)\} \quad \text{and} \quad \psi(t) = \max_{1\le i\le m} \{\psi_i(t)\},\ t \in \mathbb{T},\ t \ge 0. \tag{20}$$

Obviously, $\psi_i(t)$ are nondecreasing and $\phi_i(t) \le \psi_i(t) \le \psi(t)$ for all $1 \le i \le m$, $t \ge 0$.
The following result was given in [24].

**Lemma 2.1.** *Assume that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $g : \mathbb{T} \to \mathbb{R}$ is nonincreasing and $\phi : \mathbb{T} \to \mathbb{T}$ is nondecreasing. If $b < u$, then*

$$\int_{b}^{\sigma(u)} f(s)g(\phi(s))\Delta s \ge g(\phi(u)) \int_{b}^{\sigma(u)} f(s)\Delta s. \tag{21}$$

The following lemmas can be easily obtained from [6].

**Lemma 2.2.** *Assume that $-\sum_{i=1}^{m} p_i \in \mathcal{R}^+$. Then, we get*

$$e_{-\sum_{i=1}^{m} p_i}(t,s) \le \exp\left\{ -\int_{s}^{t} \sum_{i=1}^{m} p_i(u)\Delta u \right\},\ s \le t. \tag{22}$$

**Lemma 2.3.** *Assume that* $-\sum_{i=1}^{m} p_i \in \mathcal{R}^+$. *If*

$$x^{\Delta}(t) + x(t)\sum_{i=1}^{m} p_i(t) \leq 0, \tag{23}$$

*then, we have*

$$x(t) \leq e_{-\sum_{j=1}^{m} p_j}(t,s)\, x(s), \quad \forall t \geq s. \tag{24}$$

**Theorem 2.4.** *Assume that* (2) *holds and* $-\sum_{i=1}^{m} p_i \in \mathcal{R}^+$. *If*

$$\limsup_{t\to\infty} \int_{\psi(t)}^{\sigma(t)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(t),\phi_i(s))} \Delta s > 1, \tag{25}$$

*where* $\psi(t)$ *is defined by* (20), *then all solutions of* (1) *are oscillatory.*

*Proof.* Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of (1). If $x(t)$ is an eventually negative solution of (1), the proof the theorem can be done similarly as below. Then, there exists $t_1 > t_0$ such that $x(t),\ x(\phi_i(t)) > 0$ for all $t \geq t_1$ and $1 \leq i \leq m$. Thus, from (1) we get

$$x^{\Delta}(t) = -\sum_{i=1}^{m} p_i(t) x(\phi_i(t)) \leq 0, \quad \forall\, t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function. Considering this and $\phi_i(t) \leq \psi(t) \leq t$ for $1 \leq i \leq m$, Eq. (1) gives

$$x^{\Delta}(t) + x(t)\sum_{i=1}^{m} p_i(t) \leq 0, \quad t \geq t_1 \tag{26}$$

and so we get the following statement from Lemma 2.3.

$$x(\psi(t)) \leq e_{-\sum_{j=1}^{m} p_j}(\psi(t),\phi_i(s)) x(\phi_i(s)), \text{ for all } 1 \leq i \leq m, \psi(t) \geq \phi_i(s). \tag{27}$$

Integrating (1) from $\psi(t)$ to $\sigma(t)$ and using (27), we get

$$x(\sigma(t)) - x(\psi(t)) + \int_{\psi(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) x(\phi_i(s)) \Delta s = 0,$$

$$x(\sigma(t)) - x(\psi(t)) + \int_{\psi(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \frac{x(\psi(t))}{e_{-\sum_{j=1}^{m} p_j}(\psi(t),\phi_i(s))} \Delta s \leq 0,$$

$$x(\sigma(t)) - x(\psi(t)) + x(\psi(t)) \int_{\psi(t)}^{\sigma(t)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(t),\phi_i(s))} \Delta s \leq 0,$$

$$x(\psi(t))\left[\int_{\psi(t)}^{\sigma(t)}\sum_{i=1}^{m}\frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(t),\phi_i(s))}\Delta s-1\right]\leq 0.$$

Therefore, we get

$$\limsup_{t\to\infty}\int_{\psi(t)}^{\sigma(t)}\sum_{i=1}^{m}\frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(t),\phi_i(s))}\Delta s\leq 1,$$

which contradicts to (25). So, the proof of the theorem is completed. □

The following result is easily obtained by using the similar way in the proof of Lemma 2.3 in [22].

**Lemma 2.5.** *Suppose that* (20) *holds. Then, we get*

$$\liminf_{t\to\infty}\int_{\psi(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s=\liminf_{t\to\infty}\int_{\phi(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s,$$

*where* $\phi(t)=\max\limits_{1\leq i\leq m}\{\phi_i(t)\}$, $t\in\mathbb{T}$, $t\geq 0$.

**Theorem 2.6.** *Suppose that* (2) *holds and* $-\sum\limits_{i=1}^{m}p_i\in\mathcal{R}^+$*. If*

$$\liminf_{t\to\infty}\int_{\phi(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(s),\phi_i(s))}\Delta s>\frac{1}{e},\tag{28}$$

*where* $\psi(t)$ *is defined by* (20) *and* $\phi(t)=\max\limits_{1\leq i\leq m}\{\phi_i(t)\}$, *then all solutions of* (1) *are oscillatory.*

*Proof.* Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of (1). If $x(t)$ is an eventually negative solution of (1), the proof the theorem can be done similarly as below. Then, there exists $t_1>t_0$ such that $x(t),\ x(\phi_i(t))>0$ for all $t\geq t_1$ and $1\leq i\leq m$. Thus, from (1) we have

$$x^{\Delta}(t)=-\sum_{i=1}^{m}p_i(t)x(\phi_i(t))\leq 0,\quad \forall t\geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function. By means of this and $\phi_i(t)\leq\psi(t)\leq t$ for $1\leq i\leq m$, Eq. (1) gives

$$x^{\Delta}(t)+x(t)\sum_{i=1}^{m}p_i(t)\leq 0,\quad t\geq t_1.$$

So, we have Lemma 2.3. Then, by using this we get

$$x(\psi(t))\leq e_{-\sum\limits_{j=1}^{m}p_j}(\psi(t),\phi_i(t))\,x(\phi_i(t))\text{ for all }1\leq i\leq m,\psi(t)\geq\phi_i(t).\tag{29}$$

Hence, by taking into account that $x(t)$ is nonincreasing, $\psi(t) \le t$ and (29), from (1) we obtain

$$x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t) x(\phi_i(t)) = 0,$$

$$x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t) \frac{x(\psi(t))}{e_{-\sum_{j=1}^{m} p_j}(\psi(t), \phi_i(t))} \le 0, \tag{30}$$

$$x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t) \frac{x(t)}{e_{-\sum_{j=1}^{m} p_j}(\psi(t), \phi_i(t))} \le 0. \tag{31}$$

We define $\sum_{i=1}^{m} q_i(t) := \sum_{i=1}^{m} \frac{p_i(t)}{e_{-\sum_{j=1}^{m} p_j}(\psi(t), \phi_i(t))}$.

So from (31), we obtain

$$x^{\Delta}(t) + x(t) \sum_{i=1}^{m} q_i(t) \le 0. \tag{32}$$

Hence, by Lemma 2.3 and (32) we have

$$x\,(t) \le e_{-\sum_{i=1}^{m} q_i}(t, \psi(t))\, x(\psi(t)) \text{ for all } \ t \ge \psi(t)$$

and

$$\frac{x(\psi(t))}{x\,(t)} \ge \frac{1}{e_{-\sum_{i=1}^{m} q_i}(t, \psi(t))}. \tag{33}$$

On the other hand, we know the following one from Lemma 2.2.

$$e_{-\sum_{i=1}^{m} q_i}(t, \psi(t)) \le \exp\left\{ \int_{\psi(t)}^{t} \sum_{i=1}^{m} (-q_i(s))\, \Delta s \right\}. \tag{34}$$

Hence, from (33) and (34), we get

$$\frac{x(\psi(t))}{x\,(t)} \ge \exp\left\{ \int_{\psi(t)}^{t} \sum_{i=1}^{m} q_i(s) \Delta s \right\} = \exp\left\{ \int_{\psi(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} \Delta s \right\}. \tag{35}$$

Moreover, from (28) and Lemma 2.5, there exists a constant $c > 0$ such that

$$\int_{\psi(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} \Delta s \ge c > \frac{1}{e}, \quad t \ge t_2 \ge t_1. \tag{36}$$

Combining the inequalities (35) and (36), we have

$$\frac{x(\psi(t))}{x\,(t)} \geq e^{c}. \tag{37}$$

Since $e^{x} \geq ex$ for all $x \in \mathbb{R}$, from (37), we obtain

$$\frac{x\,(\psi(t))}{x\,(t)} \geq e^{c} \geq ec, \quad t \geq t_2, \tag{38}$$

where $ec > 1$. By using (38) in (30), it follows by induction that for any positive integer $k$, we get

$$\frac{x\,(\psi(t))}{x\,(t)} \geq (ec)^{k} \quad \text{for sufficiently large } t. \tag{39}$$

On the other hand, from (36), there exists $t^* \in [\psi(t), t)$, $t^* \in \mathbb{T}$ such that

$$\int\limits_{\psi(t)}^{\sigma(t^*)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} \Delta s \geq \frac{c}{2} \quad \text{and} \quad \int\limits_{t^*}^{\sigma(t)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} \Delta s \geq \frac{c}{2}. \tag{40}$$

Integrating (1) from $\psi(t)$ to $\sigma(t^*)$, using Lemma 2.3, we get

$$x(\sigma(t^*)) - x\,(\psi(t)) + \int\limits_{\psi(t)}^{\sigma(t^*)} \sum_{i=1}^{m} p_i(s) x\,(\phi_i(s))\,\Delta s = 0,$$

$$x(\sigma(t^*)) - x\,(\psi(t)) + \int\limits_{\psi(t)}^{\sigma(t^*)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} x\,(\psi(s))\,\Delta s \leq 0.$$

Then, from Lemma 2.1 and (40), we have

$$-x\,(\psi(t)) + x\,(\psi(t^*)) \int\limits_{\psi(t)}^{\sigma(t^*)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} \Delta s \leq 0,$$

$$x\,(\psi(t)) \geq \frac{c}{2} x\,(\psi(t^*))\,. \tag{41}$$

Integrating (1) from $t^*$ to $\sigma(t)$ and using the same facts, we obtain

$$x(\sigma(t)) - x\,(t^*) + \int\limits_{t^*}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) x\,(\phi_i(s))\,\Delta s = 0,$$

$$x(\sigma(t)) - x\,(t^*) + \int\limits_{t^*}^{\sigma(t)} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m} p_j}(\psi(s), \phi_i(s))} x\,(\psi(s))\,\Delta s \leq 0.$$

Then, from Lemma 2.1 and (40), we get

$$-x\left(t^{*}\right)+x\left(\psi(t)\right)\int_{t^{*}}^{\sigma(t)}\sum_{i=1}^{m}\frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m}p_j}\left(\psi(s),\phi_i(s)\right)}\Delta s\leq 0,$$

$$x\left(t^{*}\right)\geq\frac{c}{2}x\left(\psi(t)\right). \tag{42}$$

Combining the inequalities (41) and (42), we have

$$x\left(t^{*}\right)\geq\frac{c}{2}x(\psi(t))\geq\frac{c}{2}\frac{c}{2}x\left(\psi(t^{*})\right)$$

or

$$\frac{x\left(\psi(t^{*})\right)}{x\left(t^{*}\right)}\leq\left(\frac{2}{c}\right)^{2}<+\infty$$

and this contradicts with (39). So, the proof of the theorem is completed. □

**Example 2.7.** Let $m=2$ and $\mathbb{T}=2\mathbb{Z}=\{2k:k\in\mathbb{Z}\}$. Then, we obtain

$$\sigma(t)=t+2,\quad \mu(t)=2\ \text{ and }\ x^{\Delta}(t)=\frac{x(t+2)-x(t)}{2}$$

for $t\in\mathbb{T}$. Thus, Eq. (1) becomes

$$\frac{x(t+2)-x(t)}{2}+p_1(t)x\left(\phi_1(t)\right)+p_2(t)x\left(\phi_2(t)\right)=0,\quad t\in\{2k:k\in\mathbb{Z}\}.$$

Letting $\phi_1(t)=t-2,\phi_2(t)=t-4$, then $\phi(t)=\max\limits_{1\leq i\leq m}\{\phi_i(t)\}=\phi_1(t)=t-2$. Since $p_i(t)\in\{2k:k\in\mathbb{Z}\}$, we assume

$$p_1(t)=0.12 \text{ and } p_2(t)=0.06,\quad t=0,2,4,\ldots$$

When $\mathbb{T}=h\mathbb{Z}$, from (iii) in Theorem 1.79 [4], we have the following.

$$\int_{a}^{b}f(t)\Delta t=\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}f(kh)h\ \text{ for }\ a<b. \tag{43}$$

So, by using (43), we observe that, for $\phi(t)$, $p_i(t)\in\{2k:k\in\mathbb{Z}\}$

$$\begin{aligned}\liminf_{t\to\infty}\int_{\phi(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s &= \liminf_{t\to\infty}\sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1}\sum_{i=1}^{2}2p_i(2j)\\ &= \liminf_{t\to\infty}\sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1}\left[2p_1(2j)+2p_2(2j)\right]\\ &= \liminf_{t\to\infty}2\left[p_1(t-2)+p_2(t-2)\right]\\ &= 0.36<\frac{1}{e}.\end{aligned}$$

It means that (19) doesn't hold. However, by using (43), we have

$$\liminf_{t\to\infty}\int\limits_{\phi(t)}^{t}\sum_{i=1}^{m}\frac{p_i(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(s),\phi_i(s))}\Delta s=\liminf_{t\to\infty}\sum_{k=\frac{t-2}{2}}^{\frac{t}{2}-1}\sum_{i=1}^{m}\frac{2p_i(2k)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(2k),\phi_i(2k))}.$$

Since $\phi_1(t)=\psi(t)$,

$$e_{-\sum\limits_{j=1}^{m}p_j}(\psi(s),\phi_1(s))=1$$

then, we get

$$\int\limits_{\psi(t)}^{t}\frac{p_1(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(s),\phi_1(s))}\Delta s=\int\limits_{\psi(t)}^{t}p_1(s)\Delta s=\sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1}2p_1(2j)=2p_1(t-2)=0.24.$$

Also, we have

$$\begin{aligned}
e_{-\sum\limits_{j=1}^{m}p_j}(\psi(2k),\phi_2(2k)) &= \exp\left\{\int\limits_{\phi_2(2k)}^{\psi(2k)}\xi_{\mu(u)}\sum_{j=1}^{m}(-p_j(u))\Delta u\right\}\\
&= \exp\left\{\int\limits_{\phi_2(2k)}^{\psi(2k)}\xi_{\mu(u)}(-(p_1(u)+p_2(u)))\Delta u\right\}\\
&= \exp\left\{\sum_{i=\frac{\phi_2(2k)}{2}}^{\frac{\psi(2k)}{2}-1}\frac{2\log(1-\mu(2i)(p_1(2i)+p_2(2i)))}{\mu(2i)}\right\}\\
&= \exp\left\{\sum_{i=\frac{\phi_2(2k)}{2}}^{\frac{\psi(2k)}{2}-1}\log(1-2(p_1(2i)+p_2(2i)))\right\}\\
&= \exp\left\{\log\prod_{i=\frac{\phi_2(2k)}{2}}^{\frac{\psi(2k)}{2}-1}(1-2(p_1(2i)+p_2(2i)))\right\}\\
&= \prod_{i=\frac{\phi_2(2k)}{2}}^{\frac{\psi(2k)}{2}-1}(1-2(p_1(2i)+p_2(2i)))
\end{aligned}$$

and then,

$$\int\limits_{\psi(t)}^{t}\frac{p_2(s)}{e_{-\sum\limits_{j=1}^{m}p_j}(\psi(s),\phi_2(s))}\Delta s \quad = \quad \sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1}2p_2(2j)\prod_{i=\frac{\phi_2(2j)}{2}}^{\frac{\psi(2j)}{2}-1}\frac{1}{(1-2(p_1(2i)+p_2(2i)))}$$

$$
\begin{aligned}
&= \sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1} 2p_2(2j) \prod_{i=j-2}^{j-2} \frac{1}{(1-2(p_1(2i)+p_2(2i)))} \\
&= 2p_2(t-2)\frac{1}{(1-2(p_1(t-6)+p_2(t-6)))} \\
&= 2(0.06)\frac{1}{0.64} = 0.1875.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\liminf_{t\to\infty} \int_{\phi(t)}^{t} \sum_{i=1}^{m} \frac{p_i(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(s),\phi_i(s))} \Delta s \\
&= \liminf_{t\to\infty} \left[ \int_{\psi(t)}^{t} \frac{p_1(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(s),\phi_1(s))} \Delta s + \int_{\psi(t)}^{t} \frac{p_2(s)}{e_{-\sum_{j=1}^{m} p_j}(\psi(s),\phi_2(s))} \Delta s \right] \\
&= 0.24 + 0.1875 = 0.4275 > \frac{1}{e}.
\end{aligned}
$$

So, this implies that all conditions of Theorem 2.6 are satisfied and every solution of this dynamic equation oscillates.

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