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COINCIDENCE POINT AND FIXED POINT THEOREMS IN PARTIAL METRIC SPACES FOR CONTRACTIVE TYPE MAPPINGS WITH APPLICATIONS

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Abstract. The purpose of this article is to establish some fixed point theorems, a common fixed point theorem and a coincidence point theorem via contractive type condition in the framework of complete partial metric spaces and give some examples in support of our results. As an application to the results, we give some fixed point theorems for integral type contractive conditions. The results presented in this paper extend and generalize several results from the existing literature.

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1. Introduction

The notion of partial metric space was originally developed by Matthews ([12, 13]) to provide mechanism generalizing metric space theories. A partial metric space is an extension of metric by replacing the condition $d(x,x) = 0$ of the (usual) metric with the inequality $d(x,x) \leq d(x,y)$ for all x, y. Also, this concept provide to study denotational semantics of dataflow networks [2, 5, 9, 12, 13, 14, 17, 19]. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews extended the Banach contraction principle [4] and proved the fixed point theorem in this space.

Matthews gave some basic definitions and properties on partial metric space such as Cauchy sequence, convergent sequence etc. Due to importance of the fixed point theory it is very interesting to study some fixed point theorems on different concepts.

Now, we give some basic structures and results on the concept of partial metric space (PMS).

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Definition 1.1. ([13]) Let X be a nonempty set and $p\colon X\times X\to\mathbb{R}^+$ be such that for all $x,y,z\in X$ the followings are satisfied:

- $(P1)$ $x=y$ if and only if $p(x,x)=p(x,y)=p(y,y)$ (equality),
- $(P2)$ $p(x,x)\le p(x,y)$ (small self-distance),
- $(P3)$ $p(x,y)=p(y,x)$ (symmetry),
- $(P4)$ $p(x,y)\le p(x,z)+p(z,y)-p(z,z)$ (triangularity).

Then p is called a partial metric on X and the pair (X,p) is called a partial metric space.

Remark 1.1. It is clear that if $p(x,y)=0$, then $x=y$. But, on the contrary $p(x,x)$ need not be zero.

Example 1.2. ([3]) Let $X=\mathbb{R}^+$ and $p\colon X\times X\to\mathbb{R}^+$ given by $p(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}^+$. Then $(\mathbb{R}^+,p)$ is a partial metric space.

Example 1.3. ([3]) Let $X=\{[a,b]:a,b\in\mathbb{R},a\le b\}$. Then $p\Big([a,b],[c,d]\Big)=\max\{b,d\}-\min\{a,c\}$ defines a partial metric p on X.

Various applications of this space has been extensively investigated by many authors (see [10], [18] for details).

Remark 1.2. ([8]) Let (X,p) be a partial metric space.

- $(A1)$ The function $d^s\colon X\times X\to\mathbb{R}^+$ defined as $d^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$ is a (usual) metric on X and (X,d^s) is a (usual) metric space.
- $(A2)$ The function $d^t\colon X\times X\to\mathbb{R}^+$ defined as $d^t(x,y)=\max\{p(x,y)-p(x,x),p(x,y)-p(y,y)\}$ is a (usual) metric on X and (X,d^t) is a (usual) metric space.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x,\varepsilon):x\in X,\varepsilon>0\}$ where $B_p(x,\varepsilon)=\{y\in X:p(x,y)\le p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

Definition 1.4. ([13]) Let (X,p) be a partial metric space. Then

- $(B1)$ a sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty}p(x_n,x)$,
- $(B2)$ a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty}p(x_m,x_n)$ exists and finite,
- $(B3)$ (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$ with respect to τ_p. Furthermore,
$$\lim_{m,n\to\infty}p(x_m,x_n)=\lim_{n\to\infty}p(x_n,x)=p(x,x).$$

Definition 1.5. ([15]) Let (X,p) be a partial metric space. Then

- $(C1)$ a sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{m,n\to\infty}p(x_m,x_n)=0$,
- $(C2)$ (X,p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$ such that $p(x,x)=0$.

Definition 1.6. Let (X, p) be a partial metric space. A point $y \in X$ is called point of coincidence of two self mappings T and f on X if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 1.7. ([12, 13]) *Let (X, p) be a partial metric space. Then*

(i) *a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d^s),*

(ii) *(X, p) is complete if and only if the metric space (X, d^s) is complete,*

(iii) *a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.*

Lemma 1.8. ([1]) *Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

The purpose of this paper is to establish some fixed point theorems, a common fixed point theorem and a coincidence point theorem via contractive type mappings in the setting of partial metric spaces. Our results extend several results from the existing literature.

2. Main Results

Now, we are in a position to introduce some fixed point theorems, a coincidence point theorems and a common fixed point theorems in a partial metric spaces.

2.1. Fixed point theorems.

Theorem 2.1. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping satisfying the inequality*

$$\begin{aligned} p(Tx, Ty) \quad &\leq \quad \lambda \max\{p(x, y), p(x, Tx), p(y, Ty)\} \\ &\quad\quad +\mu\, [p(x, Ty) + p(y, Tx)], \end{aligned} \tag{1}$$

for all $x, y \in X$, where $\lambda, \mu \geq 0$ are constants such that $0 \leq \lambda + 2\mu < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. We construct the iterative sequence $\{x_n\}$ which is defined as $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$, then $x_n = T^n x_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (1) and (P_4), we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \lambda \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\} \\ &\quad +\mu\, [p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})] \\ &= \lambda \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \\ &\quad +\mu\, [p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ &\leq \lambda \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned}$$

$$
\begin{aligned}
& +\mu\ [p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)] \\
= \ & \lambda\ \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \\
& +\mu\ [p(x_{n-1}, x_n) + p(x_n, x_{n+1})]. \qquad (2)
\end{aligned}
$$

(i) If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$, then from (2), we obtain

$$
\begin{aligned}
p(x_n, x_{n+1}) &\leq \lambda\ p(x_n, x_{n+1}) + \mu\ [p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\
&= (\lambda + \mu)\ p(x_n, x_{n+1}) + \mu\ p(x_{n-1}, x_n).
\end{aligned}
$$

The above inequality implies

$$(1 - \lambda - \mu)p(x_n, x_{n+1}) \leq \mu\ p(x_{n-1}, x_n),$$

that is,

$$p(x_n, x_{n+1}) \leq \Big(\frac{\mu}{1 - \lambda - \mu}\Big)p(x_{n-1}, x_n)$$

or

$$p(x_n, x_{n+1}) \leq t\ p(x_{n-1}, x_n), \qquad (3)$$

where $t = \Big(\frac{\mu}{1-\lambda-\mu}\Big) < 1$ since by hypothesis $0 \leq \lambda + 2\mu < 1$.

(ii) If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$, then from (2), we obtain

$$
\begin{aligned}
p(x_n, x_{n+1}) &\leq \lambda\ p(x_{n-1}, x_n) + \mu\ [p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\
&= (\lambda + \mu)\ p(x_{n-1}, x_n) + \mu\ p(x_n, x_{n+1}).
\end{aligned}
$$

The above inequality implies

$$(1 - \mu)p(x_n, x_{n+1}) \leq (\lambda + \mu)\ p(x_{n-1}, x_n),$$

that is,

$$p(x_n, x_{n+1}) \leq \Big(\frac{\lambda + \mu}{1 - \mu}\Big)p(x_{n-1}, x_n)$$

or

$$p(x_n, x_{n+1}) \leq t'\ p(x_{n-1}, x_n), \qquad (4)$$

where $t' = \Big(\frac{\lambda+\mu}{1-\mu}\Big) < 1$ since by hypothesis $0 \leq \lambda + 2\mu < 1$.

Let $q = \max\{t, t'\} < 1$. Then from above two cases, we obtain

$$p(x_n, x_{n+1}) \leq q\ p(x_{n-1}, x_n), \qquad (5)$$

where $q = \lambda + 2\mu < 1$.

Set $D_n = p(x_n, x_{n+1})$, then from (5), we obtain

$$D_n \leq qD_{n-1} \leq q^2 D_{n-2} \leq \cdots \leq q^n D_0.$$

Now we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$. Then

$$
\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\
&\quad -p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\
&\leq q^n p(x_0, x_1) + q^{n+1} p(x_0, x_1) + \cdots + q^{n+m-1} p(x_0, x_1)
\end{aligned}
$$

$$
\begin{aligned}
&= q^n[p(x_0,x_1)+qp(x_0,x_1)+\cdots+q^{m-1}p(x_0,x_1)]\\
&= q^n[1+q+\cdots+q^{m-1}]D_0\\
&\leq q^n\Big(\frac{1-q^{m-1}}{1-q}\Big)D_0.
\end{aligned}
$$

Taking $n, m \to \infty$ in the above inequality, we get $p(x_n, x_m) \to 0$ since $0 < q < 1$, and hence $\{x_n\}$ is a Cauchy sequence in X. Thus by Lemma 1.7 this sequence will also Cauchy in (X, d^s). In addition, since (X, p) is complete, (X, d^s) is also complete. Thus there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Moreover by Lemma 1.8,

$$p(x,x)=\lim_{n\to\infty}p(x,x_n)=\lim_{n,m\to\infty}p(x_n,x_m)=0, \tag{6}$$

implies

$$\lim_{n\to\infty}d^s(x,x_n)=0. \tag{7}$$

Now, we show that x is a fixed point of T. Notice that due to (6), we have $p(x, x) = 0$. As

$$
\begin{aligned}
p(x,Tx) &\leq p(x,x_{n+1})+p(x_{n+1},Tx)-p(x_{n+1},x_{n+1})\\
&= p(x,x_{n+1})+p(Tx_n,Tx)-p(x_{n+1},x_{n+1})\\
&\leq p(x,x_{n+1})+\lambda\max\{p(x_n,x),p(x_n,Tx_n),p(x,Tx)\}\\
&\quad +\mu\,[p(x_n,Tx)+p(x,Tx_n)]\\
&= p(x,x_{n+1})+\lambda\ \max\{p(x_n,x),p(x_n,x_{n+1}),p(x,Tx)\}\\
&\quad +\mu\,[p(x_n,Tx)+p(x,x_{n+1})].
\end{aligned} \tag{8}
$$

Taking $n \to \infty$ in equation (8) and using equation (6) and Lemma 1.8, we obtain

$$
\begin{aligned}
p(x,Tx) &\leq \lambda\, p(x,Tx)+\mu\, p(x,Tx)\\
&= (\lambda+\mu)p(x,Tx),
\end{aligned}
$$

which implies

$$(1-\lambda-\mu)p(x,Tx) \quad \leq 0.$$

Hence $p(x, Tx) = 0$ and $x = Tx$, This shows that x is a fixed point of T.

Now we show that the fixed point of T is unique. For this, we suppose that u is another fixed point of T, that is, $u = Tu$ with $u \neq x$. Then from (1) and (6), we have

$$
\begin{aligned}
p(x,u) &\leq p(Tx,Tu)\\
&\leq \lambda\ \max\{p(x,u),p(x,Tx),p(u,Tu)\}\\
&\quad +\mu\,[p(x,Tu)+p(u,Tx)]\\
&= \lambda\ \max\{p(x,u),p(x,x),p(u,u)\}\\
&\quad +\mu\,[p(x,u)+p(u,x)],\\
&\leq (\lambda+2\mu)p(x,u),
\end{aligned}
$$

$$< \quad p(x,u).$$

This is a contraction. Therefore, we have $p(x,u)=0$ and $x=u$. This shows that the fixed point of T is unique. This completes the proof. □

Theorem 2.2. *Let (X,p) be a complete partial metric space and let $T\colon X\to X$ be a mapping satisfying the inequality for some positive integer n,*

$$\begin{aligned} p(T^n x, T^n y) \quad &\leq \quad \lambda\ \max\{p(x,y), p(x,T^n x), p(y,T^n y)\} \\ &\quad\quad +\mu\ [p(x,T^n y)+p(y,T^n x)], \end{aligned} \tag{9}$$

for all $x,y\in X$, where $\lambda,\mu\geq 0$ are constants such that $0\leq\lambda+2\mu<1$. Then T has a unique fixed point in X.

Proof. Let $F=T^n$. Then from (9), we have

$$\begin{aligned} p(Fx, Fy) \quad &\leq \quad \lambda\ \max\{p(x,y), p(x,Fx), p(y,Fy)\} \\ &\quad\quad +\mu\ [p(x,Fy)+p(y,Fx)], \end{aligned}$$

for all $x,y\in X$. So by Theorem 2.1, F has a unique fixed point, that is, T^n has a unique fixed point u_0. Since $T^n(Tu_0)=T(T^n u_0)=Tu_0$, Tu_0 is also a fixed point of T^n. Hence $Tu_0=u_0$, this means that u_0 is a fixed point of T. Since the fixed point of T is also a fixed point of T^n, so the fixed point of T is unique. This completes the proof. □

If we take $\max\{p(x,y),p(x,Tx),p(y,Ty)\}=p(x,y)$ and $\mu=0$ in Theorem 2.1, then we obtain the following result as corollary due to Banach contraction mapping principle [4] in a partial metric space.

Corollary 2.3. *Let (X,p) be a complete partial metric space and let $T\colon X\to X$ be a mapping satisfying the inequality:*

$$p(Tx,Ty)\leq\lambda\, p(x,y)$$

for all $x,y\in X$, where $0\leq\lambda<1$ is a constant. Then T has a unique fixed point in X.

If we take $\lambda=0$ in Theorem 2.1, then we obtain the following result as corollary due to Chatterjae [7] in a partial metric space.

Corollary 2.4. *Let (X,p) be a complete partial metric space and let $T\colon X\to X$ be a mapping satisfying the inequality:*

$$p(Tx,Ty)\leq\mu\ [p(x,Ty)+p(y,Tx)]$$

for all $x,y\in X$, where $0\leq\mu<\frac{1}{2}$ is a constant. Then T has a unique fixed point in X.

The following results are obtain from Theorem 2.1.

Corollary 2.5. *Let (X, p) be a complete partial metric space and let $T\colon X \to X$ be a mapping satisfying the inequality*

$$p(Tx, Ty) \leq \lambda \,\max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(y, Tx)]\Big\}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. Follows from Theorem 2.1, by noting that

$$\begin{aligned} & \lambda \,\max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(y, Tx)]\Big\} \\ \leq \quad & \lambda \max\Big\{p(x, y), p(x, Tx), p(y, Ty)\Big\} + 2\mu\Big(\frac{p(x, Ty) + p(y, Tx)}{2}\Big). \end{aligned}$$

□

Corollary 2.6. *Let (X, p) be a complete partial metric space and let $T\colon X \to X$ be a mapping satisfying the inequality*

$$p(Tx, Ty) \leq r_1\, p(x, y) + r_2\, p(x, Tx) + r_3\, p(y, Ty) + \frac{r_4}{2}[p(x, Ty) + p(y, Tx)]$$

for all $x, y \in X$, where $r_1, r_2, r_3, r_4 \geq 0$ are constants such that $r_1+r_2+r_3+r_4 < 1$. Then T has a unique fixed point in X.

Proof. Follows from Corollary 2.5, by noting that

$$\begin{aligned} & r_1\, p(x, y) + r_2\, p(x, Tx) + r_3\, p(y, Ty) + \frac{r_4}{2}[p(x, Ty) + p(y, Tx)] \\ \leq \quad & (r_1 + r_2 + r_3 + r_4) \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \\ & \qquad \frac{1}{2}[p(x, Ty) + p(y, Tx)]\Big\}. \end{aligned}$$

□

2.2. Coincidence point theorems.

Theorem 2.7. *Let T and f be two self-maps on a complete partial metric space X satisfying the inequality*

$$\begin{aligned} p(Tx, Ty) \quad \leq \quad & \lambda \,\max\{p(fx, fy), p(fx, Tx), p(fy, Ty)\} \\ & +\mu\, [p(fx, Ty) + p(fy, Tx)], \end{aligned} \tag{10}$$

for all $x, y \in X$, where $\lambda, \mu \geq 0$ are constants such that $0 \leq \lambda + 2\mu < 1$. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then T and f have a coincidence point.

Proof. Let $x_0 \in X$ and choose a point x_1 in X such that $Tx_0 = fx_1, \ldots, Tx_n = fx_{n+1}$. Then from (10) and (P_4), we get

$$\begin{aligned} p(fx_n, fx_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \lambda \max\{p(fx_{n-1}, fx_n), p(fx_{n-1}, Tx_{n-1}), p(fx_n, Tx_n)\} \\ &+ \mu\,[p(fx_{n-1}, Tx_n) + p(fx_n, Tx_{n-1})] \\ &= \lambda \max\{p(fx_{n-1}, fx_n), p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\} \\ &+ \mu\,[p(fx_{n-1}, fx_{n+1}) + p(fx_n, fx_n)] \\ &\leq \lambda \max\{p(fx_{n-1}, fx_n), p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\} \\ &+ \mu\,[p(fx_{n-1}, fx_n) + p(fx_n, fx_{n+1}) - p(fx_n, fx_n) + p(fx_n, fx_n)] \\ &\leq \lambda \max\{p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\} \\ &+ \mu\,[p(fx_{n-1}, fx_n) + p(fx_n, fx_{n+1})]. \end{aligned} \tag{11}$$

(i) If $\max\{p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\} = p(fx_n, fx_{n+1})$, then from (11), we obtain

$$\begin{aligned} p(fx_n, fx_{n+1}) &\leq \lambda\, p(fx_n, fx_{n+1}) + \mu\,[p(fx_{n-1}, fx_n) + p(fx_n, fx_{n+1})] \\ &= (\lambda + \mu)\, p(fx_n, fx_{n+1}) + \mu\, p(fx_{n-1}, fx_n). \end{aligned}$$

The above inequality implies

$$(1 - \lambda - \mu) p(fx_n, fx_{n+1}) \leq \mu\, p(fx_{n-1}, fx_n),$$

that is,

$$p(fx_n, fx_{n+1}) \leq \Big(\frac{\mu}{1 - \lambda - \mu}\Big) p(fx_{n-1}, fx_n)$$

or

$$p(fx_n, fx_{n+1}) \leq t\, p(fx_{n-1}, fx_n), \tag{12}$$

where $t = \Big(\frac{\mu}{1-\lambda-\mu}\Big) < 1$ since by hypothesis $0 \leq \lambda + 2\mu < 1$.

(ii) If $\max\{p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\} = p(fx_{n-1}, fx_n)$, then from (11), we obtain

$$\begin{aligned} p(fx_n, fx_{n+1}) &\leq \lambda\, p(fx_{n-1}, fx_n) + \mu\,[p(fx_{n-1}, fx_n) + p(fx_n, fx_{n+1})] \\ &= (\lambda + \mu)\, p(fx_{n-1}, fx_n) + \mu\, p(fx_n, fx_{n+1}). \end{aligned}$$

The above inequality implies

$$(1 - \mu) p(fx_n, fx_{n+1}) \leq (\lambda + \mu)\, p(fx_{n-1}, fx_n),$$

that is,

$$p(fx_n, fx_{n+1}) \leq \Big(\frac{\lambda + \mu}{1 - \mu}\Big) p(fx_{n-1}, fx_n)$$

or

$$p(fx_n, fx_{n+1}) \leq t'\, p(fx_{n-1}, fx_n), \tag{13}$$

where $t' = \Big(\frac{\lambda+\mu}{1-\mu}\Big) < 1$ since by hypothesis $0 \leq \lambda + 2\mu < 1$.

Let $q = \max\{t, t'\} < 1$. Then from above two cases, we obtain

$$p(fx_n, fx_{n+1}) \quad \leq \quad q\, p(fx_{n-1}, fx_n), \tag{14}$$

where $q = \lambda + 2\mu < 1$.

Let $H_n = p(fx_n, fx_{n+1})$. Then from (14), we obtain

$$H_n \leq qH_{n-1} \leq q^2 H_{n-2} \leq \cdots \leq q^n H_0.$$

Now we show that $\{fx_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$. Then

$$\begin{aligned} p(fx_n, fx_m) &\leq p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) + \cdots + p(fx_{n+m-1}, fx_m) \\ &\quad - p(fx_{n+1}, fx_{n+1}) - p(fx_{n+2}, fx_{n+2}) \\ &\quad - \cdots - p(fx_{n+m-1}, fx_{n+m-1}) \\ &\leq q^n p(fx_0, fx_1) + q^{n+1} p(fx_0, fx_1) + \cdots + q^{n+m-1} p(fx_0, fx_1) \\ &= q^n [p(fx_0, fx_1) + qp(fx_0, fx_1) + \cdots + q^{m-1} p(fx_0, fx_1)] \\ &= q^n [1 + q + \cdots + q^{m-1}] H_0 \\ &\leq q^n \Big(\frac{1 - q^{m-1}}{1 - q} \Big) H_0. \end{aligned}$$

Taking $n, m \to \infty$ in the above inequality, we get $p(fx_n, fx_m) \to 0$ since $0 < q < 1$, and hence $\{fx_n\}$ is a Cauchy sequence in X. Thus by Lemma 1.7 this sequence also Cauchy in (X, d^s). In addition, since (X, p) is complete, (X, d^s) is also complete. Thus there exists $x \in X$ such that $x_n \to x$, it implies that $fx_n \to fx$ as $n \to \infty$, since $f(X)$ is a complete subspace of X.

Moreover by Lemma 1.8

$$p(fx, fx) = \lim_{n\to\infty} p(fx, fx_n) = \lim_{n,m\to\infty} p(fx_n, fx_m) = 0, \tag{15}$$

implies

$$\lim_{n\to\infty} d^s(fx, fx_n) = 0. \tag{16}$$

Now, we show that x is a coincidence point of T and f. Notice that due to (15), we have $p(fx, fx) = 0$. Note that,

$$\begin{aligned} p(fx, Tx) \quad &\leq \quad p(fx, fx_{n+1}) + p(fx_{n+1}, Tx) - p(fx_{n+1}, fx_{n+1}) \\ &= \quad p(fx, fx_{n+1}) + p(Tx_n, Tx) - p(fx_{n+1}, fx_{n+1}) \\ &\leq \quad p(fx, fx_{n+1}) + \lambda \ \max\{p(fx_n, fx), p(fx_n, Tx_n), p(fx, Tx)\} \\ &\qquad +\mu\, [p(fx_n, Tx) + p(fx, Tx_n)] - p(fx_{n+1}, fx_{n+1}) \\ &\leq \quad p(fx, fx_{n+1}) + \lambda \ \max\{p(fx_n, fx), p(fx_n, fx_{n+1}), p(fx, Tx)\} \\ &\qquad +\mu\, [p(fx_n, Tx) + p(fx, fx_{n+1})]. \end{aligned} \tag{17}$$

Taking $n \to \infty$ in equation (17) and using (15) and Lemma 1.8, we obtain

$$\begin{aligned} p(fx, Tx) \quad &\leq \quad \lambda\, p(fx, Tx) + \mu\, p(fx, Tx) \\ &= \quad (\lambda + \mu)\, p(fx, Tx), \end{aligned}$$

which implies

$$(1-\lambda-\mu)p(fx,Tx) \leq 0.$$

Hence we have, $p(fx,Tx) = 0$, and hence $fx = Tx$. This shows that x is a coincidence point of T and f. This completes the proof. □

Remark 2.1. If we take $f = I$, the identity map and T is the single valued map in Theorem 2.7, then we get Theorem 2.1 of this paper.

Theorem 2.8. *Let T and f be two self-maps on a complete partial metric space X satisfying the inequality (10), where $\lambda, \mu \geq 0$ are as in Theorem 2.7. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then T and f have at most a unique point of coincidence.*

Proof. Let $v_1, v_2 \in X$ be such that $v_1 = Tw_1 = fw_1$ and $v_2 = Tw_2 = fw_2$ for some $w_1, w_2 \in X$. Using (10), (15) and $(P3)$, we obtain that

$$\begin{aligned}
p(v_1,v_2) &\leq p(Tw_1,Tw_2) \\
&\leq \lambda \max\{p(fw_1,fw_2), p(fw_1,Tw_1), p(fw_2,Tw_2)\} \\
&\quad +\mu\,[p(fw_1,Tw_2)+p(fw_2,Tw_1)] \\
&= \lambda \max\{p(v_1,v_2), p(v_1,v_1), p(v_2,v_2)\} \\
&\quad +\mu\,[p(v_1,v_2)+p(v_2,v_1)] \\
&= \lambda \max\{p(v_1,v_2),0,0\} + 2\mu\, p(v_1,v_2) \\
&= \lambda\, p(v_1,v_2) + 2\mu\, p(v_1,v_2) = (\lambda+2\mu)p(v_1,v_2) \\
&< p(v_1,v_2),
\end{aligned}$$

which is a a contradiction. Hence $p(v_1, v_2) = 0$, that is, $v_1 = v_2$. This completes the proof. □

2.3. Common fixed point theorems.

Theorem 2.9. *Let S and T be two self-maps on a complete partial metric space X satisfying the inequality*

$$\begin{aligned}
p(Sx,Ty) &\leq \lambda \max\{p(x,y), p(x,Sx), p(y,Ty)\} \\
&\quad +\mu\,[p(x,Ty)+p(y,Sx)], \qquad (18)
\end{aligned}$$

for all $x, y \in X$, where $\lambda, \mu \geq 0$ are constants such that $0 \leq \lambda + 2\mu < 1$. Then S and T have a unique common fixed point in X.

Proof. For each $x_0 \in X$. Put $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for $n = 0, 1, 2, \ldots$. We prove that $\{x_n\}$ is a Cauchy sequence in (X, p). It follows from (18) that

$$\begin{aligned}
&p(x_{2n},x_{2n+1}) \\
&\quad = p(Sx_{2n-1},Tx_{2n}) \\
&\quad \leq \lambda \max\{p(x_{2n-1},x_{2n}), p(x_{2n-1},Sx_{2n-1}), p(x_{2n},Tx_{2n})\}
\end{aligned}$$

$$
\begin{aligned}
&+\mu\,[p(x_{2n-1},Tx_{2n})+p(x_{2n},Sx_{2n-1})]\\
&\le\lambda\ \max\{p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}\\
&+\mu\,[p(x_{2n-1},x_{2n+1})+p(x_{2n},x_{2n})]\\
\le&\ \lambda\ \max\{p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}\\
&+\mu\,[p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1})-p(x_{2n},x_{2n})+p(x_{2n},x_{2n})]\\
=&\ \lambda\ \max\{p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}\\
&+\mu\,[p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1})]\\
=&\ \lambda\ \max\{p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}\\
&+\mu\,[p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1})].
\end{aligned}
\tag{19}
$$

By similar arguments as in Theorem 2.1, we obtain

$$p(x_{2n},x_{2n+1})\le q\,p(x_{2n-1},x_{2n}),\tag{20}$$

where $q=\max\{t,t'\}<1$, $t=\left(\frac{\mu}{1-\lambda-\mu}\right)$, $t'=\left(\frac{\lambda+\mu}{1-\mu}\right)$ and $0\le q=\lambda+2\mu<1$.

Let $U_{2n}=p(x_{2n},x_{2n+1})$. Then from (20), we obtain

$$U_{2n}\le qU_{2n-1}\le q^2U_{2n-2}\le\cdots\le q^{2n}U_0.$$

For $n,m\in\mathbb{N}$ with $m>n$, by repeated use of $(P4)$, we have that

$$
\begin{aligned}
p(x_n,x_m)&\le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\cdots+p(x_{n+m-1},x_m)\\
&\quad-p(x_{n+1},x_{n+1})-p(x_{n+2},x_{n+2})-\cdots-p(x_{n+m-1},x_{n+m-1})\\
&\le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\cdots+p(x_{n+m-1},x_m)\\
&\le q^n[1+q+q^2+\cdots+q^{m-1}]U_0\\
&\le q^n\Big(\frac{1-q^{m-1}}{1-q}\Big)U_0.
\end{aligned}
$$

Taking $n,m\to\infty$ in the above inequality, we get $p(x_n,x_m)\to 0$ since $0<q<1$, and hence $\{x_n\}$ is a Cauchy sequence in X. Thus by Lemma 1.7 this sequence also Cauchy in (X,d^s). In addition, since (X,p) is complete, (X,d^s) is also complete. Thus there exists $v\in X$ such that $x_n\to v$ as $n\to\infty$. Moreover by Lemma 1.8,

$$p(v,v)=\lim_{n\to\infty}p(v,x_n)=\lim_{n,m\to\infty}p(x_n,x_m)=0,\tag{21}$$

implies

$$\lim_{n\to\infty}d^s(v,x_n)=0.\tag{22}$$

Now, we show that v is a common fixed point of S and T. Notice that due to (21), we have $p(v,v)=0$. Note that,

$$
\begin{aligned}
p(v,Tv)&\le p(v,x_{2n+2})+p(x_{2n+2},Tv)-p(x_{2n+2},x_{2n+2})\\
&=p(v,x_{2n+2})+p(Sx_{2n+1},Tv)-p(x_{2n+2},x_{2n+2})\\
&\le p(v,x_{2n+2})+p(Sx_{2n+1},Tv)
\end{aligned}
$$

$$\begin{aligned}
&\leq p(v, x_{2n+2}) + \lambda \max\{p(x_{2n+1}, v), p(x_{2n+1}, Sx_{2n+1}), p(v, Tv)\} \\
&\quad + \mu\,[p(x_{2n+1}, Tv) + p(v, Sx_{2n+1})] \\
&= p(v, x_{2n+2}) + \lambda\ \max\{p(x_{2n+1}, v), p(x_{2n+1}, x_{2n+2}), p(v, Tv)\} \\
&\quad + \mu\,[p(x_{2n+1}, Tv) + p(v, x_{2n+2})]. \qquad (23)
\end{aligned}$$

Taking $n \to \infty$ in equation (23) and using equation (21) and Lemma 1.8, we obtain

$$\begin{aligned}
p(v, Tv) &\leq \lambda\, p(v, Tv) + \mu\, p(v, Tv) \\
&= (\lambda + \mu) p(v, Tv),
\end{aligned}$$

which implies

$$(1 - \lambda - \mu) p(v, Tv) \leq 0.$$

Hence, we have $p(v, Tv) = 0$ and $v = Tv$. This shows that v is a fixed point of T.

Similarly, we can show that $v = Sv$. Thus v is a common fixed point of S and T. The uniqueness of the common fixed point of S and T follows from Theorem 2.1. This completes the proof. □

If we take $\max\big\{p(x, y), p(x, Sx), p(y, Ty)\big\} = p(x, y)$ and $\mu = 0$ in Theorem 2.9, then we have the following result.

Corollary 2.10. ([11], Corollary 3.4) *Let S and T be two self-maps on a complete partial metric space X satisfying the inequality*

$$p(Sx, Ty) \leq \lambda\, p(x, y), \qquad (24)$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ is a constant. Then S and T have a unique common fixed point in X.

Remark 2.2. ([11]) The above Corollary 2.10 can not be deduced from similar result of metric spaces. Actually the contractive condition (24) for a pair of mappings $S, T \colon X \to X$ on a metric space (X, d), that is,

$$d(Sx, Ty) \leq \lambda\, d(x, y),$$

for all $x, y \in X$, is not attainable. Because $S \neq T$ implies that $Su \neq Tu$ for some $u \in X$, then $d(Su, Tu) > 0 = \lambda\, d(u, u)$.

Condition (24) is not satisfied for $x = y = u$. However the same condition in partial metric space is feasible to find common fixed point result for a pair of mappings. This fact can be seen again in Example 3.6.

3. Illustrations

Example 3.1. Let $X = [0, 1]$. Define $p \colon X \times X \to \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$ with $T \colon X \to X$ by $T(x) = \frac{x}{2}$. Then, clearly (X, p) is a partial metric space. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Tx, Ty) = \frac{y}{2}$,

$p(x,y) = y$, $p(x,Tx) = x$, $p(y,Ty) = y$, $p(x,Ty) = \frac{y}{2}$, $p(y,Tx) = y$ and $\max\{p(x,y), p(x,Tx), p(y,Ty)\} = \max\{y,x,y\} = y$. Now, we consider

$$p(Tx,Ty) = \frac{y}{2} \leq \lambda\, y + \mu \left(\frac{y}{2} + y\right)$$

or

$$\frac{1}{2} \leq \lambda + \frac{3}{2}\mu. \tag{25}$$

Then we know that:

(1) Inequality (25) satisfied for (i) $\lambda = \frac{3}{4}$ and $\mu = 0$ (ii) $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{6}$ (iii) $\lambda = 0$ and $\mu = \frac{1}{3}$. Thus T satisfies the conditions of Theorem 2.1. Hence T has a unique fixed point. It is seen that 0 is the unique fixed point of T. Therefore the sequence $\{T^n x\} = \{\frac{x}{2^n}\}$ converges to the fixed point $z = 0$ of the operator T for every $x \in X$

(2) If $\lambda = \frac{3}{4}$ and $\mu = 0$, then T satisfies the conditions of Corollary 2.3. Hence, by applying Corollary 2.3, the operator T has a unique fixed point $0 \in X$.

(3) If $\mu = \frac{1}{3}$ and $\lambda = 0$, then T satisfies the conditions of Corollary 2.4. Hence, by applying Corollary 2.4, the operator T has a unique fixed point $0 \in X$.

(4)

$$p(Tx,Ty) = \frac{y}{2} \leq \lambda \max\left\{y, x, y, \frac{1}{2}(\frac{y}{2} + y)\right\} = \lambda\, y$$

or

$$\lambda \geq \frac{1}{2}.$$

If we take $0 < \lambda < 1$, then all the conditions of Corollary 2.5 are satisfied. Hence, by applying Corollary 2.5, T has a unique fixed point $0 \in X$.

Example 3.2. Let $X = [0,1]$ be endowed with the partial metric $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then (X,p) is a 0-complete partial metric space. Define the mapping $T\colon X \to X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & \text{otherwise } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then, we distinguish the following cases.

Case 1. If $x, y \in [0, \frac{1}{2}]$ with $x \leq y$, then we have $p(Tx,Ty) = 0$, $p(x,y) = y$, $p(x,Tx) = p(x,0) = x$, $p(y,Ty) = p(y,0) = y$, $p(x,Ty) = p(x,0) = x$, $p(y,Tx) = p(y,0) = y$ and $\max\{p(x,y), p(x,Tx), p(y,Ty)\} = \max\{y,x,y\} = y$. And also, we have

$$p(Tx,Ty) = 0 \leq \lambda\, y + \mu\,(x + y) = \mu\, x + (\lambda + \mu)\, y,$$

where $\lambda, \mu \geq 0$. Thus the inequality (1) of Theorem 2.1 is satisfied and $0 (\in X)$ is the unique fixed point of T.

Case 2. If $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$ with $x \leq y$, then we have $p(Tx, Ty) = \frac{1}{2}$, $p(x, y) = y$, $p(x, Tx) = p(x, 0) = x$, $p(y, Ty) = p(y, 1/2) = y$, $p(x, Ty) = p(x, 1/2) = x$, $p(y, Tx) = p(y, 0) = y$ and $\max\{p(x, y), p(x, Tx), p(y, Ty)\} = \max\{y, x, y\} = y$. And also, we have

$$p(Tx, Ty) = \frac{1}{2} \leq \lambda \cdot 1 + \mu\, (1 + y) = \mu\, y + (\lambda + \mu),$$

where $\lambda, \mu \geq 0$ with $\lambda + 2\mu < 1$. Thus the inequality (1) of Theorem 2.1 is satisfied and $0 (\in X)$ is the unique fixed point of T.

Case 3. If $x, y \in (\frac{1}{2}, 1]$ with $x \leq y$, then we have $p(Tx, Ty) = \frac{1}{2}$, $p(x, y) = y$, $p(x, Tx) = p(x, 1/2) = x$, $p(y, Ty) = p(y, 1/2) = y$, $p(x, Ty) = p(x, \frac{1}{2}) = x$, $p(y, Tx) = p(y, \frac{1}{2}) = y$ and $\max\{p(x, y), p(x, Tx), p(y, Ty)\} = \max\{y, x, y\} = y$. And also, we have

$$p(Tx, Ty) = \frac{1}{2} \leq \lambda\, y + \mu\, (x + y) = \mu\, x + (\lambda + \mu) y.$$

Putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{2} \leq \lambda + \frac{3}{2}\mu.$$

The above inequality is satisfied for (i) $\lambda = 0$ and $\mu = 1/3$ (ii) $\lambda = 1/2$ and $\mu = 0$ and (iii) $\lambda = 1/3$ and $\mu = 1/6$ with $\lambda + 2\mu < 1$. Hence the inequality (1) of Theorem 2.1 is satisfied and $0 (\in X)$ is the unique fixed point of T.

Thus in all the above cases inequality (1) of Theorem 2.1 is satisfied and $0 (\in X)$ is the unique fixed point of T.

Example 3.3. Let $X = \{0, 1, 2, 3, \dots\}$. Define $p \colon X \times X \to \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$. Let $T, f \colon X \to X$ be defined respectively as follows: $f(x) = x$ for all $x \in X$ and

$$T(x) = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then (X, p) is a partial metric space. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Tx, Ty) = y - 1$, $p(fx, fy) = y$, $p(fx, Tx) = x$, $p(fy, Ty) = y$, $p(fx, Ty) = x$, $p(fy, Tx) = y$ and $\max\{p(fx, fy), p(fx, Tx),$
$p(fy, Ty)\} = \max\big\{y, x, y\big\} = y$. Now, we consider

$$p(Tx, Ty) = y - 1 \leq \lambda\, y + \mu\, (x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$0 \leq \lambda + \frac{3}{2}\mu.$$

The above inequality is satisfied for all $\lambda, \mu \geq 0$ with $\lambda + 2\mu < 1$. Then T and f have the properties mentioned in Theorem 2.7. Hence the conditions of Theorem 2.7 are satisfied. Therefore it is seen that 0 is the unique point of coincidence, that is, $f(x) = 0 = T(x)$.

Example 3.4. Let $X = \{1, 2, 3, 4\}$ and $p\colon X \times X \to \mathbb{R}$ be defined by

$$p(x, y) = \begin{cases} |x - y| + \max\{x, y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in X$. Then (X, p) is a complete partial metric space.

Define the mapping $T\colon X \to X$ by

$$T(1) = 1,\ T(2) = 1,\ T(3) = 2,\ T(4) = 2.$$

Then, we have

$$p(T(1), T(2)) = p(1, 1) = 0 \leq \frac{3}{4} \cdot 3 = \frac{3}{4} p(1, 2),$$

$$p(T(1), T(3)) = p(1, 2) = 3 \leq \frac{3}{4} \cdot 5 = \frac{3}{4} p(1, 3),$$

$$p(T(1), T(4)) = p(1, 2) = 3 \leq \frac{3}{4} \cdot 7 = \frac{3}{4} p(1, 4),$$

$$p(T(2), T(3)) = p(1, 2) = 3 \leq \frac{3}{4} \cdot 4 = \frac{3}{4} p(2, 3),$$

$$p(T(2), T(4)) = p(1, 2) = 3 \leq \frac{3}{4} \cdot 6 = \frac{3}{4} p(2, 4),$$

$$p(T(3), T(4)) = p(2, 2) = 2 \leq \frac{3}{4} \cdot 5 = \frac{3}{4} p(3, 4).$$

Thus, T satisfies all the conditions of Corollary 2.3 with $\lambda = \frac{3}{4} < 1$. Now by Corollary 2.3, T has a unique fixed point, which in this case is 1.

Example 3.5. Let $X = [0, 1]$. Define $p\colon X \times X \to \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$ and let $S, T\colon X \to X$ be defined respectively by $S(x) = \frac{x}{2}$ and $T(x) = 0$ for all $x \in X$. Then (X, p) is a partial metric space. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Sx, Ty) = \frac{x}{2}$, $p(x, y) = y$, $p(x, Sx) = x$, $p(y, Ty) = y$, $p(x, Ty) = x$, $p(y, Sx) = y$ and $\max\{p(x, y), p(x, Sx), p(y, Ty)\} = \max\big\{y, x, y\big\} = y$. Now, we consider

$$p(Sx, Ty) = \frac{x}{2} \leq \lambda\, y + \mu\,(x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{4} \leq \lambda + \frac{3}{2} \mu.$$

Then the above inequality is satisfied for (i) $\lambda = 0$ and $\mu = 1/6$ (ii) $\lambda = 1/4$ and $\mu = 0$, (iii) $\lambda = 1/3$ and $\mu = 1/6$ and (iv) $\lambda = 1/5$ and $\mu = 1/10$ with $\lambda + 2\mu < 1$. Thus S and T satisfy all the conditions of Theorem 2.9. Hence by applying Theorem 2.9, S and T have a unique common fixed point $0 (\in X)$ of S and T.

Example 3.6. ([11]) Let $X = [0,1]$. Define $p\colon X \times X \to \mathbb{R}^+$ as $p(x,y) = \max\{x,y\}$ and let $S,T\colon X \to X$ be defined by $S(x) = \frac{x}{8}$ and $T(x) = \frac{3x}{8}$. Then clearly (X,p) is a partial metric space. Now, let $x \le y$. Then, we have

$$p(Sx,Ty) = \max\left\{\frac{x}{8}, \frac{3y}{8}\right\} = \frac{1}{8}\max\{x,3y\}$$

and

$$p(Sx,Ty) = \frac{1}{8}\max\{x,3y\} \le \frac{5}{11}\max\{x,y\} = \lambda\, p(x,y).$$

Therefore, for $\lambda = \frac{5}{11}$ all the conditions of Corollary 2.10 are satisfied to find common fixed point of S and T. However, note that for any metric d on X

$$d(S(1),T(1)) = d\Big(\frac{1}{8}, \frac{3}{8}\Big) > \lambda\, d(1,1) = 0 \ \text{for any } \lambda \in [0,1).$$

Therefore common fixed points of S and T can not be obtained from a corresponding metric fixed point theorem.

4. Applications

As an application of our results, we introduce some fixed point theorems of integral type.

Denote Φ the set of functions $\phi\colon [0,+\infty) \to [0,+\infty)$ satisfying the following hypothesis:

($\mathcal{H}1$) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0,+\infty)$;
($\mathcal{H}2$) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \phi(s)ds > 0$.

Corollary 4.1. *Let (X,p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the following inequality:*

$$\begin{aligned}\int_0^{p(Tx,Ty)} \psi(s)ds \quad &\le \quad \lambda \int_0^{\max\left\{p(x,y),p(x,Tx),p(y,Ty)\right\}} \psi(s)ds \\ &\quad + \mu \int_0^{[p(x,Ty)+p(y,Tx)]} \psi(s)ds\end{aligned}$$

for all $x,y \in X$, where $\lambda,\mu \ge 0$ are constants such that $0 \le \lambda + 2\mu < 1$ and $\psi \in \Phi$. Then T has a unique fixed point in X.

Proof. Follows from Theorem 2.1 by taking

$$t = \int_0^t \psi(s)ds.$$

□

Corollary 4.2. *Let (X,p) be a complete partial metric space. Let $S,T\colon X \to X$ be two mappings satisfying the following inequality:*

$$\int_0^{p(Sx,Ty)} \psi(s)ds \quad \le \quad \lambda \int_0^{\max\left\{p(x,y),p(x,Sx),p(y,Ty)\right\}} \psi(s)ds$$

$$+\mu \int_0^{[p(x,Ty)+p(y,Sx)]} \psi(s)ds$$

for all $x, y \in X$, where $\lambda, \mu \geq 0$ are constants such that $0 \leq \lambda + 2\mu < 1$ and $\psi \in \Phi$. Then S and T have a unique common fixed point in X.

Proof. Follows from Theorem 2.1 by taking

$$t = \int_0^t \psi(s)ds.$$

□

If we take $\lambda = 0$ in Corollary 4.1, then we obtain the following result due to Chatterjae [7].

Corollary 4.3. *Let (X, p) be a complete partial metric space. Let $T \colon X \to X$ be a mapping satisfying the following inequality:*

$$\int_0^{p(Tx,Ty)} \psi(s)ds \quad \leq \quad \mu \int_0^{[p(x,Ty)+p(y,Tx)]} \psi(s)ds$$

for all $x, y \in X$, where $\mu \in [0, \frac{1}{2})$ is a constant and $\psi \in \Phi$. Then T has a unique fixed point in X.

Remark 4.1. Corollary 4.3 extends the corresponding result of Chatterjae [7] from complete metric space to the setting of complete partial metric space for integral type contractive condition.

If we take $\mu = 0$ and $\max\Big\{p(x,y), p(x,Tx), p(y,Ty)\Big\} = p(x,y)$ in Corollary 4.1, then we obtain the following result due to Branciari [6].

Corollary 4.4. *Let (X, p) be a complete partial metric space. Let $T \colon X \to X$ be a mapping satisfying the following inequality:*

$$\int_0^{p(Tx,Ty)} \psi(s)ds \quad \leq \quad \lambda \int_0^{p(x,y)} \psi(s)ds$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ is a constant and $\psi \in \Phi$. Then T has a unique fixed point in X.

Remark 4.2. Corollary 4.4 extends Theorem 2.1 of Branciari [6] from complete metric space to the setting of complete partial metric space.

5. Conclusion

In this article, we establish some fixed point theorems, a common fixed point theorem and a coincidence point theorem in the setting of complete partial metric spaces and we obtain the well-known Banach contraction principle and Chatterjae contraction as corollaries to our results. Also we support our results by some examples and give some applications to our results.

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