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UPHILL ZAGREB INDICES OF SOME GRAPH OPERATIONS FOR CERTAIN GRAPHS

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Abstract. The topological indices are numerical parameters which determined the biological, physical and chemical properties based on the structure of the chemical compounds. One of the recently topological indices is the uphill Zagreb indices. In this paper, the formulae of some uphill Zagreb indices for a few graph operations of some graphs have been derived. Furthermore, the precise formulae of those indices for the honeycomb network have been found along with their graphical profiles.

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1. Introduction

In this research, by graphs, we mean undirected finite simple graph. We denote $G = (V, E)$ for a graph, where V is the set of vertices and E is the set of edges. For a vertex $v \in V(G)$ the degree of v, $d(v)$ is the number of edges incident with v. Any terminology or notation which, we did not mention its definition, we refer the reader to [2].
Topological indices have a widespread position specifically in pharmacology, chemistry, networks and many others, (see[5, 6, 9, 10, 16, 11, 14, 20, 21, 22]). Most of the indices of contemporary are interesting in mathematical chemistry are introduced based on vertex degrees of the chemical graph. The two well-known topological indices of graphs are the Zagreb indices that have been introduced by Gutman and Trinajstic by their work in [12], and described as $M_1(G) = \sum_{u\in V(G)} (d(u))^2$ and $M_2(G) = \sum_{uv\in E(G)} d(u)\, d(v)$, respectively. The forgotten topological index was introduced by Furtula and Gutman [7] as $F(G) = \sum_{u\in V(G)} (d(u))^3$.

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Zagreb indices were studied considerably due to their numerous applications inside the area of present chemical methods which want extra time and more charges. Many new reformulated and prolonged versions of the Zagreb indices have been delivered for several similar reasons, (cf. [1, 3, 8, 15, 18, 23, 24, 25]). Recently in [19], Anwar Saleh et al., have introduced four new topological indices; first uphill, second uphill, modified uphill and forgotten uphill indices. Graph operations are very important to constricting new graphs and they play a vital role in the design and analysis of networks. In this research work motivated by the uphill indices and the importance of graph operations we find exact formulae of graph operations; join, corona product and Cartesian product for certain graphs and finally for the honeycomb network.

2. Some Results on the Uphill Zagreb Indices of Graphs

Definition 2.1. [4] For any graph $G = (V, E)$. A path $u - v$ is a sequence of vertices in G, initialing with u and terminal at v, such that sequential vertices are adjacent, and no vertex is repeated. A path $\Pi = v_1, v_2, ...v_{k+1}$ in G is an uphill path if for every i,$1 \leq i \leq k$, $deg(v_i) \leq deg(v_{i+1})$.
For any vertices u and v in G,if there is an uphill path from u to v we say that u is uphill adjacent to v.

Definition 2.2. [19] A vertex v is uphill dominates a vertex u i a graph G if v uphill adjacent to u. An uphill neighborhood of the vertex v is denoted by $N_{up}(v)$ and described as: $N_{up}(v) = \{u : v \text{ is uphill adjacent to } u\}$. The uphill degree of the vertex v, denoted by $d_{up}(v)$, is the number of vertices which v is uphill adjacent to them, which means $d_{up}(v) = |N_{up}(v)|$.
The uphill closed neighborhood, $N_{up}[v]$, of the vertex v is the uphill open neighborhood of v together with the vertex v.
The maximum and minimum uphill degrees in the graph are denoted by $\Delta_{up}(G)$ and $\delta_{up}(G)$, respectively. The vertex with uphill degree equals to zero is called uphill isolated vertex.

In this paper by $E_{x,y}$, we mean that $E_{x,y} = \{uv \in E(G) : d_{up}(u) = x$ and $d_{up}(v) = y\}$.

Definition 2.3. [19]. For any graph $G = (V, E)$ the first uphill Zagreb, second uphill Zagreb, forgotten uphill Zagreb index and modified first uphill Zagreb are defined as:

$$UPM_1(G) = \sum_{v \in V(G)} (d_{up}(v))^2,$$

$$UPM_2(G) = \sum_{vu \in E(G)} d_{up}(v) d_{up}(u),$$

$$UPM_1^*(G) = \sum_{vu \in E(G)} (d_{up}(v) + d_{up}(u))$$

and

$$UPF(G) = \sum_{v \in V(G)} (d_{up}(v))^3.$$

A join $G = G_1 + G_2$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$[13].

Theorem 2.4. *Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. Then,*

$$UPM_1(G) = \begin{cases} 8m^3 - 40m^2 + 114m - 96 & if \quad s = 0; \\ 5m^3 - 19m^2 + 32m - 30 & if \quad s = 1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 5m^3 - 7m^2 s - 18m^2 + 4ms^2 + 18ms + 30m - s^3 - 6s^2 - 17s - 20$.

Proof. Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. We have three cases:

Case 1. If $s = 0$. The graph G has four vertices of uphill degree $2m - 1$ and $2(m - 2)$ vertices of uphill degree $2m - 5$. Then,

$$\begin{aligned} UPM_1(G) &= 4(2m - 1)^2 + 2(m - 2)(2m - 5)^2 \\ &= 8m^3 - 40m^2 + 114m - 96. \end{aligned}$$

Case 2. If $s = 1$. Then there are two vertices of uphill degree $2m-3$, m vertices of uphill degree $2m - 4$ and $m - 3$ vertices of uphill degree $m - 4$. So,

$$\begin{aligned} UPM_1(G) &= 2(2m - 3)^2 + m(2m - 4)^2 + (m - 3)(m - 4)^2 \\ &= 5m^3 - 19m^2 + 32m - 30. \end{aligned}$$

Case 3. If $s \geq 2$. There are two vertices of uphill degree $2m - s - 2$, $m - 2$ vertices of uphill degree $2m - s - 3$, two vertices of uphill degree $m - s - 2$ and $m - s - 2$ vertices of uphill degree $m - s - 3$. Then,

$$\begin{aligned} UPM_1(G) &= 2(2m-s-2)^2 + (m-2)(2m-s-3)^2 + 2(m-s-2)^2 + (m-s-2)(m-s-3)^2 \\ &= 5m^3 - 7m^2 s - 18m^2 + 4ms^2 + 18ms + 30m - s^3 - 6s^2 - 17s - 20. \end{aligned}$$

□

Theorem 2.5. *Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. Then,*

$$UPM_2(G) = \begin{cases} 4m^4 - 12m^3 + 9m^2 + 42m - 66 & if \quad s = 0; \\ 2m^4 - 5m^3 - 6m^2 + 38m - 48 & if \quad s = 1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 2m^4 - 5m^3 s - 4m^3 + 4m^2 s^2 + 8m^2 s - 8m^2 + 9ms + 30m - ms^3 - 2ms^2 - s^3 - 6s^2 - 19s - 26$.

Proof. Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. We have three cases:

Case 1. If $s = 0$. In this case, there are three types of edges

TABLE 1. Edge partition of $P_m + P_{m-s}$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m-1,2m-1}$	4
$E_{2m-1,2m-5}$	$4(m-1)$
$E_{2m-5,2m-5}$	$m^2 - 2m - 2$

Now, by using the edge partition in Table 1, we get

$$UPM_2(G)) = 4(2m-1)^2 + 4(m-1)(2m-1)(2m-5) + (m^2 - 2m - 2)(2m-5)^2$$

$$= 4m^4 - 12m^3 + 9m^2 + 42m - 66.$$

Case 2. If $s = 1$. In this case, there are six types of edges

TABLE 2. Edge partition of $P_m + P_{m-s}$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m-4,2m-4}$	$3m - 7$
$E_{m-4,m-4}$	$m - 4$
$E_{2m-3,2m-4}$	6
$E_{2m-3,m-4}$	$2(m-3)$
$E_{2m-4,m-4}$	$(m-2)(m-3) + 2$

Now, by using the edge partition given in Table 2, we get

$$UPM_2(G)) = (3m-7)(2m-4)^2 + (m-4)^3 + 6(2m-3)(2m-4)$$

$$+ 2(m-3)(2m-3)(m-4) + ((m-2)(m-3) + 2)(2m-4)(m-4)$$

$$= 2m^4 - 5m^3 - 6m^2 + 38m - 48.$$

Case 3. If $s \geq 2$. In this case, there are eight types of edges

TABLE 3. Edge partition of $P_m + P_{m-s}$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m-s-2,m-s-2}$	4
$E_{2m-s-2,m-s-3}$	$2(m-s-2)$
$E_{2m-s-3,m-s-2}$	$2(m-2)$
$E_{2m-s-3,m-s-3}$	$(m-2)(m-s-2)$
$E_{2m-s-2,2m-s-3}$	2
$E_{2m-s-3,2m-s-3}$	$m-3$
$E_{m-s-2,m-s-3}$	2
$E_{m-s-3,m-s-3}$	$m-s-3$

Now, by using the edge partition in Table 3, we get

$$\begin{aligned}
&UPM_2(G)\\
&= 4(2m-s-2)(m-s-2) + 2(m-s-2)(2m-s-2)(m-s-3)\\
&+ 2(m-2)(2m-s-3)(m-s-2) + (m-2)(m-s-2)(2m-s-3)(m-s-3)\\
&+ 2(2m-s-2)(2m-s-3) + (m-3)(2m-s-3)^2\\
&+ 2(m-s-2)(m-s-3) + (m-s-3)^3\\
&= 2m^4 - 5m^3s - 4m^3 + 4m^2s^2 + 8m^2s - 8m^2 + 9ms + 30m - ms^3 - 2ms^2\\
&- s^3 - 6s^2 - 19s - 26.
\end{aligned}$$

□

Theorem 2.6. *Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. Then,*

$$UPM_1^*(G) = \begin{cases} 4m^3 - 2m^2 - 12m + 36 & if \quad s = 0;\\ 3m^3 - 7m^2 + 8m & if \quad s = 1;\\ 3m^3 - 14m + 2s^2 + 8s - 5m^2s + 2ms^2 + 16 & if \quad s \geq 2. \end{cases}$$

Proof. In the same way as Theorem 2.5. □

Theorem 2.7. *Let $G \cong P_m + P_{m-s}$, where $m - s \geq 3$ be a join graph of $2m - s$ vertices. Then,*

$$UPF(G) = \begin{cases} 16m^4 - 120m^3 + 492m^2 - 826m + 496 & if \quad s = 0; \\ 9m^4 - 47m^3 + 108m^2 - 164m + 138 & if \quad s = 1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 9m^4 - 16m^3s - 45m^3 + 12m^2s^2 + 63m^2s + 111m^2 - 5ms^3 - 36ms^2 - 117ms - 144m + s^4 + 9s^3 + 39s^2 + 87s + 76.$

Proof. Similarly as in Theorem 2.4. □

Proposition 2.8. *Let $G \cong C_m + C_n$, where $3 \leq n < m$ be a join graph of $n + m$ vertices. Then,*

i. $UPM_1(G) = 2m^2n + m^3 - 2m^2 + mn^2 - 2mn + m + n^3 - 2n^2 + n,$

ii. $UPM_2(G) = m^3 + m^2n^2 + m^2n - 2m^2 + mn^3 - mn^2 - mn + m + n^3 - 2n^2 + n,$

iii. $UPM_1^*(G) = 2m^2 - 2m + 2n^2 - 2n + m^2n + 2mn^2,$

iv. $UPF(G) = m^4 + 3m^3n - 3m^3 + 3m^2n2 - 6m^2n + 3m^2 + mn^3 - 3mn^2 + 3mn - m + n^4 - 3n^3 + 3n^2 - n.$

Proof. Let $G \cong C_m + C_n$. There are m vertices of uphill degree $m + n - 1$ and n vertices of uphill degree $n - 1$. Then,

$$UPM_1(G) = m(m + n - 1)^2 + n(n - 1)^2$$
$$= 2m^2n + m^3 - 2m^2 + mn^2 - 2mn + m + n^3 - 2n^2 + n.$$

Similarly, we get

$$UPF(G) = m(m + n - 1)^3 + n(n - 1)^3$$
$$= m^4+3m^3n-3m^3+3m^2n2-6m^2n+3m^2+mn^3-3mn^2+3mn-m+n^4-3n^3+3n^2-n.$$

TABLE 4. Edge partition of $C_m + C_n$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{m+n-1,m+n-1}$	m
$E_{n-1,n-1}$	n
$E_{m+n-1,n-1}$	nm

Now, by using the edge partition in Table 4, we get

$$UPM_2(G)) = (m + n - 1)^2m + (n - 1)^2n + (n - 1)(m + n - 1)nm$$

$$= m^3 + m^2n^2 + m^2n - 2m^2 + mn^3 - mn^2 - mn + m + n^3 - 2n^2 + n.$$

Similarly, we get

$$UPM_1^*(G) = 2m^2 - 2m + 2n^2 - 2n + m^2n + 2mn^2$$

□

Theorem 2.9. *Let $G \cong C_{m+s} + P_m$, where $m \geq 3$ be a join graph of $2m+s$ vertices. Then,*

$$UPM_1(G) = \begin{cases} 8m^3 - 24m^2 + 26m - 10 & if \quad s = 0; \\ 5m^3 + 4m^2 + 21m - 18 & if \quad s = 1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 5m^3 + 8sm^2 - 10m^2 + 5ms^2 + 14m - 6ms + s^3 + s - 2s^2 - 10.$

Proof. Let $G \cong C_{m+s} + P_m$, where $m \geq 3$ be a join graph of $2m+s$ vertices. We have three cases:
Case 1. If $s = 0$. There are $2m-2$ vertices of uphill degree $2m-3$ and two vertices of uphill degree $2m-2$. So,

$$UPM_1(G) = (2m-2)(2m-3)^2 + 2(2m-2)^2$$

$$= 8m^3 - 24m^2 + 26m - 10.$$

Case 2. If $s = 1$. There are $m+3$ vertices of uphill degree $2m$ and $m-2$ vertices of uphill degree $m-3$. Then,

$$UPM_1(G) = (m+3)(2m)^2 + (m-2)(m-3)^2$$

$$= 5m^3 + 4m^2 + 21m - 18.$$

Case 3. If $s \geq 2$. There are $m+s$ vertices of uphill degree $2m+s-1$, two vertices of uphill degree $m-2$ and $m-2$ vertices of uphill degree $m-3$. Then,

$$UPM_1(G) = (m+s)(2m+s-1)^2 + 2(m-2)^2 + (m-2)(m-3)^2$$

$$= 5m^3 + 8m^2s - 10m^2 + 5ms^2 + 14m - 6ms + s^3 + s - 2s^2 - 10.$$

□

Theorem 2.10. *Let $G \cong C_{m+s} + P_m$, where $m \geq 3$ be a join graph of $2m+s$ vertices. Then,*

$$UPM_2(G) = \begin{cases} 4m^4 - 4m^3 - 15m^2 + 28m - 15 & if \quad s = 0; \\ 2m^4 + 8m^3 + 6m^2 & if \quad s = 1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 2m^4 + 3m^3s - 2m^3 + m^2s^2 - 2m^2s - 4m^2 + 2ms^2 + 3ms + 16m + s^3 - s - 15.$

Proof. Let $G \cong C_{m+s} + P_m$, where $m \geq 3$ be a join graph of $2m + s$ vertices. We have three cases:

Case 1. If $s = 0$. In this case, there are two types of edges

TABLE 5. Edge partition of $C_{m+s} + P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m-3,2m-3}$	$m^2 - 3$
$E_{2m-3,2m-2}$	$2m + 2$

Now, by using the partition in Table 5, we get

$$UPM_2(G) = (m^2 - 3)(2m - 3)^2 + (2m + 2)(2m - 3)(2m - 2)$$

$$= 4m^4 - 4m^3 - 15m^2 + 28m - 15.$$

Case 2. If $s = 1$. In this case, there are three types of edges

TABLE 6. Edge partition of $C_{m+s} + P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m,2m}$	$3m + 3$
$E_{m-3,m-3}$	$m - 3$
$E_{2m,m-3}$	$(m + 1)(m - 2) + 2$

Now, by using the partition in Table 6, we get

$$UPM_2(G) = 2m^2(m - 3)(m - 1)$$

$$= 2m^4 + 8m^3 + 6m^2.$$

Case 3. If $s \geq 2$. In this case, there are five types of edges

TABLE 7. Edge partition of $C_{m+s}+P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{2m+s-1,2m+s-1}$	$m+s$
$E_{m-2,m-3}$	2
$E_{m-3,m-3}$	$m-3$
$E_{2m+s-1,m-2}$	$2(m+s)$
$E_{2m+s-1,m-3}$	$(m-2)(m+s)$

Now, by using the edge partition in Table 7, we get

$$UPM_2(G) = (m+s)(2m+s-1)^2+2(m-2)(m-3)+(m-3)^3+2(m+s)(2m+s-1)(m-2)$$
$$+(m-2)(m+s)(2m+s-1)(m-3)$$
$$= 2m^4+3m^3s-2m^3+m^2s^2-2m^2s-4m^2+2ms^2+3ms+16m+s^3-s-15.$$

□

Theorem 2.11. *Let $G \cong C_{m+s}+P_m$, where $m \geq 3$ be a join graph of $2m+s$ vertices. Then,*

$$UPM_1^*(G) = \begin{cases} 4m^3+2m^2-14m+8 & if \quad s=0; \\ 3m^3+8m^2+3m+18 & if \quad s=1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 3m^3+2m^2+2ms-8m+2s^2+ms^2+4m^2s+8$.

Proof. Similarly as in Theorem 2.10. □

Theorem 2.12. *Let $G \cong C_{m+s}+P_m$, where $m \geq 3$ be a join graph of $2m+s$ vertices. Then,*

$$UPF(G) = \begin{cases} 16m^4-72m^3+132m^2-114m+38 & if \quad s=0; \\ 9m^4+13m^3+45m^2-81m+54 & if \quad s=1; \\ \Gamma & if \quad s \geq 2. \end{cases}$$

where, $\Gamma = 9m^4+20m^3s-21m^3+18m^2s^2+39m^2-24m^2s+7ms^3+9ms-15ms^2-58m+s^4+3s^2+38-3s^3-s$.

Proof. Similarly as in Theorem 2.9. □

Proposition 2.13. *Let $G \cong C_n+P_m$, where $n, m \geq 3$ be a join graph of $n+m$ vertices, if $n < m$. Then,*

i. $UPM_1(G) = n^3+n^2m-2n^2+2nm^2+5n-6nm+m^3+13m-6m^2-10,$

ii. $UPM_2(G) = n^3 - 3(n+m-3)^2 + 2n^2 - 11n + m(n+m-3)^2 + 7nm + 2m^2 - 10m + n^3m + n^2m^2 - nm^2 - 4n^2m + 12$,

iii. $UPM_1^*(G) = 2n^2 - 2n - 8m + nm^2 - 2nm + 2m^2 + 2n^2m + 8$,

iv. $UPF(G) = n^4 + n^3m - 3n^3 + 3n^2m^2 + 9n^2 - 9n^2m + 3nm^3 + 39nm - 18nm^2 - 31n + m^4 + 33m^2 + 38 - 9m^3 - 57m$.

Proof. Let $G \cong C_n + P_m$. The graph G has n vertices of uphill degree $n-1$, two vertices of uphill degree $n+m-2$ and $m-2$ vertices of uphill degree $n+m-3$. Then,

$$UPM_1(G) = n(n-1)^2 + 2(n+m-2)^2 + (m-2)(n+m-3)^2$$
$$= n^3 + n^2m - 2n^2 + 2nm^2 + 5n - 6nm + m^3 + 13m - 6m^2 - 10.$$

Similarly,

$$UPF(G) = n(n-1)^3 + 2(n+m-2)^3 + (m-2)(n+m-3)^3$$
$$= n^3 + n^2m - 2n^2 + 2nm^2 + 5n - 6nm + m^3 + 13m - 6m^2 - 10.$$

TABLE 8. Edge partition of $C_n + P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{n-1,n-1}$	n
$E_{n+m-2,n+m-3}$	2
$E_{n+m-3,n+m-3}$	$m-3$
$E_{n-1,n+m-2}$	$2n$
$E_{n-1,n+m-3}$	$n(m-2)$

Now, by using the edge partition in Table 8, we get

$$\begin{aligned} &UPM_2(G) \\ &= n(n-1)^2 + 2(n+m-2)(n+m-3) + (m-3)(n+m-3)^2 \\ &+ 2n(n-1)(n+m-2) + n(m-2)(n-1)(n+m-3) \\ &= n^3 - 3(n+m-3)^2 + 2n^2 - 11n + m(n+m-3)^2 + 7nm + 2m^2 - 10m \\ &+ n^3m + n^2m^2 - nm^2 - 4n^2m + 12. \end{aligned}$$

In the same way,

$$UPM_1^*(G) = 2n^2 - 2n - 8m + nm^2 - 2nm + 2m^2 + 2n^2m + 8.$$

□

The Corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the ith-copy of H with the ith-vertex of G.[13]

Proposition 2.14. *Let $G \cong C_n \circ P_m$, where $n \geq 3$ and $m \geq 2$ be a corona product graph of $n + nm$ vertices. Then,*

i. $UPM_1(G) = 2n(m+n-2)^2 + n(m-2)(m+n-3)^2 + n(n-1)^2$,

ii. $UPM_2(G) = 3n^3 - 10n^2 + 11n + n^2m^2 + n^3m + nm^2 - 7nm + nm\,(m+n-3)^2 - 3n\,(m+n-3)^2$,

iii. $UPM_1^*(G) = 8n - 12nm + 3nm^2 + 4n^2m$,

iv. $UPF(G) = 2n(m+n-2)^3 + n(m-2)(m+n-3)^3 + n(n-1)^3$.

Proof. Let $G \cong C_n \circ P_m$, where $n \geq 3$ and $m \geq 2$ be a corona product graph of $n + nm$ vertices. Then there are $2n$ vertices of uphill degree $m+n-2$, $n(m-2)$ vertices of uphill degree $m+n-3$ and n vertices of uphill degree $n-1$. So,

$$UPM_1(G) = 2n(m+n-2)^2 + n(m-2)(m+n-3)^2 + n(n-1)^2.$$

Then clearly,

$$UPF(G) = 2n(m+n-2)^3 + n(m-2)(m+n-3)^3 + n(n-1)^3.$$

Table 9. Edge partition of $C_n \circ P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{n-1,n-1}$	n
$E_{m+n-2,m+n-3}$	$2n$
$E_{m+n-3,m+n-3}$	$n(m-3)$
$E_{m+n-2,n-1}$	$2n$
$E_{m+n-3,n-1}$	$n(m-2)$

Now, by using the edge partition in Table 9, we get

$$\begin{aligned}
&UPM_2(G)\\
&= n(n-1)^2 + 2n(m+n-2)(m+n-3) + n(m-3)(m+n-3)^2\\
&+ 2n(m+n-2)(n-1) + n(m-2)(m+n-3)(n-1)\\
&= 3n^3 - 10n^2 + 11n + n^2m^2 + n^3m + nm^2 - 7nm + nm\,(m+n-3)^2\\
&- 3n\,(m+n-3)^2.
\end{aligned}$$

Similarly we get,

$$UPM_1^*(G) = 8n - 12nm + 3nm^2 + 4n^2m.$$

□

The Cartesian product G of two graphs G_1 and G_2, denoted $G_1 \times G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$, and two distinct vertices (a, b) and (c, d) of $G_1 \times G_2$ are adjacent if either $a = c$ and $bd \in E(G_2)$, or $b = d$ and $ac \in (G_1)$.

Proposition 2.15. *Let $G \cong P_m \square P_m$, where $m \geq 3$ be a Cartesian product graph of m^2 vertices. Then,*

i. $UPM_1(G) = 4m^2(m-2)^2 + 4(m-2)(m^2-3m+1)^2 + (m-2)^2(m^2-4m+3)^2$,

ii. $UPM_2(G) = 8(m^2 - 2m)(m^2 - 3m + 1) + 4(m - 3)(m^2 - 3m + 1)^2 + 4(m - 2)(m^2 - 3m + 1)(m^2 - 4m + 3) + (m - 3)(m - 2)(m^2 - 4m + 3)^2$,

iii. $UPM_1^*(G) = 2m^4 - 2m^3 - 18m^2 + 34m - 12$,

iv. $UPF(G) = 4m^3(m-2)^3 + 4(m-2)(m^2-3m+1)^3 + (m-2)^2(m^2-4m+3)^3$.

Proof. Let $G \cong P_m \square P_m$, where $m \geq 3$ be a Cartesian product graph of m^2 vertices. In Figure 1, we can see the graph G has four vertices are labeled by $(v_{1,1}, v_{1,m}, v_{m,1}, v_{m,m})$ of uphill degree $m(m-2)$, $4(m-2)$ vertices are labeled by $(v_{1,2}, v_{1,3}, ..., v_{1,m-1})$, $(v_{m,2}, v_{m,3}, ..., v_{m,m-1})$, $(v_{2,1}, v_{3,1}, ..., v_{m-1,1})$ and $(v_{2,m}, v_{3,m}, ..., v_{m-1,m})$ of uphill degree m^2-3m+1, $(m-2)^2$ vertices are labeled by $(v_{2,2}, v_{2,3}, ..., v_{2,m-1})$, $(v_{3,2}, v_{3,3}, ..., v_{3,m-1})$, ..., $(v_{m-1,2}, v_{m-1,3}, ..., v_{m-1,m-1})$ of uphill degree $m^2 - 4m + 3$. Then,

$UPM_1(G) = 4m^2(m-2)^2 + 4(m-2)(m^2 - 3m + 1)^2 + (m-2)^2(m^2 - 4m + 3)^2$.

Similarly,

$UPF(G) = 4m^3(m-2)^3 + 4(m-2)(m^2 - 3m + 1)^3 + (m-2)^2(m^2 - 4m + 3)^3$.

There are 4 types of edges.

TABLE 10. Edge partition of $P_m \square P_m$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{m(m-2),m^2-3m+1}$	8
E_{m^2-3m+1,m^2-3m+1}	$4(m-3)$
E_{m^2-3m+1,m^2-4m+3}	$4(m-2)$
E_{m^2-4m+3,m^2-4m+3}	$(m-3)(m-2)$

In Figure 1, the types of edges, $E_{m(m-2),m^2-3m+1}$, E_{m^2-3m+1,m^2-3m+1}, E_{m^2-3m+1,m^2-4m+3} and E_{m^2-4m+3,m^2-4m+3} are colored by red, blue, green and yellow respectively.

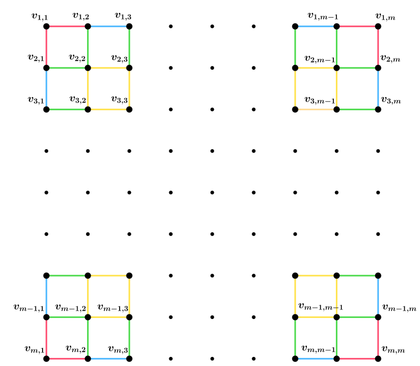

FIGURE 1. Cartesian product graph $P_m \square P_m$

Now, by using the partition in Table 10, we get

$$UPM_2(G) = 8(m^2-2m)(m^2-3m+1)+4(m-3)(m^2-3m+1)^2+4(m-2)(m^2-3m+1)(m^2-4m+3)+(m-3)(m-2)(m^2-4m+3)^2.$$

Similarly,

$$UPM_1^*(G) = 2m^4 - 2m^3 - 18m^2 + 34m - 12.$$

□

Proposition 2.16. *For any ladder graph $L_n = P_2 \square P_n$ with $2n$ vertices and $3n-2$ edges, where $n \geq 3$. Then,*

i. $UPM_1(L_n) = 8n^3 - 40n^2 + 82n - 64$,

ii. $UPM_2(L_n) = 12n^3 - 68n^2 + 147n - 122$,

iii. $UPM_1^*(L_n) = 12n^2 - 38n + 36$,

iv. $UPF(L_n) = 16n^4 - 120n^3 + 396n^2 - 634n + 392$.

Proof. For any ladder graph $L_n = P_2 \square P_n$ with $2n$ vertices and $3n-2$ edges, where $n \geq 3$. The graph L_n has four vertices of uphill degree $2n-3$ and $2n-4$ vertices of uphill degree $2n-5$. Then,

$$UPM_1(L_n) = 8n^3 - 40n^2 + 82n - 64.$$

Similarly,

$$UPF(L_n) = 16n^4 - 120n^3 + 396n^2 - 634n + 392.$$

There are three different types of edges; two edges of the type $E_{2n-3,2n-3}$, four edges of the type $E_{2n-3,2n-5}$ and $3n-8$ edges of the type $E_{2n-5,2n-5}$.Then,

$$UPM_2(L_n) = 12n^3 - 68n^2 + 147n - 122.$$

In the same way,

$$UPM_1^*(L_n) = 12n^2 - 38n + 36.$$

□

Proposition 2.17. *Let $G \cong C_m \square P_n$, where $m, n \geq 3$ be a stacked prism graph of mn vertices. Then,*

i. $UPM_1(G) = 2m(mn-m-1)^2 + (mn-2m)(mn-2m-1)^2$,

ii. $UPM_2(G) = 2m(mn-m-1)^2 + 2m(mn-m-1)(mn-2m-1) + (m(n-2) + m(n-3))(mn-2m-1)^2$,

iii. $UPM_1^*(G) = 4m^2n^2 - 10m^2n + 10m^2 - 4mn + 2m$

iv. $UPF(G) = 2m(mn-m-1)^3 + (mn-2m)(mn-2m-1)^3$.

Proof. Let $G \cong C_m \square P_n$, where $m, n \geq 3$ be a stacked prism graph of mn vertices. The graph G has $2m$ vertices which are based in the inner and outer circle of uphill degree $mn-m-1$ and all the other vertices $mn-2m$ has uphill degree $mn-2m-1$. Hence,

$$UPM_1(G) = 2m(mn-m-1)^2 + (mn-2m)(mn-2m-1)^2.$$

Similarly,

$$UPF(G) = 2m(mn-m-1)^3 + (mn-2m)(mn-2m-1)^3.$$

There are three types of edges.

Table 11. Edge partition of $C_m \square P_n$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{mn-m-1,mn-m-1}$	$2m$
$E_{mn-m-1,mn-2m-1}$	$2m$
$E_{mn-2m-1,mn-2m-1}$	$m(n-2)+m(n-3)$

In Figure 2, the types of edges, $E_{mn-m-1,mn-m-1}, E_{mn-m-1,mn-2m-1}$ and $E_{mn-2m-1,mn-2m-1}$ are colored by red, blue and green respectively.

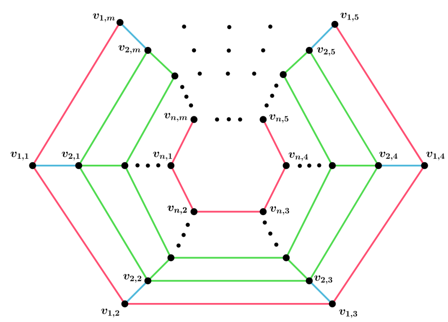

FIGURE 2. Stacked prism graph $C_m \square P_m$

Now, by using the partition in Table 11, we get

$UPM_2(G)$

$= 2m(mn-m-1)^2 + 2m(mn-m-1)(mn-2m-1) + (m(n-2)$

$+ m(n-3))(mn-2m-1)^2.$

Similarly,

$$UPM_1^*(G) = 4m^2n^2 - 10m^2n + 10m^2 - 4mn + 2m.$$

□

Honeycomb Networks[17]

Built recursively using the hexagon tessellation, honeycomb networks are widely used in computer graphics, cellular phone base stations, image processing, and in chemistry as the representation of benzenoid hydrocarbons.

Definition 2.18. [17] Honeycomb network $HC(p)$ is obtained from $HC(p-1)$ by adding a layer of hexagons around the boundary of $HC(p-1)$. The parameter p of $HC(p)$ is determined as the number of hexagons between the center and boundary of $HC(p)$. The number of vertices and edges of $HC(p)$ are $6p^2$ and $9p^2-3p$ respectively. In honeycomb network, there are $6p$ vertices of degree two and the remaining vertices are of degree 3. There are three types of edges based on the degree of the vertices of each edge.

Theorem 2.19. *Let G be the honeycomb network $HC(p)$ of dimension p with $6p^2$ vertices and $9p^2-3p$ of edges. Then,*

i. $UPM_1(G) = 12(6p^2-6p+1)^2+(6p-12)(6p^2-6p)^2+(6p^2-6p)(6p^2-6p-1)^2,$

ii. $UPM_2(G) = 6(6p^2-6p+1)^2+12(6p^2-6p+1)(6p^2-6p-1)+(9p^2-15p+6)(6p^2-6p-1)^2$,

iii. $UPM_1^*(G) = 108p^4-288p^3+450p^2-258p$,

iv. $UPF(G) = 12(6p^2-6p+1)^3+(6p-12)(6p^2-6p)^3+(6p^2-6p)(6p^2-6p-1)^3$.

Proof. Let G be the honeycomb network $HC(p)$ of dimension p with $6p^2$ vertices and $9p^2-3p$ of edges. The graph G has $2p$ lines are named by $L_1, L_2, ..., L_{2p}$ witch are based from up to down, it is clearly to see, L_1 symmetric with L_{2p}, L_2 symmetric with L_{2p-1} ... L_p symmetric with L_{p+1}. There are 12 vertices of uphill degree $6p^2-6p+1$, $6p-12$ vertices of uphill degree $6p^2-6p$ and $6p^2-6p$ of uphill degree $6p^2-6p-1$. Hence,

$$UPM_1(G) = 12(6p^2-6p+1)^2+(6p-12)(6p^2-6p)^2+(6p^2-6p)(6p^2-6p-1)^2.$$

In the same way we get,

$$UPF(G) = 12(6p^2-6p+1)^3+(6p-12)(6p^2-6p)^3+(6p^2-6p)(6p^2-6p-1)^3.$$

In the following table there are four types of edges.

TABLE 12. Edge partition of $HC(p)$ graph based on uphill degree of end vertices.

Type	Number of edges
$E_{6p^2-6p+1,6p^2-6p+1}$	6
$E_{6p^2-6p+1,6p^2-6p-1}$	12
$E_{6p^2-6p,6p^2-6p-1}$	$12p-24$
$E_{6p^2-6p-1,6p^2-6p-1}$	$9p^2-15p+6$

In Figure 3, the types of edges, $E_{6p^2-6p+1,6p^2-6p+1}$, $E_{6p^2-6p+1,6p^2-6p-1}$, $E_{6p^2-6p,6p^2-6p-1}$ and $E_{6p^2-6p-1,6p^2-6p-1}$ are colored by red, blue, green and black respectively.

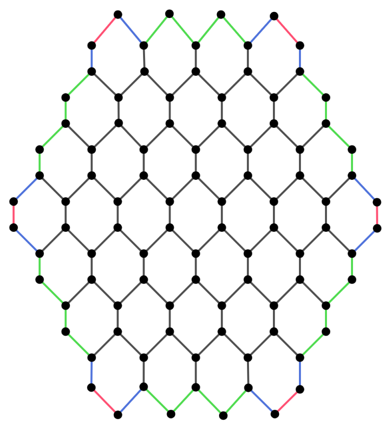

FIGURE 3. Honeycomb network $HC(4)$

Now, by using the partition in Table 12, we get

$$\begin{aligned}&UPM_2(G)\\&= 6(6p^2 - 6p + 1)^2 + 12(6p^2 - 6p + 1)(6p^2 - 6p - 1)\\&+ (9p^2 - 15p + 6)(6p^2 - 6p - 1)^2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}UPM_1^*(G) = &\sum_{vu\in E(G)} (d_{up}(v) + d_{up}(u))\\&= 6(12p^2 - 12p + 2) + 12(12p^2 - 12p) + (12p - 24)(12p^2 - 12p - 1)\\&+ (9p^2 - 15p + 6)(12p^2 - 12p - 2).\end{aligned}$$

Hence,

$$UPM_1^*(G) = 108p^4 - 288p^3 + 450p^2 - 258p.$$

We outline the graphical profiles of the first uphill Zagreb index, second uphill Zagreb index, forgotten uphill Zagreb index and modified uphill Zagreb index for the honeycomb network of dimension p in Figure 4.

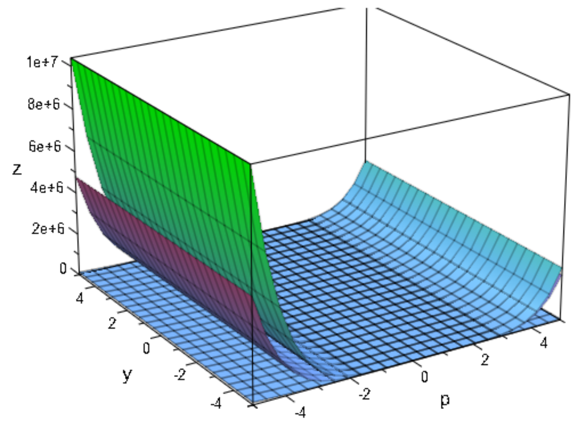

FIGURE 4. Graphical profile of uphill indices for $HC(p)$

□

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