J. Appl. Math. & Informatics Vol. **40***(2022), No. 5 - 6, pp. 979 - 982*
https://doi.org/10.14317/jami.2022.979

NEW INEQUALITIES CONNECTED WITH TRACES OF MATRICES

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Abstract. This paper covers some important operator inequalities connected with traces of matrices, from the classical inequalities we obtained new inequalities connected with traces of matrices which are better than the others.

AMS Mathematics Subject Classification : 15A60, 47A63, 47B15.
Key words and phrases : Trace marix inequalities, positie semidefinite matrices.

1. Introduction

If $X > 0$ and $Y > 0$, then

$$n(detX.detY)^{\frac{m}{n}} \leq tr(X^m Y^n)$$

for any positive integer m, n.
Also, Ali M. Farah proved that If $B, D \in H_n$, then

$$\lambda_1(\frac{B+D}{2}) \leq \frac{1}{2}[\lambda_1(B) + \lambda_1(D)],$$

$$\lambda_n(\frac{B+D}{2}) \leq \frac{1}{2}[\lambda_1(B) + \lambda_n(D)].$$

In this article by using some inequalities in [3, 4, 5, 6, 7, 8, 9] we obtained new inequalities connected with traces of matrices which are better than the others.or Introduction

Received October 6, 2021. Revised March 6, 2022. Accepted May 16, 2022. *Corresponding author.

2. Classical Inequalities

Theorem 2.1. [3] *If* $X > 0$ *and* $Y > 0$*, then* $n(detX.detY)^{\frac{m}{n}} \leq tr(X^m Y^n)$ *for any positive integer* m, n*.*

Corollary 2.2. [3] *Let* X *and* Z *be positive definite*$n \times n$*-matrices, such that ,detZ=1 then* $n(detX)^{\frac{1}{n}} \leq tr(XZ)$*.*

Theorem 2.3. [1] *If* $B, D \in H_n$*, then* $\lambda_1(\frac{B+D}{2}) \leq \frac{1}{2}[\lambda_1(B)+\lambda_1(D)]$, $\lambda_n(\frac{B+D}{2}) \leq \frac{1}{2}[\lambda_1(B) + \lambda_n(D)]$.

Theorem 2.4. [4] *Let* Y *and* Z *be positive semi-definite matrices of the same order. Then for n=1, 2, ...,* $0 \leq tr(YZ)^{2n} \leq (trY)^2(trY^2)^{n-1}(trZ^2)^n$*,*
and $0 \leq tr(YZ)^{2n+1} \leq tr(Y)tr(Z)tr(Y^2)^n tr(Z^2)^n$*.*

Theorem 2.5. [2] *The sum of eigen values of matrix equals its trace* $\lambda_1 + \lambda_2 + ... + \lambda_n = trB = b_{11} + b_{22} + ... + b_{mm}$*. The product of the eigenvalues equals its determinant* $\lambda_1\lambda_2...\lambda_n = detB$*.*

Corollary 2.6. [4] *If* X *and* Y *are defined in Theorem (4), then* $0 \leq tr(XY)^n \leq tr(X)^n tr(Y)^n$*.*

3. New Results

Theorem 3.1. *If* $X > 0$*, then* $n(detX)^{\frac{m}{n}} \leq tr(X^m)$ *for any positive integer* m, n*.*

Proof. Letting $Y = I$ in Theorem 2.1. Then, we have $n(detX.detI)^{\frac{m}{n}} \leq tr(X^m I^n)$, $n(detX)^{\frac{m}{n}} \leq tr(X^m)$, and we get the result. □

Theorem 3.2. *Let* X *be positive definite* $n \times n$ *matrices. Then* $n(detX)^{\frac{1}{n}} \leq tr(X)$*.*

Proof. Letting $X = I$ in Corollary (2.2). Then, we have $n(detX)^{\frac{1}{n}} \leq tr(XI) \leq tr(X)$. □

Theorem 3.3. *If* $B \in H_n$ *and* $D = B + 1$*, then,* $\lambda_1(\frac{2B+I}{2}) \leq \lambda_1(B) + \frac{1}{2}$ *and* $\lambda_n(\frac{2B+I}{2}) \leq \lambda_n(B) + \frac{1}{2}$

Proof. By applying Theorem(2) and $D = B + 1$, then we have,

$$\begin{gathered}\lambda_1(\tfrac{B+B+I}{2}) = \lambda_1(\tfrac{2B+I}{2}),\\ \leq \lambda_1(B) + \lambda_1(\tfrac{1}{2})\\ \leq \lambda_1(B) + (\tfrac{1}{2})\end{gathered}$$

□

Similarly proof for λ_n

Proof. By applying Theorem(2) and $D = B + 1$, then we have,

$$\begin{gathered}\lambda_n(\tfrac{B+B+I}{2}) = \lambda_n(\tfrac{2B+I}{2}),\\ \leq \lambda_n(B) + \lambda_n(\tfrac{1}{2})\\ \leq \lambda_n(B) + (\tfrac{1}{2})\end{gathered}$$

□

Theorem 3.4. *If X is positive definite matrices of the order n,then*

$$0 \leq tr(X)^n \leq tr(X)^n (n)^n,$$

Proof. Let $Y = I$ on Corollary (2.6). Then

$$\begin{gathered}0 \leq tr(XI)^n \leq tr(X)^n (trI)^n,\\ = tr(X)^n (n)^n\end{gathered}$$

□

Theorem 3.5. *Let Y be positive semi-definite matrices of the order n,then for $n = 1, 2, ...$*

$$0 \leq tr(Y)^{2n} \leq tr(Y)^2 tr(Y^2)^{n-1} (n)^n,$$

and

$$0 \leq tr(Y)^{2n+1} \leq tr(Y) tr(Y^2)^n (n)^{n+1}.$$

Proof. Let $Z = I$ in Theorem (3). Then we have

$$\begin{gathered}0 \leq tr(YI)^{2n} \leq tr(Y)^2 tr(Y^2)^{n-1} (trI^2)^n,\\ = tr(Y)^2 tr(Y^2)^{n-1} (n)^n\end{gathered}$$

□

Also,

$$\begin{gathered}0 \leq tr(Y)^{2n+1} \leq ntr(Y) tr(Y^2)^n (n)^n.\\ = tr(Y) tr(Y^2)^n (n)^{n+1}.\end{gathered}$$

So; we get the results.

Remark 3.1. The new results are sometimes better than the classical results for some cases.

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