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EVALUATION OF THE CONVOLUTION SUMS $\sum_{ak+bl+cm=n}\sigma(k)\sigma(l)\sigma(m)$, $\sum_{al+bm=n} l\sigma(l)\sigma(m)$ AND $\sum_{al+bm=n}\sigma_3(l)\sigma(m)$ FOR DIVISORS a, b, c OF $10^{\dagger}$

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ABSTRACT. The generating functions of the divisor function $\sigma_s(n) = \sum_{0<d|n} d^s$ are quasimodular forms. In this paper, we find the basis of the space of quasimodular forms of weight 6 on $\Gamma_0(10)$ consisting of Eisenstein series and η-quotients. Then we evaluate the convolution sum $\sum_{ak+bl+cm=n}\sigma(k)\sigma(l)\sigma(m)$ with $\operatorname{lcm}(a,b,c)=10$ and $\sum_{al+bm=n} l\sigma(l)\sigma(m)$ and $\sum_{al+bm=n}\sigma_3(l)\sigma(m)$ with $\operatorname{lcm}(a,b)=10$.

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1. Introduction

For a non-negative integer n we denote $\sigma_s(n)$ by the classical divisor function $\sigma_s(n) := \sum_{0<d|n} d^s$. We also denote $W_{a,b}(n)$ by the convolution sum of divisor functions defined by

$$W_{a,b}(n) := \sum_{\substack{l,m\geq 1\\ al+bm=n}} \sigma(l)\sigma(m),$$

where $\sigma(n) := \sigma_1(n)$. The evaluation of $W_{a,b}$ is expression as a linear combination of well-known elementary functions, and it has been studied by Ramanujan and many mathematicians for $1 \leq \operatorname{lcm}(a,b) \leq 36$, except $\operatorname{lcm}(a,b) = 21, 29, 31, 34$ and 35 ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 20, 21, 22, 26, 27, 28, 31, 32, 33, 34]). Recently, $W_{1,41}, W_{1,47}, W_{1,59}, W_{1,71}$ were evaluated by Cho([10]).

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The convolution sum

$$W_{a,b,c}(n) := \sum_{\substack{k,l,m\geq 1\\ ak+bl+cm=n}} \sigma(k)\sigma(l)\sigma(m) \qquad (\mathrm{lcm}(a,b,c)\leq 9)$$

of three divisor functions was studied in [23, 25]. For positive integers a_1, a_2, a_3, a_4 with $\mathrm{lcm}(a_1,a_2,a_3,a_4) \leq 4$,

$$W_{a_1,a_2,a_3,a_4}(n) := \sum_{\substack{m_1,m_2,m_3,m_4\geq 1\\ a_1m_1+a_2m_2+a_3m_3+a_4m_4}} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4)$$

was considered in [17] by Lee and the author.

Lahiri used $S[(r_1,\dots,r_t),(s_1,\dots,s_t),(a_1,\dots,a_t)]$ to consider more general convolution sums[16]:

$$\begin{aligned} &S[(r_1,\dots r_t),(s_1,\dots,s_t),(a_1,\dots,a_t)](n)\\ := &\sum_{\substack{m_1,\dots,m_t\geq 1\\ a_1m_1+\cdots+a_tm_t=n}} m_1^{r_1}\cdots m_t^{r_t}\sigma_{s_1}(m_1)\cdots\sigma_{s_t}(m_t),\end{aligned}$$

where t, r_j, s_j, a_j are non-negative integers satisfying $t>0, a_j>0$ and s_j is odd. One can evaluate $S[(r_1,\dots,r_t),(s_1,\dots,s_t),(a_1,\dots,a_t)]$ because we know that it is the nth Fourier coefficient of a certain quasimodular form of weight $t+s_1+\cdots+s_t+2(r_1+\cdots r_t)$ and a level depending on $a_1,\dots,a_t$.

When t, r_j, s_j and a_j are small, the evaluation of

$$S[(r_1,\dots r_t),(s_1,\dots,s_t),(a_1,\dots,a_t)]$$

can be obtained by elementary functions. For any t, r_j, s_j, a_j, one can evaluate this general convolution sum by using the theory of modular form.

In detail, $S[(r_1,\dots r_t),(s_1,\dots,s_t),(a_1,\dots,a_t)]$ can be evaluated by finding a basis of the respective space of quasimodular forms in terms of well-known functions, such as the Eisenstein series and η-quotients.

In this paper, we evaluate the following convolution sums.

$$\sum_{ak+bl+cm=n} \sigma(k)\sigma(l)\sigma(m) \qquad (\mathrm{lcm}(a,b,c)=10),$$

$$\sum_{al+bm=n} l\sigma(l)\sigma(m) \qquad (\mathrm{lcm}(a,b)=10),$$

and

$$\sum_{al+bm=n} \sigma_3(l)\sigma(m) \qquad (\mathrm{lcm}(a,b)=10).$$

For $ab\leq 9$, the second sums are evaluated in [24]. Here, we need the modular form of weight 6 and level 10. Before stating our main work precisely, we introduce some necessary functions below.

Definition 1.1. Let $\mathcal{H}$ be the complex upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $q = \exp(2\pi i z)$ for $z \in \mathcal{H}$. We define the following q-series.

$$\begin{aligned}
\delta_5(z) &:= \eta^4(z)\eta^4(5z) = \sum_{n=1}^{\infty} b_5(n)q^n, \\
\delta_{10}(z) &:= \frac{\eta^2(2z)\eta^8(5z)}{\eta^2(10z)} = \sum_{n=1}^{\infty} b_{10}(n)q^n, \\
\Delta_5(z) &:= 6\left(G_2(z) - 5G_2(5z)\right)\delta_5(z) = \sum_{n=1}^{\infty} c_5(n)q^n, \\
\Delta_{10,1}(z) &:= 24\left(G_2(z) - 2G_2(2z)\right)\delta_{10}(z) = \sum_{n=1}^{\infty} c_{10,1}(n)q^n, \\
\Delta_{10,2}(z) &:= 6\left(G_2(z) - 5G_2(5z)\right)\delta_{10}(z) = \sum_{n=1}^{\infty} c_{10,2}(n)q^n, \\
\Delta_{10,3}(z) &:= 8\left(G_2(z) - 10G_2(10z)\right)\delta_{10}(z) = \sum_{n=1}^{\infty} c_{10,3}(n)q^n,
\end{aligned}$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind η-function and $G_2(z) = -1/24 + \sum_{n=1}^{\infty}\sigma(n)q^n$ is the generating function of $\sigma(n)$.

We state our main results as given below.

Theorem 1.2. *For a non-negative integer n, let $W(n)$ be the one of the three functions :*

(1) *with* $lcm(a,b,c) = 10$,

$$S[(0,0,0),(1,1,1),(a,b,c)](n) = \sum_{ak+bl+cm=n} \sigma(k)\sigma(l)\sigma(m),$$

(2) *with* $lcm(a,b) = 10$,

$$S[(1,0),(1,1),(a,b)](n) = \sum_{al+bm=n} l\sigma(l)\sigma(m),$$

(3) *with* $lcm(a,b) = 10$,

$$S[(0,0),(3,1),(a,b)](n) = \sum_{al+bm=n} \sigma_3(l)\sigma(m).$$

Then, $W(n)$ can be written as a linear combination of $\sigma(n/d), \sigma_3(n/d), \sigma_5(n/d)$, $b_5(n)$, $b_5(n/2), b_{10}(n)$, $c_5(n), c_5(n/2), c_{10,1}(n), c_{10,2}(n)$ and $c_{10,3}(n)$ for positive integers n, d, s with $d \mid 10$.

This paper is organized as follows. In §2, we find a basis for the space of quasimodular forms of weight 6 and level 10. In §3, we state the explicit form of Theorem 1.2 separately in Theorems 3.1, 3.4 and 3.6. We used a MAPLE program to perform all calculations in this paper.

2. Basis of the vector spaces $M_k(\Gamma_0(10))$ of modular forms with $k = 2, 4, 6$

In this section, we state a brief theory of quasimodular forms. For further details, the reader is invited to refer to the works of Royer[29] or Kaneko and Zagier[15]. Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

For a positive integer N, the congruence subgroup $\Gamma_0(N)$ of level N is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, b, c, d \in \mathbb{Z} \text{ and } N|c \right\}.$$

A *modular form of weight k and level N* is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that:

(1) for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and any $z \in \mathcal{H}$,

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = f(z),$$

(2) f is holomorphic on the set $\mathbb{Q} \cup \{\infty\}$.

We write $M_k(\Gamma_0(N))$ for the space of modular forms of weight k on $\Gamma_0(N)$. It is a finite dimensional vector space over $\mathbb{C}$. When $f \in M_k(\Gamma_0(N))$ vanishes at all $x \in \mathbb{Q} \cup \{\infty\}$, we call f *a cuspform* and write $S_k(\Gamma_0(N))$ for the space of cuspforms.

Let G_k be the Eisenstein series of weight k defined by

$$G_k(z) := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number. We write $E_k(\Gamma_0(N))$ for the vector space generated by

$$\begin{cases} < G_k(dz) : d|N >, & \text{if } 2|k \geq 4 \\ < \sum_{d|N} c_d G_2(dz) : \sum c_d/d = 0 >, & \text{if } k = 2. \end{cases}$$

Then, $M_k(\Gamma_0(N)) = S_k(\Gamma_0(N)) \oplus E_k(\Gamma_0(N))$.

The quasimodular form is defined as follows. Denote by D the differential operator

$$D := \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}.$$

Definition 2.1. Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function, k and $s \geq 0$ be integers. The function f is a *quasimodular form* of weight k and depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, \ldots, f_s$ over $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j$$

for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and any $z \in \mathcal{H}$. Here, f_s is not identically zero.

We write $\widetilde{M}_k^{(\leq s)}(\Gamma_0(N))$ for the space of quasimodular forms of weight k and depth s on $\Gamma_0(N)$ and $\widetilde{M}_k(\Gamma_0(N))$ for $\widetilde{M}_k^{(\leq k/2)}(\Gamma_0(N))$.

The space $\widetilde{M}_k(\Gamma_0(N))$ is a vector space with the following structure([15]).

$$\widetilde{M}_k(\Gamma_0(N)) \quad = \quad \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(\Gamma_0(N)) \oplus < D^{k/2-1} G_2(z) > .$$

We need the space of quasimodular formrs of weight 6 and level 10. By the above structure formula, we have

$$\widetilde{M}_6(\Gamma_0(10)) = M_6(\Gamma_0(10)) \oplus DM_4(\Gamma_0(10)) \oplus < D^2 G_2(dz) : d = 1, 2, 5, 10 >$$

because there is no cuspform of weight 2 and level 10. We find a basis of $\widetilde{M}_6(\Gamma_0(10))$ in terms of η-quotients and the Eisenstein series.

Lemma 2.2. (1) *For $k = 2, 4$ and 6, let $\mathcal{B}_k$ be the set defined by*

$$\begin{aligned}
\mathcal{B}_2 &= \{G_2(z) - 2G_2(2z), G_2(z) - 5G_2(5z), G_2(z) - 10G_2(10z)\}, \\
\mathcal{B}_4 &= \{G_4(z), G_4(2z), G_4(5z), G_4(10z), \delta_5(z), \delta_5(2z), \delta_{10}(z)\}, \\
\mathcal{B}_6 &= \{G_6(q), G_6(2z), G_6(5z), G_6(10z), \Delta_5(z), \Delta_5(2z), \Delta_{10,1}(z), \Delta_{10,2}(z), \Delta_{10,3}(z)\}.
\end{aligned}$$

Then, $\mathcal{B}_k$ $(k = 2, 4, 6)$ is a basis for the space $M_k(\Gamma_0(10))$.

(2) *Let $\mathcal{B}'_6$ be the subset of $\mathcal{B}_6$ defined by*

$$\mathcal{B}'_6 := \{\Delta_5(z), \Delta_5(2z), \Delta_{10,1}(z), \Delta_{10,2}(z), \Delta_{10,3}(z)\}.$$

Then, the newforms $f_{6,10,j}$ $(j = 1, 2, 3)$ of weight 6 ([19] or [28]) can be written as combinations of the elements of $\mathcal{B}'_6$.

$$\begin{aligned}
f_{6,10,1}(z) &= -\frac{3}{4}\Delta_5(z) + 8\Delta_5(2z) - \frac{1}{8}\Delta_{10,1}(z) - \frac{15}{4}\Delta_{10,2}(z) + \frac{15}{8}\Delta_{10,3}(z), \\
f_{6,10,2}(z) &= -\frac{3}{4}\Delta_5(z) + 8\Delta_5(2z) - \frac{7}{6}\Delta_{10,1}(z) + \frac{35}{12}\Delta_{10,2}(z), \\
f_{6,10,3}(z) &= -\frac{1}{4}\Delta_5(z) - 8\Delta_5(2z) + \frac{5}{8}\Delta_{10,1}(z) - \frac{5}{4}\Delta_{10,2}(z) + \frac{5}{8}\Delta_{10,3}(z).
\end{aligned}$$

Proof. (1) By [30, Proposition 6.1], we know that the dimensions of spaces are as follows.

$$\begin{aligned}
\dim_{\mathbb{C}} S_2(\Gamma_0(10)) &= 0, & \dim_{\mathbb{C}} E_2(\Gamma_0(10)) &= 3, \\
\dim_{\mathbb{C}} S_4(\Gamma_0(10)) &= 3, & \dim_{\mathbb{C}} E_4(\Gamma_0(10)) &= 4,
\end{aligned}$$

$$\dim_{\mathbb{C}} S_6(\Gamma_0(10)) = 5, \qquad \dim_{\mathbb{C}} E_6(\Gamma_0(10)) = 4.$$

We now prove that $\mathcal{B}_2$ is a basis of the space $M_2(\Gamma_0(10))$. Because the Eisenstein series $G_2(z)$ satisfies

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \frac{c(cz+d)}{4\pi i}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $G_2(z) - mG_2(mz)$ is a modular form of weight k on $\Gamma_0(10)$ when m is a positive divisor of 10. By using q-expansion, we can obtain that $\mathcal{B}_2$ is linearly independent. Thus, $\mathcal{B}_2$ is a basis of $M_2(\Gamma_0(10))$.

We also consider the set $\mathcal{B}_4$. In [11, 18] (respectively, [12]) the function $\delta_5(z)$ (respectively, $\delta_{10}(z)$) is used to calculate the convolution sums of divisor sums of weight 4 and level 5 (respectively, 10). That is, $\delta_5(z), \delta_5(2z)$ and $\delta_{10}(z)$ are modular forms of weight 4 on $\Gamma_0(10)$. Because these three functions are cuspforms which are linearly independent and the dimension of $S_4(\Gamma_0(10)) = 3$, $\mathcal{B}_4$ is a basis of $M_4(\Gamma_0(10))$.

To prove that $\mathcal{B}_6$ is a basis of $M_6(10)$, it suffices to show that $\mathcal{B}_6'$ is a basis for $S_6(\Gamma_0(10))$. In the definition, all elements of B_6' are cuspforms of weight 6 on $\Gamma_0(10)$, because they are defined as the product of cuspforms of weight 4 and one of the elements of $\mathcal{B}_2$.

By referring to the following q-expansions of the elements of $\mathcal{B}_6'$,

$$\begin{aligned}
\Delta_5(z) &= q + 2\,q^2 - 4\,q^3 - 28\,q^4 + 25\,q^5 - 8\,q^6 + 192\,q^7 - 120\,q^8 - 227\,q^9 + O\left(q^{10}\right), \\
\Delta_5(2z) &= q^2 + 2\,q^4 - 4\,q^6 - 28\,q^8 + O\left(q^{10}\right), \\
\Delta_{10,1}(z) &= q + 24\,q^2 + 22\,q^3 + 48\,q^4 - 25\,q^5 - 80\,q^6 - 166\,q^7 - 320\,q^8 - 527\,q^9 + O\left(q^{10}\right), \\
\Delta_{10,2}(z) &= q + 6\,q^2 + 16\,q^3 + 12\,q^4 + 5\,q^5 - 56\,q^6 - 76\,q^7 - 104\,q^8 - 155\,q^9 + O\left(q^{10}\right) \\
&\text{and} \\
\Delta_{10,3}(z) &= 3\,q + 8\,q^2 + 18\,q^3 + 16\,q^4 + 5\,q^5 - 48\,q^6 - 98\,q^7 - 192\,q^8 - 205\,q^9 + O\left(q^{10}\right),
\end{aligned}$$

we can check that they are linearly independent by finding echelon forms as follows.

$$\begin{aligned}
&\Delta_5(z) = q + 2\,q^2 - 4\,q^3 - 28\,q^4 + 25\,q^5 - 8\,q^6 + O\left(q^7\right), \\
&\Delta_5(2z) = q^2 + 2\,q^4 - 4\,q^6 - 28\,q^8 + O\left(q^{10}\right), \\
&\Delta_{10,1}(z) - \Delta_5(z) - 22\Delta_5(2z) = 26\,q^3 + 32\,q^4 - 50\,q^5 + 16\,q^6 + O\left(q^7\right), \\
&\Delta_{10,2}(z) - \frac{3}{13}\Delta_5(z) + \frac{168}{13}\Delta_5(2z) - \frac{10}{13}\,\Delta_{10,1}(z) \\
&= \frac{96}{13}q^4 + \frac{240}{13}q^5 - \frac{576}{13}q^6 + O\left(q^7\right), \\
&\Delta_{10,3}(z) - 80\Delta_5(2z) + 5\Delta_{10,1}(z) - 8\Delta_{10,2}(z) = -160\,q^5 + 320\,q^6 + O\left(q^7\right).
\end{aligned}$$

Hence, $\mathcal{B}_6'$ is proven be a basis of $S_6(\Gamma_0(10))$.

(2) The set $\{\Delta_{10,1}(z), \Delta_{10,2}(z), \Delta_{10,3}(z)\}$ cannot generate the space of newforms of weight 6 and level 10. By using the q-expansions of newforms $f_{6,10,j}$ $(j = 1, 2, 3)$ of weight 6 in [19] or [28], we obtain the following.

$$\begin{aligned} f_{6,10,1}(z) &= q - 4\,q^2 - 26\,q^3 + 16\,q^4 - 25\,q^5 + 104\,q^6 - 22\,q^7 - 64\,q^8 + 433\,q^9 + O\left(q^{10}\right), \\ f_{6,10,2}(z) &= q - 4\,q^2 + 24\,q^3 + 16\,q^4 + 25\,q^5 - 96\,q^6 - 172\,q^7 - 64\,q^8 + 333\,q^9 + O\left(q^{10}\right), \\ f_{6,10,3}(z) &= q + 4\,q^2 + 6\,q^3 + 16\,q^4 - 25\,q^5 + 24\,q^6 - 118\,q^7 + 64\,q^8 - 207\,q^9 \\ &\quad -100\,q^{10} + O\left(q^{10}\right). \end{aligned}$$

Using this result together with the q-expansions of the elements of the set $\mathcal{B}_6'$ mentioned above completes the proof. □

The following lemma is clearly obtained from Lemma 2.2 (1), but it plays a major role in proving the main theorem.

Lemma 2.3. *Define*

$$\mathcal{B} = \mathcal{B}_6 \cup \{Df : f \in \mathcal{B}_4\} \cup \left\{D^2 G_2(dz) : 0 < d|10\right\}.$$

Then $\mathcal{B}$ is a basis of $\widetilde{M}_6(\Gamma_0(10))$.

3. Proofs of Theorem 1.2

Theorem 1.2 evaluates three convolution sums. Here, we separate each type of convolution sums in Theorem 3.1, 3.4 and 3.6. We also use the theory of modular forms and basis $\mathcal{B}$ found in Lemma 2.3.

Theorem 3.1. (1)

$$\begin{aligned} &\sum_{k+l+10m=n} \sigma(k)\sigma(l)\sigma(m) \\ &= \frac{65}{187488}\sigma_5(n) + \frac{13}{11718}\sigma_5\left(\frac{n}{2}\right) + \frac{3125}{374976}\sigma_5\left(\frac{n}{5}\right) \\ &+ \frac{625}{23436}\sigma_5\left(\frac{n}{10}\right) + \frac{44-15n}{2496}\sigma_3(n) + \frac{1-3n}{936}\sigma_3\left(\frac{n}{2}\right) \\ &+ \frac{25-75n}{3744}\sigma_3\left(\frac{n}{5}\right) + \frac{25-75n}{936}\sigma_3\left(\frac{n}{10}\right) + \frac{5-33n+12n^2}{1440}\sigma(n) \\ &+ \frac{1-12n+24n^2}{576}\sigma\left(\frac{n}{10}\right) + \frac{1-3n}{12480}b_5(n) + \frac{3n-1}{936}b_5\left(\frac{n}{2}\right) \\ &+ \frac{3n-1}{576}b_{10}(n) + \frac{631}{1249920}c_5(n) - \frac{13}{11718}c_5\left(\frac{n}{2}\right) - \frac{11}{24192}c_{10,1}(n) \\ &- \frac{37}{8064}c_{10,2}(n) + \frac{1}{8064}c_{10,3}(n), \end{aligned}$$

(2)

$$\sum_{k+2l+5m=n} \sigma(k)\sigma(l)\sigma(m)$$

$$
\begin{aligned}
&= \frac{65}{187488}\sigma_5(n) + \frac{13}{11718}\sigma_5\left(\frac{n}{2}\right) + \frac{3125}{374976}\sigma_5\left(\frac{n}{5}\right) + \frac{625}{23436}\sigma_5\left(\frac{n}{10}\right)\\
&+\frac{320-261n}{74880}\sigma_3(n) + \frac{45-31n}{3120}\sigma_3\left(\frac{n}{2}\right) + \frac{100-175n}{4992}\sigma_3\left(\frac{n}{5}\right)\\
&+\frac{25-75n}{1872}\sigma_3\left(\frac{n}{10}\right) + \frac{5-21n+12n^2}{2880}\sigma(n) + \frac{5-36n+24n^2}{2880}\sigma\left(\frac{n}{2}\right)\\
&+\frac{1-9n+12n^2}{576}\sigma\left(\frac{n}{5}\right) + \frac{29n-11}{8320}b_5(n) + \frac{93n-31}{9360}b_5\left(\frac{n}{2}\right) + \frac{1-3n}{1152}b_{10}(n)\\
&-\frac{659}{499968}c_5(n) - \frac{781}{117180}c_5\left(\frac{n}{2}\right) + \frac{247}{241920}c_{10,1}(n) - \frac{67}{16128}c_{10,2}(n)\\
&+\frac{23}{16128}c_{10,3}(n),
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\sum_{k+2l+10m=n}\sigma(k)\sigma(l)\sigma(m)\\
&= \frac{13}{187488}\sigma_5(n) + \frac{65}{46872}\sigma_5\left(\frac{n}{2}\right) + \frac{625}{374976}\sigma_5\left(\frac{n}{5}\right) + \frac{3125}{93744}\sigma_5\left(\frac{n}{10}\right)\\
&+\frac{90-31n}{24960}\sigma_3(n) + \frac{565-261n}{37440}\sigma_3\left(\frac{n}{2}\right) + \frac{50-75n}{14976}\sigma_3\left(\frac{n}{5}\right) + \frac{75-175n}{2496}\sigma_3\left(\frac{n}{10}\right)\\
&+\frac{5-18n+6n^2}{2880}\sigma(n) + \frac{5-33n+12n^2}{2880}\sigma\left(\frac{n}{2}\right) + \frac{1-9n+12n^2}{576}\sigma\left(\frac{n}{10}\right)\\
&+\frac{2-3n}{49920}b_5(n) + \frac{33n-16}{18720}b_5\left(\frac{n}{2}\right) + \frac{3n-2}{2304}b_{10}(n) + \frac{127}{999936}c_5(n)\\
&-\frac{649}{937440}c_5\left(\frac{n}{2}\right) - \frac{23}{483840}c_{10,1}(n) - \frac{31}{32256}c_{10,2}(n) + \frac{5}{32256}c_{10,3}(n),
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\sum_{k+5l+10m=n}\sigma(k)\sigma(l)\sigma(m)\\
&= \frac{5}{374976}\sigma_5(n) + \frac{1}{23436}\sigma_5\left(\frac{n}{2}\right) + \frac{1625}{187488}\sigma_5\left(\frac{n}{5}\right) + \frac{325}{11718}\sigma_5\left(\frac{n}{10}\right)\\
&+\frac{20-7n}{24960}\sigma_3(n) + \frac{5-3n}{9360}\sigma_3\left(\frac{n}{2}\right) + \frac{352-261n}{14976}\sigma_3\left(\frac{n}{5}\right) + \frac{17-31n}{624}\sigma_3\left(\frac{n}{10}\right)\\
&+\frac{25-45n+12n^2}{14400}\sigma(n) + \frac{5-33n+12n^2}{2880}\sigma\left(\frac{n}{5}\right) + \frac{5-36n+24n^2}{2880}\sigma\left(\frac{n}{10}\right)\\
&+\frac{9n-35}{124800}b_5(n) + \frac{3n-5}{9360}b_5\left(\frac{n}{2}\right) + \frac{3n-5}{5760}b_{10}(n) + \frac{443}{12499200}c_5(n)\\
&-\frac{1}{23436}c_5\left(\frac{n}{2}\right) - \frac{1}{241920}c_{10,1}(n) - \frac{5}{16128}c_{10,2}(n) + \frac{23}{80640}c_{10,3}(n),
\end{aligned}
$$

(5)

$$
\begin{aligned}
&\sum_{k+10l+10m=n}\sigma(k)\sigma(l)\sigma(m)\\
&= \frac{1}{374976}\sigma_5(n) + \frac{5}{93744}\sigma_5\left(\frac{n}{2}\right) + \frac{325}{187488}\sigma_5\left(\frac{n}{5}\right) + \frac{1625}{46872}\sigma_5\left(\frac{n}{10}\right)\\
&+\frac{10-3n}{37440}\sigma_3(n) + \frac{10-3n}{9360}\sigma_3\left(\frac{n}{2}\right) + \frac{50-15n}{7488}\sigma_3\left(\frac{n}{5}\right) + \frac{55-75n}{1248}\sigma_3\left(\frac{n}{10}\right)
\end{aligned}
$$

$$+\frac{25-30n+6n^2}{14400}\,\sigma(n)+\frac{5-33n+12n^2}{1440}\sigma\left(\frac{n}{10}\right)+\frac{10-3n}{124800}\,b_5(n)$$
$$+\frac{3n-10}{9360}b_5\left(\frac{n}{2}\right)+\frac{3n-10}{5760}\,b_{10}(n)+\frac{349}{12499200}\,c_5(n)-\frac{5}{93744}c_5\left(\frac{n}{2}\right)$$
$$+\frac{1}{60480}\,c_{10,1}(n)-\frac{11}{26880}\,c_{10,2}(n)+\frac{17}{40320}\,c_{10,3}(n),$$

(6)

$$\sum_{2k+2l+5m=n}\sigma(k)\sigma(l)\sigma(m)$$
$$=\frac{13}{187488}\sigma_5(n)+\frac{65}{46872}\sigma_5\left(\frac{n}{2}\right)+\frac{625}{374976}\sigma_5\left(\frac{n}{5}\right)+\frac{3125}{93744}\sigma_5\left(\frac{n}{10}\right)$$
$$+\frac{2-3n}{7488}\sigma_3(n)+\frac{23-15n}{1248}\sigma_3\left(\frac{n}{2}\right)+\frac{50-75n}{7488}\sigma_3\left(\frac{n}{5}\right)+\frac{50-75n}{1872}\sigma_3\left(\frac{n}{10}\right)$$
$$+\frac{5-21n+12n^2}{1440}\sigma\left(\frac{n}{2}\right)+\frac{1-6n+6n^2}{576}\sigma\left(\frac{n}{5}\right)+\frac{15n-10}{4992}b_5(n)$$
$$+\frac{93n-62}{9360}b_5\left(\frac{n}{2}\right)+\frac{2-3n}{1152}b_{10}(n)-\frac{913}{499968}c_5(n)-\frac{7811}{468720}c_5\left(\frac{n}{2}\right)$$
$$+\frac{11}{6048}c_{10,1}(n)-\frac{103}{16128}c_{10,2}(n)+\frac{17}{8064}c_{10,3}(n),$$

(7)

$$\sum_{2k+5l+5m=n}\sigma(k)\sigma(l)\sigma(m)$$
$$=\frac{5}{374976}\sigma_5(n)+\frac{1}{23436}\sigma_5\left(\frac{n}{2}\right)+\frac{1625}{187488}\sigma_5\left(\frac{n}{5}\right)+\frac{325}{11718}\sigma_5\left(\frac{n}{10}\right)$$
$$+\frac{5-3n}{18720}\sigma_3(n)+\frac{5-3n}{4680}\sigma_3\left(\frac{n}{2}\right)+\frac{20-25n}{832}\sigma_3\left(\frac{n}{5}\right)+\frac{25-15n}{936}\sigma_3\left(\frac{n}{10}\right)$$
$$+\frac{25-60n+24n^2}{14400}\sigma\left(\frac{n}{2}\right)+\frac{5-21n+12n^2}{1440}\sigma\left(\frac{n}{5}\right)+\frac{3n-5}{2496}b_5(n)$$
$$+\frac{93n-155}{23400}\,b_5\left(\frac{n}{2}\right)-\frac{5-3n}{2880}\,b_{10}(n)-\frac{151}{1249920}c_5(n)+\frac{4219}{292950}c_5\left(\frac{n}{2}\right)$$
$$-\frac{19}{24192}c_{10,1}(n)+\frac{11}{13440}c_{10,2}(n)+\frac{1}{40320}c_{10,3}(n),$$

(8)

$$\sum_{2k+5l+10m=n}\sigma(k)\sigma(l)\sigma(m)$$
$$=\frac{1}{374976}\sigma_5(n)+\frac{5}{93744}\sigma_5\left(\frac{n}{2}\right)+\frac{325}{187488}\sigma_5\left(\frac{n}{5}\right)+\frac{1625}{46872}\sigma_5\left(\frac{n}{10}\right)$$
$$+\frac{10-3n}{74880}\sigma_3(n)+\frac{15-7n}{12480}\sigma_3\left(\frac{n}{2}\right)+\frac{34-31n}{4992}\sigma_3\left(\frac{n}{5}\right)+\frac{329-261n}{7488}\sigma_3\left(\frac{n}{10}\right)$$
$$+\frac{25-45n+12n^2}{14400}\sigma\left(\frac{n}{2}\right)+\frac{5-18n+6n^2}{2880}\sigma\left(\frac{n}{5}\right)+\frac{5-21n+12n^2}{2880}\sigma\left(\frac{n}{10}\right)$$
$$+\frac{3n-10}{9984}b_5(n)+\frac{111n-340}{93600}b_5\left(\frac{n}{2}\right)+\frac{10-3n}{11520}b_{10}(n)-\frac{251}{4999680}c_5(n)$$
$$+\frac{10817}{4687200}c_5\left(\frac{n}{2}\right)-\frac{11}{96768}c_{10,1}(n)+\frac{11}{161280}c_{10,2}(n)+\frac{1}{32256}c_{10,3}(n).$$

Lemma 3.2. *The following eight functions are quasimodular forms of weight 6 and level 10. We rewrite each function as a linear combination of the basis found in the previous section.*

(1)

$$\begin{aligned}
&G_2^2(z)G_2(10z)\\
&=\frac{1}{120}D^2G_2(z)+\frac{25}{6}D^2G_2(10z)-\frac{5}{832}DG_4(z)-\frac{1}{156}DG_4(2z)-\frac{125}{1248}DG_4(5z)\\
&-\frac{125}{156}DG_4(10z)-\frac{1}{4160}D\delta_5(z)+\frac{1}{156}D\delta_5(2z)+\frac{1}{192}D\delta_{10}(z)+\frac{65}{187488}G_6(z)\\
&+\frac{13}{11718}G_6(2z)+\frac{3125}{374976}G_6(5z)+\frac{625}{23436}G_6(10z)+\frac{631}{1249920}\Delta_5(z)\\
&-\frac{13}{11718}\Delta_5(2z)-\frac{11}{24192}\Delta_{10,1}(z)-\frac{37}{8064}\Delta_{10,2}(z)+\frac{1}{8064}\Delta_{10,3}(z),
\end{aligned}$$

(2)

$$\begin{aligned}
&G_2(z)G_2(2z)G_2(5z)\\
&=\frac{1}{240}D^2G_2(z)+\frac{1}{30}D^2G_2(2z)+\frac{25}{48}D^2G_2(5z)-\frac{29}{8320}DG_4(z)-\frac{31}{1560}DG_4(2z)\\
&-\frac{875}{4992}DG_4(5z)-\frac{125}{312}DG_4(10z)+\frac{29}{8320}D\delta_5(z)+\frac{31}{1560}D\delta_5(2z)-\frac{1}{384}D\delta_{10}(z)\\
&+\frac{65}{187488}G_6(z)+\frac{13}{11718}G_6(2z)+\frac{3125}{374976}G_6(5z)+\frac{625}{23436}G_6(10z)-\frac{659}{499968}\Delta_5(z)\\
&-\frac{781}{117180}\Delta_5(2z)+\frac{247}{241920}\Delta_{10,1}(z)-\frac{67}{16128}\Delta_{10,2}(z)+\frac{23}{16128}\Delta_{10,3}(z),
\end{aligned}$$

(3)

$$\begin{aligned}
&G_2(z)G_2(2z)G_2(10z)\\
&=\frac{1}{480}D^2G_2(z)+\frac{1}{60}D^2G_2(2z)+\frac{25}{12}D^2G_2(10z)-\frac{31}{24960}DG_4(z)-\frac{29}{2080}DG_4(2z)\\
&-\frac{125}{4992}DG_4(5z)-\frac{875}{1248}DG_4(10z)-\frac{1}{16640}D\delta_5(z)+\frac{11}{3120}D\delta_5(2z)+\frac{1}{768}D\delta_{10}(z)\\
&+\frac{13}{187488}G_6(z)+\frac{65}{46872}G_6(2z)+\frac{625}{374976}G_6(5z)+\frac{3125}{93744}G_6(10z)+\frac{127}{999936}\Delta_5(z)\\
&-\frac{649}{937440}\Delta_5(2z)-\frac{23}{483840}\Delta_{10,1}(z)-\frac{31}{32256}\Delta_{10,2}(z)+\frac{5}{32256}\Delta_{10,3}(z),
\end{aligned}$$

(4)

$$\begin{aligned}
&G_2(z)G_2(5z)G_2(10z)\\
&=\frac{1}{1200}D^2G_2(z)+\frac{5}{48}D^2G_2(5z)+\frac{5}{6}D^2G_2(10z)-\frac{7}{24960}DG_4(z)-\frac{1}{1560}DG_4(2z)\\
&-\frac{145}{1664}DG_4(5z)-\frac{155}{312}DG_4(10z)+\frac{3}{41600}D\delta_5(z)+\frac{1}{1560}D\delta_5(2z)+\frac{1}{1920}D\delta_{10}(z)\\
&+\frac{5}{374976}G_6(z)+\frac{1}{23436}G_6(2z)+\frac{1625}{187488}G_6(5z)+\frac{325}{11718}G_6(10z)+\frac{443}{12499200}\Delta_5(z)\\
&-\frac{1}{23436}\Delta_5(2z)-\frac{1}{241920}\Delta_{10,1}(z)-\frac{5}{16128}\Delta_{10,2}(z)+\frac{23}{80640}\Delta_{10,3}(z),
\end{aligned}$$

(5)

$$\begin{aligned}
&G_2(z)G_2^2(10z)\\
&=\frac{1}{2400}D^2G_2(z)+\frac{5}{6}D^2G_2(10z)-\frac{1}{12480}DG_4(z)-\frac{1}{1560}DG_4(2z)-\frac{25}{2496}DG_4(5z)\\
&-\frac{125}{208}DG_4(10z)-\frac{1}{41600}D\delta_5(z)+\frac{1}{1560}D\delta_5(2z)+\frac{1}{1920}D\delta_{10}(z)+\frac{1}{374976}G_6(z)\\
&+\frac{5}{93744}G_6(2z)+\frac{325}{187488}G_6(5z)+\frac{1625}{46872}G_6(10z)+\frac{349}{12499200}\Delta_5(z)-\frac{5}{93744}\Delta_5(2z)\\
&+\frac{1}{60480}\Delta_{10,1}(z)-\frac{11}{26880}\Delta_{10,2}(z)+\frac{17}{40320}\Delta_{10,3}(z),
\end{aligned}$$

(6)

$$\begin{aligned}
&G_2^2(2z)G_2(5z)\\
&=\frac{1}{30}D^2G_2(2z)+\frac{25}{96}D^2G_2(5z)-\frac{1}{2496}DG_4(z)-\frac{5}{208}DG_4(2z)-\frac{125}{2496}DG_4(5z)\\
&-\frac{125}{312}DG_4(10z)+\frac{5}{1664}D\delta_5(z)+\frac{31}{1560}D\delta_5(2z)-\frac{1}{384}D\delta_{10}(z)+\frac{13}{187488}G_6(z)\\
&+\frac{65}{46872}G_6(2z)+\frac{625}{374976}G_6(5z)+\frac{3125}{93744}G_6(10z)-\frac{913}{499968}\Delta_5(z)-\frac{7811}{468720}\Delta_5(2z)\\
&+\frac{11}{6048}\Delta_{10,1}(z)-\frac{103}{16128}\Delta_{10,2}(z)+\frac{17}{8064}\Delta_{10,3}(z),
\end{aligned}$$

(7)

$$\begin{aligned}
&G_2(2z)G_2^2(5z)\\
&=\frac{1}{150}D^2G_2(2z)+\frac{5}{24}D^2G_2(5z)-\frac{1}{6240}DG_4(z)-\frac{1}{780}DG_4(2z)-\frac{125}{832}DG_4(5z)\\
&-\frac{25}{156}DG_4(10z)+\frac{1}{832}D\delta_5(z)+\frac{31}{3900}D\delta_5(2z)-\frac{1}{960}D\delta_{10}(z)+\frac{5}{374976}G_6(z)\\
&+\frac{1}{23436}G_6(2z)+\frac{1625}{187488}G_6(5z)+\frac{325}{11718}G_6(10z)-\frac{151}{1249920}\Delta_5(z)+\frac{4219}{292950}\Delta_5(2z)\\
&-\frac{19}{24192}\Delta_{10,1}(z)+\frac{11}{13440}\Delta_{10,2}(z)+\frac{1}{40320}\Delta_{10,3}(z),
\end{aligned}$$

(8)

$$\begin{aligned}
&G_2(2z)G_2(5z)G_2(10z)\\
&=\frac{1}{300}D^2G_2(2z)+\frac{5}{96}D^2G_2(5z)+\frac{5}{12}D^2G_2(10z)-\frac{1}{24960}DG_4(z)-\frac{7}{6240}DG_4(2z)\\
&-\frac{155}{4992}DG_4(5z)-\frac{145}{416}DG_4(10z)+\frac{1}{3328}D\delta_5(z)+\frac{37}{15600}D\delta_5(2z)-\frac{1}{3840}D\delta_{10}(z)\\
&+\frac{1}{374976}G_6(z)+\frac{5}{93744}G_6(2z)+\frac{325}{187488}G_6(5z)+\frac{1625}{46872}G_6(10z)-\frac{251}{4999680}\Delta_5(z)\\
&+\frac{10817}{4687200}\Delta_5(2z)-\frac{11}{96768}\Delta_{10,1}(z)+\frac{11}{161280}\Delta_{10,2}(z)+\frac{1}{32256}\Delta_{10,3}(z).
\end{aligned}$$

Proof. The functions $G_2(z), G_2(2z), G_2(5z)$ and $G_2(10z)$ are quasimodular forms of weight 2 and level 10. Note that if $f(z) \in \widetilde{M_k}(\Gamma_0(N))$ and $g(z) \in \widetilde{M_{k'}}(\Gamma_0(N))$, then $f(z)g(z) \in \widetilde{M_{k+k'}}(\Gamma_0(N))$. Hence $G_2(az)G_2(bz)G_2(cz) \in \widetilde{M_6}(\Gamma_0(10))$ for positive integers a, b, c of 10, and it is wirtten as the linear combinations of a basis of $\widetilde{M_6}(\Gamma_0(10))$ in detail,

$$\left\{D^2G_2(dz), DG_4(dz), G_6(dz) : d = 1, 2, 5, 10\right\}$$
$$\cup \left\{D\delta_5(z), D\delta_5(2z), D\delta_{10}(z), \Delta_5(z), \Delta_5(2z), \Delta_{10,1}(z), \Delta_{10,2}(z), \Delta_{10,3}(z)\right\}.$$

□

Lemma 3.3. (1)

$$\sum_{l+m=n} \sigma(l)\sigma(m) = \frac{5}{12}\sigma_3(n) + \frac{1-6n}{12}\sigma(n),$$

(2)

$$\sum_{l+2m=n} \sigma(l)\sigma(m) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \frac{1-3n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{2}\right),$$

(3)

$$\sum_{l+5m=n} \sigma(l)\sigma(m) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{5-6n}{120}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{5}\right) - \frac{1}{130}b_5(n),$$

(4)

$$\begin{aligned}&\sum_{l+10m=n} \sigma(l)\sigma(m)\\ = \;& \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{10}\right) + \frac{5-3n}{120}\sigma(n)\\ &+ \frac{1-6n}{24}\sigma\left(\frac{n}{10}\right) + \frac{1}{1040}b_5(n) - \frac{1}{78}b_5\left(\frac{n}{2}\right) - \frac{1}{48}b_{10}(n),\end{aligned}$$

(5)

$$\begin{aligned}&\sum_{2l+5m=n} \sigma(l)\sigma(m)\\ = \;& \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{10}\right) + \frac{5-6n}{120}\sigma\left(\frac{n}{2}\right)\\ &+ \frac{1-3n}{24}\sigma\left(\frac{n}{5}\right) - \frac{5}{208}b_5(n) - \frac{31}{390}b_5\left(\frac{n}{2}\right) + \frac{1}{48}b_{10}(n).\end{aligned}$$

Proof. (1) is in [8, 13, 27], (2) in [14], (3) in [18, 11], (4) in [28] and (5) in [12]. □

Now we are ready to prove Theorem 3.1 (1), because $W_{a,b}(n) = W_{b,a}(n)$ and $W_{da,db}(n) = W_{a,b}(n/d)$.

Proof of Theorem 3.1. One can easily obtain the following.

$$\begin{aligned} & G_2(az)G_2(bz)G_2(cz) \\ = \quad & -\frac{1}{24^3} + \frac{1}{24^2}\sum_{n\geq 1}\left\{\sigma\left(\frac{n}{a}\right) + \sigma\left(\frac{n}{b}\right) + \sigma\left(\frac{n}{c}\right)\right\} q^n \\ & -\frac{1}{24}\sum_{n\geq 1}\left\{W_{a,b}(n) + W_{b,c}(n) + W_{c,a}(n)\right\} q^n + \sum_{n\geq 1} W_{a,b,c}(n) q^n. \end{aligned}$$

Given that $G_2(az)$ is a quasimodular form of weight 2 and level N' for any multiple N' of a,

$$G_2(az)G_2(bz)G_2(cz) \in \widetilde{M}_6(\Gamma_0(N))$$

with $N = \operatorname{lcm}(a,b,c)$. By equating coefficients of q^n in Lemma 3.2 (1) and Lemma 3.3, we find the first formula (1). The other formulas (2)-(8) are obtained by (2)-(8) of Lemma 3.2, respectively. ☐

Lemma 3.4. (1)

$$\begin{aligned} & DG_2(z)G_2(10z) \\ & = -\frac{1}{60}D^2G_2(z) - \frac{25}{3}D^2G_2(10z) + \frac{1}{624}DG_4(z) + \frac{1}{78}DG_4(2z) + \frac{125}{624}DG_4(5z) \\ & + \frac{125}{78}DG_4(10z) + \frac{1}{2080}D\delta_5(z) - \frac{1}{78}D\delta_5(2z) - \frac{1}{96}D\delta_{10}(z) - \frac{1}{960}\Delta_5(z) \\ & - \frac{1}{576}\Delta_{10,1}(z) + \frac{1}{576}\Delta_{10,2}(z) - \frac{1}{192}\Delta_{10,3}(z), \end{aligned}$$

(2)

$$\begin{aligned} & DG_2(2z)G_2(5z) \\ & = -\frac{1}{15}D^2G_2(2z) - \frac{25}{48}D^2G_2(5z) + \frac{1}{1248}DG_4(z) + \frac{1}{156}DG_4(2z) + \frac{125}{1248}DG_4(5z) \\ & + \frac{125}{156}DG_4(10z) - \frac{5}{832}D\delta_5(z) - \frac{31}{780}D\delta_5(2z) + \frac{1}{192}D\delta_{10}(z) - \frac{1}{384}\Delta_5(z) - \frac{1}{15}\Delta_5(2z) \\ & + \frac{5}{1152}\Delta_{10,1}(z) - \frac{11}{1152}\Delta_{10,2}(z) + \frac{1}{384}\Delta_{10,3}(z), \end{aligned}$$

(3)

$$\begin{aligned} & DG_2(5z)G_2(2z) \\ & = -\frac{1}{75}D^2G_2(2z) - \frac{5}{12}D^2G_2(5z) + \frac{1}{3120}DG_4(z) + \frac{1}{390}DG_4(2z) + \frac{25}{624}DG_4(5z) \\ & + \frac{25}{78}DG_4(10z) - \frac{1}{416}D\delta_5(z) - \frac{31}{1950}D\delta_5(2z) + \frac{1}{480}D\delta_{10}(z) + \frac{1}{960}\Delta_5(z) + \frac{2}{75}\Delta_5(2z) \\ & - \frac{1}{576}\Delta_{10,1}(z) + \frac{11}{2880}\Delta_{10,2}(z) - \frac{1}{960}\Delta_{10,3}(z), \end{aligned}$$

(4)

$$DG_2(10z)G_2(z)$$

$$= -\frac{1}{1200}D^2G_2(z) - \frac{5}{3}D^2G_2(10z) + \frac{1}{6240}DG_4(z) + \frac{1}{780}DG_4(2z) + \frac{25}{1248}DG_4(5z) + \frac{25}{156}DG_4(10z) + \frac{1}{20800}D\delta_5(z) - \frac{1}{780}D\delta_5(2z) - \frac{1}{960}D\delta_{10}(z) + \frac{1}{9600}\Delta_5(z) + \frac{1}{5760}\Delta_{10,1}(z) - \frac{1}{5760}\Delta_{10,2}(z) + \frac{1}{1920}\Delta_{10,3}(z).$$

Theorem 3.5. (1)

$$\sum_{l+10m=n} l\sigma(l)\sigma(m) = \frac{n}{624}\sigma_3(n) + \frac{n}{156}\sigma_3\left(\frac{n}{2}\right) + \frac{25n}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{25n}{156}\sigma_3\left(\frac{n}{10}\right) + \frac{5n-2n^2}{120}\sigma(n) - \frac{n^2}{12}\sigma\left(\frac{n}{10}\right) + \frac{n}{2080}b_5(n) - \frac{n}{156}b_5\left(\frac{n}{2}\right) - \frac{n}{96}b_{10}(n) - \frac{1}{960}c_5(n) - \frac{1}{576}c_{10,1}(n) + \frac{1}{576}c_{10,2}(n) - \frac{1}{192}c_{10,3}(n),$$

(2)

$$\sum_{2l+5m=n} l\sigma(l)\sigma(m) = \frac{n}{1248}\sigma_3(n) + \frac{n}{312}\sigma_3\left(\frac{n}{2}\right) + \frac{25n}{1248}\sigma_3\left(\frac{n}{5}\right) + \frac{25n}{312}\sigma_3\left(\frac{n}{10}\right) + \frac{5n-4n^2}{240}\sigma\left(\frac{n}{2}\right) - \frac{n^2}{48}\sigma\left(\frac{n}{5}\right) - \frac{5n}{832}b_5(n) - \frac{31n}{1560}b_5\left(\frac{n}{2}\right) + \frac{1}{192}b_{10}(n) - \frac{1}{384}c_5(n) - \frac{1}{15}c_5\left(\frac{n}{2}\right) + \frac{5}{1152}c_{10,1}(n) - \frac{11}{1152}c_{10,2}(n) + \frac{1}{384}c_{10,3}(n),$$

(3)

$$\sum_{5l+2m=n} l\sigma(l)\sigma(m) = \frac{n}{3120}\sigma_3(n) + \frac{n}{780}\sigma_3\left(\frac{n}{2}\right) + \frac{5n}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{5n}{156}\sigma_3\left(\frac{n}{10}\right) - \frac{n^2}{300}\sigma\left(\frac{n}{2}\right) + \frac{n-2n^2}{120}\sigma\left(\frac{n}{5}\right) - \frac{n}{416}b_5(n) - \frac{31n}{3900}b_5\left(\frac{n}{2}\right) + \frac{n}{480}b_{10}(n) + \frac{1}{960}c_5(n) + \frac{2}{75}c_5\left(\frac{n}{2}\right) - \frac{1}{576}c_{10,1}(n) + \frac{11}{2880}c_{10,2}(n) - \frac{1}{960}c_{10,3}(n),$$

(4)

$$\sum_{10l+m=n} l\sigma(l)\sigma(m) = \frac{n}{6240}\sigma_3(n) + \frac{n}{1560}\sigma_3\left(\frac{n}{2}\right) + \frac{5n}{1248}\sigma_3\left(\frac{n}{5}\right) + \frac{5n}{312}\sigma_3\left(\frac{n}{10}\right) - \frac{n^2}{1200}\sigma(n) + \frac{n-4n^2}{240}\sigma\left(\frac{n}{10}\right) + \frac{n}{20800}b_5(n) - \frac{n}{1560}b_5\left(\frac{n}{2}\right) - \frac{n}{960}b_{10}(n) + \frac{1}{9600}c_5(n) + \frac{1}{5760}c_{10,1}(n) - \frac{1}{5760}c_{10,2}(n) + \frac{1}{1920}c_{10,3}(n).$$

Proof of Theorem 3.5. Note that

$$DG_2(az)G_2(bz) = \sum_{n\geq 1}\left\{-\frac{1}{24}\frac{n}{a}\sigma\left(\frac{n}{a}\right) + W_{a,b}^{(1)}(n)\right\} q^n.$$

Given that $DG_2(az)G_2(bz) \in \widetilde{M}_6(\Gamma_0(N))$ with $N = \mathrm{lcm}(a,b)$, equating the coefficients of q^n in Lemma 3.4 suffices to prove the theorem. □

Lemma 3.6. (1)

$$\begin{aligned}&G_4(z)G_2(10z)\\ &= -\frac{1}{80}DG_4(z) + \frac{13}{15624}G_6(z) + \frac{26}{9765}G_6(2z) + \frac{625}{31248}G_6(5z) + \frac{125}{1953}G_6(10z)\\ &-\frac{1}{26040}\Delta_5(z) - \frac{26}{9765}\Delta_5(2z) - \frac{1}{315}\Delta_{10,1}(z) - \frac{1}{112}\Delta_{10,2}(z) - \frac{1}{168}\Delta_{10,3}(z),\end{aligned}$$

(2)

$$\begin{aligned}&G_4(2z)G_2(5z)\\ &= -\frac{1}{20}DG_4(2z) + \frac{13}{78120}G_6(z) + \frac{13}{3906}G_6(2z) + \frac{125}{31248}G_6(5z) + \frac{625}{7812}G_6(10z)\\ &-\frac{391}{52080}\Delta_5(z) - \frac{4687}{39060}\Delta_5(2z) + \frac{193}{20160}\Delta_{10,1}(z) - \frac{3}{112}\Delta_{10,2}(z) + \frac{11}{1344}\Delta_{10,3}(z),\end{aligned}$$

(3)

$$\begin{aligned}&G_4(5z)G_2(2z)\\ &= -\frac{5}{16}DG_4(5z) + \frac{1}{31248}G_6(z) + \frac{1}{9765}G_6(2z) + \frac{325}{15624}G_6(5z) + \frac{130}{1953}G_6(10z)\\ &+\frac{5}{5208}\Delta_5(z) + \frac{130}{1953}\Delta_5(2z) - \frac{1}{252}\Delta_{10,1}(z) + \frac{11}{1680}\Delta_{10,2}(z) - \frac{1}{840}\Delta_{10,3}(z),\end{aligned}$$

(4)

$$\begin{aligned}&G_4(10z)G_2(z)\\ &= -\frac{5}{4}DG_4(10z) + \frac{1}{156240}G_6(z) + \frac{1}{7812}G_6(2z) + \frac{65}{15624}G_6(5z) + \frac{325}{3906}G_6(10z)\\ &+\frac{1}{5208}\Delta_5(z) - \frac{1}{7812}\Delta_5(2z) + \frac{1}{4032}\Delta_{10,1}(z) - \frac{1}{840}\Delta_{10,2}(z) + \frac{11}{6720}\Delta_{10,3}(z).\end{aligned}$$

Theorem 3.7. (1)

$$\begin{aligned}&\sum_{l+10m=n}\sigma_3(l)\sigma(m)\\ &= \frac{13}{15624}\sigma_5(n) + \frac{26}{9765}\sigma_5\left(\frac{n}{2}\right) + \frac{625}{31248}\sigma_5\left(\frac{n}{5}\right) + \frac{125}{1953}\sigma_5\left(\frac{n}{10}\right) + \frac{10-3n}{240}\sigma_3(n)\\ &-\frac{1}{240}\sigma\left(\frac{n}{10}\right) - \frac{1}{26040}c_5(n) - \frac{26}{9765}c_5\left(\frac{n}{2}\right) - \frac{1}{315}c_{10,1}(n) - \frac{1}{112}c_{10,2}(n)\end{aligned}$$

$$-\frac{1}{168}c_{10,3}(n),$$

(2)

$$\sum_{2l+5m=n}\sigma_3(l)\sigma(m)$$
$$=\frac{13}{78120}\sigma_5(n)+\frac{13}{3906}\sigma_5\left(\frac{n}{2}\right)+\frac{125}{31248}\sigma_5\left(\frac{n}{5}\right)+\frac{625}{7812}\sigma_5\left(\frac{n}{10}\right)+\frac{5-3n}{120}\sigma_3\left(\frac{n}{2}\right)$$
$$-\frac{1}{240}\sigma\left(\frac{n}{5}\right)-\frac{391}{52080}c_5(n)-\frac{4687}{39060}c_5\left(\frac{n}{2}\right)+\frac{193}{20160}c_{10,1}(n)-\frac{3}{112}c_{10,2}(n)$$
$$+\frac{11}{1344}c_{10,3}(n),$$

(3)

$$\sum_{5l+2m=n}\sigma_3(l)\sigma(m)$$
$$=\frac{1}{31248}\sigma_5(n)+\frac{1}{9765}\sigma_5\left(\frac{n}{2}\right)+\frac{325}{15624}\sigma_5\left(\frac{n}{5}\right)+\frac{130}{1953}\sigma_5\left(\frac{n}{10}\right)+\frac{2-3n}{48}\sigma_3\left(\frac{n}{5}\right)$$
$$-\frac{1}{240}\sigma\left(\frac{n}{2}\right)+\frac{5}{5208}c_5(n)+\frac{130}{1953}c_5\left(\frac{n}{2}\right)-\frac{1}{252}c_{10,1}(n)+\frac{11}{1680}c_{10,2}(n)-\frac{1}{840}c_{10,3}(n),$$

(4)

$$\sum_{10l+m=n}\sigma_3(l)\sigma(m)$$
$$=\frac{1}{156240}\sigma_5(n)+\frac{1}{7812}\sigma_5\left(\frac{n}{2}\right)+\frac{65}{15624}\sigma_5\left(\frac{n}{5}\right)+\frac{325}{3906}\sigma_5\left(\frac{n}{10}\right)+\frac{1-3n}{24}\sigma_3\left(\frac{n}{10}\right)$$
$$-\frac{1}{240}\sigma(n)+\frac{1}{5208}c_5(n)-\frac{1}{7812}c_5\left(\frac{n}{2}\right)+\frac{1}{4032}c_{10,1}(n)-\frac{1}{840}c_{10,2}(n)+\frac{11}{6720}c_{10,3}(n).$$

Proof of Theorem 3.7. Because

$$G_4(az)G_2(bz)$$
$$=-\frac{1}{5760}+\sum_{n\geq 1}\left\{-\frac{1}{24}\sigma_3\left(\frac{n}{a}\right)+\frac{1}{240}\sigma\left(\frac{n}{b}\right)+S[(0,0),(3,1),(a,b)](n)\right\}q^n$$

and $G_4(az)G_2(bz)\in\widetilde{M}_6(\Gamma_0(N))$ with $N=\text{lcm}(a,b)$, we obtain our final theorem. □

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