

DYNAMICS OF A PREY-PREDATOR INTERACTION WITH HASSELL-VARLEY TYPE FUNCTIONAL RESPONSE AND HARVESTING OF PREY

ANINDITA BHATTACHARYYA, ASHOK MONDAL, A.K. PAL* AND NIKHITA SINGH

ABSTRACT. This article aims to study the dynamical behaviours of a two species model in which non-selective harvesting of a prey-predator system by using a reasonable catch-rate function instead of usual catch-per-unit-effort hypothesis is used. A system of two ordinary differential equations(ODE's) has been proposed and analyzed with the predator functional response to prey density is considered as Hassell-Varley type functional responses to study the dynamics of the system. Positivity and boundedness of the system are studied. We have discussed the existence of different equilibrium points and stability of the system at these equilibrium points. We also analysed the system undergoes a Hopf-bifurcation around interior equilibrium point for a various parametric values which has very significant ecological impacts in this work. Computer simulation are carried out to validate our analytical findings. The biological implications of analytical and numerical findings are discussed critically.

AMS Mathematics Subject Classification : 34D20, 34C23, 34F10.

Key words and phrases : Hassell-Varley type functional response, stability, harvesting, Hopf-bifurcation.

1. Introduction

It is well recognized that the theoretical study of predator-prey systems in mathematical ecology has an elongated history starting with the revolutionary work of Lotka in 1925 [10] and Volterra in 1926 [26] on one prey and one predator species. These principles of prey- predator system have remained valid till today and many theoretical ecologists adhere to these principles. In population dynamics, a functional response plays an important role which involves the change in the density of prey attached per unit time per predator as the prey density changes. Throughout last 10 years, there has been a wide analysis on

Received May 31, 2021. Revised July 31, 2021. Accepted October 28, 2021. *Corresponding author.

© 2022 KSCAM.

the dynamics of predator–prey models with the diverse functional responses in the literature, (see [3, 5, 8, 9, 12, 13, 14, 16] and references therein)

Much different progress has been made in this direction by Mondal et al. [15, 17, 18, 22], Wang et al. [27] and Pathak et al [23]. Most of these research works comprise of ordinary differential equations. Naji and Mustaf [21] used Holling Type II response function to describe the dynamics of eco- epidemiological model with non- linear incidence rate where the uniqueness, boundedness and existence of the system was evaluated. Rao et al. [24] discussed the effect of environmental noise on the dynamical behaviour of a Hassell- Varley type prey- predator model with stochastic perturbations on the death rate of predator’s population and on the growth rate of prey’s population. Here, the stochastic differential equations were developed and analysed that stochastic perturbations with sufficiently large noise intensity may harm the species population but stochastic perturbation with small noise intensity is manageable.

There are increasing data which verifies that harvesting has a significant impact on dynamic evolution of a population species. Some of them are the research works done by S. Chakraborty et al. [2] and Das et al [4]. Hassell in 1969 introduced the mutual interference between the prey and predator population. The dynamical behaviour of non- selective harvesting prey- predator model with Hassell- Varley type functional response incorporating the impulsive effects by using the fixed-point theory based on monotone operator was studied by Li et al. [11]. Wang in 2014 [28] founded the modelling of prey- predator model with non-selective harvesting and Hassell- Varley type functional response including the time delay. One of the classical prey- predator model was proposed by Leslie-Gower which was modified by Yuan et al. [29] with saddle node- Hopf bifurcation involving time- delay and prey- harvesting. In 2019, Raymond et al. [25] proposed and analysed a mathematical model to study the dynamical behaviour of two- prey and one- predator fishery model incorporating the Holling Type II response function. Here, all the three species encountered harvesting and it was investigated that if the harvesting rate of prey’s species is greater than their intrinsic growth rate, then the three species would be extinct with time. Thus, to achieve the sustainability of the population, one should never exceed the harvesting rate of species than their intrinsic growth rate. In population dynamics, the most vital element is functional response (or trophic function) because of its far-reaching implications. The predator’s functional response is defined as the number of preys eaten per predator per unit of time. Michaelis and Menten in 1913, proposed and analyzed the functional response, $g(X) = \frac{cX}{(m+X)}$ where X represents the prey density at time t , $c(> 0)$ and $m(> 0)$ is the maximal growth rate of the species and half capturing saturation constant, respectively. Holling in 1959 [7] used the same function as one of the predator functional response. Hence, it is now known as Holling Type II function or Michaelis- Menten function. During the last ten years, there has been extensive investigation on

dynamical behavior of prey- predator system with Holling Type II response function. A predator- dependent functional response can be roughly defined as the per capita rate of predation depends on both the numbers of prey and predator. A predator- dependent functional response is called ratio- dependent only when it is a function of the ratio of prey to predator's densities. Arditi and Ginzburg [1] formulated a Holling Type II response [19, 30, 31, 32] function of the form:

$$g(X, Y) = \frac{c(\frac{X}{Y})}{m + (\frac{X}{Y})} = \frac{cX}{(mY + X)} \quad (1)$$

where X and Y represents the prey and predator population, respectively. The positive constants c and m denote the capturing rate and half capturing saturation constant, respectively. Hassell and Varley [6] investigated a general prey- predator system where the predator- dependent functional response is in a different way:

$$g(X, Y) = \frac{cX}{(mY^\gamma + X)} \quad (2)$$

This equation is known as Hassell- Varley type functional response and $\gamma \in (0, 1)$ is called the Hassell- Varley constant. In the typical prey- predator interaction where predators do not form a group and the system is ratio dependent prey- predator system, $\gamma = 1$. For terrestrial predators that form a tight number of groups $\gamma = \frac{1}{2}$ and for aquatic predators that form a tight number of groups $\gamma = \frac{1}{3}$. It can also be observed that ratio- dependent model and prey- dependent model with $\gamma = 1$ or $\gamma = 0$ are the special cases of Hassell- Varley type functional response, respectively.

In this paper, we give the detailed hopf bifurcation analysis by calculating the bifurcation near the coexistence equilibrium point for system (4) with respect to Hassel-Verley constant, harvesting coefficient and intra-specific competition of predator as bifurcating parameters. We derive the parameter conditions for the existence of predator free and coexistence equilibria and the parameter regions in which both the predator-extinction equilibrium and coexistence equilibrium are simultaneously stabilized. This paper is organized as follows. Section 2, the basic mathematical model is introduced together with basic considerations. Boundedness and positivity of the solutions of the proposed model are established in Section 3. The extinction criteria of the predator-prey population are discussed in Section 4. Section 5 deals with all possible equilibrium points of the model and their feasibility conditions. Stability of the model at various equilibrium points is discussed in Section 6. In Section 7, a detailed study of the Hopf- bifurcation around the interior equilibrium is carried out. Computer simulations are carried out to validate our analytical findings numerically in Section 8. Section 9 contains the general discussion and biological significance of our analytical findings.

2. The basic mathematical model

Before we introduce the model, we have made the following assumptions to construct the proposed mathematical model which also indicates the biological relevance of it:

- (1) We have proposed a system consisting of a single prey and single predator species with biomass $P(T)$ and $Q(T)$ respectively.
- (2) It is known that in the absence of predation and harvesting, then the prey population biomass grows according to logistic law. So, the term $RP(1 - \frac{P}{K})$ is the growth of the prey, where R is the intrinsic growth rate of the prey.
- (3) The effect of the predation is to reduce the prey growth rate by a term proportional to the prey and predator populations; this is the $A_1P(T)Q(T)$ term incorporated by Holling Type II response function with Hassell- Varley type predation model where A_1 is the capturing rate of prey by predator.
- (4) The prey's contribution to the predator growth rate is $A_1P(T)Q(T)$ which is proportional to the available prey as well as the size of the predator population incorporated by Holling Type II response function with Hassell- Varley type predation model.
- (5) In the absence of any prey for sustenance, the predator's death rate results in inverse decay, that is the term DQ where D is the death rate and δQ^2 is the death rate of predator due to intra-specific competition.

By considering the underlying assumptions of the incorporated populations, the dynamics of the model can be represented by the following set of non-linear ordinary differential equations:

$$\begin{aligned} \frac{dP}{dT} &= RP\left(1 - \frac{P}{K}\right) - \frac{1}{\beta} \frac{A_1PQ}{MQ^\gamma + P} - \frac{q_1EP}{M_1E + M_2P} \\ \frac{dQ}{dT} &= \frac{A_1PQ}{MQ^\gamma + P} - DQ - \delta Q^2 \end{aligned} \quad (3)$$

with $P(0) = P_0 > 0, Q(0) = Q_0 > 0$.

All the model parameters are assumed to be positive constants with following interpretation:

- R : is the intrinsic growth rate of prey.
- K : denotes the carrying capacity of prey.
- A_1 : is the capturing rate of prey by predator.
- q_1 : denotes the catchability coefficient.
- E : is the effort applied to harvest the prey.
- M : stands for half capturing saturation constant.
- M_1 and M_2 : stands for suitable constants.

- β : denotes the yield constant.
- γ : is called Hassell- Varley constant.
- D : is the death rate of predator population.
- δ : represents the death rate due to intra- species competition.

Let us non-dimensionalize the system (3) with the following scaling:

$$x = \frac{P}{K}, \quad Q = y \left(\frac{K}{M} \right)^{\frac{1}{\gamma}} \quad \text{and} \quad t = RT$$

Then the system (3) takes the form (after some simplification):

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x) - \frac{axy}{x + y^\gamma} - \frac{hx}{g + x} \\ \frac{dy}{dt} &= \frac{bxy}{x + y^\gamma} - dy - cy^2 \end{aligned} \tag{4}$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0$,
 where the parameters are as follows:

$$a = \frac{A_1 \left(\frac{K}{M} \right)^{\frac{1}{\gamma}}}{KR\beta}, \quad b = \frac{A_1}{R}, \quad c = \frac{\delta \left(\frac{K}{M} \right)^{\frac{1}{\gamma}}}{R}, \quad d = \frac{D}{R}, \quad g = \frac{M_1E}{M_2K}, \quad h = \frac{q_1E}{RKM_2}.$$

3. Positivity and boundedness

Positivity and boundedness of a model guarantee that the model is biologically well behaved. For positivity of the system (4), we have the following theorem.

Theorem 3.1. *All solutions of the system (4) that start in \mathbb{R}_+^2 remain positive forever.*

Proof. From the first equation of system (4), we get

$$x(t) = x(0) \exp \left[\int_0^t \left\{ 1 - x(\theta) - \frac{ay(\theta)}{y^\gamma(\theta) + x(\theta)} - \frac{h}{g + x(\theta)} \right\} d\theta \right] \Rightarrow x(t) > 0$$

From the second equation of system (4), we get

$$y(t) = y(0) \exp \left[\int_0^t \left\{ \frac{bx(\theta)}{y^\gamma(\theta) + x(\theta)} - d - cy(\theta) \right\} d\theta \right] \Rightarrow y(t) > 0$$

This proves the theorem. □

Theorem 3.2. *All solutions of the system (4) that start in \mathbb{R}_+^2 are uniformly bounded.*

Proof. Since

$$\frac{dx}{dt} \leq x(1-x)$$

we have

$$\limsup_{t \rightarrow \infty} x(t) \leq 1$$

Now we assume,

$$\tilde{W}_1 = x + \frac{a}{b}y$$

Therefore

$$\begin{aligned} \frac{d\tilde{W}_1}{dt} &= \left\{ x(1-x) - \frac{axy}{x+y^\gamma} - \frac{hx}{g+x} \right\} + \frac{a}{b} \left\{ \frac{bxy}{x+y^\gamma} - dy - cy^2 \right\} \\ &\leq -(x + \frac{ad}{b}y) + 2x \\ \therefore \frac{d\tilde{W}_1}{dt} &\leq 2x - R\tilde{W}_1, \text{ where } R = \min\{1, d\}, \end{aligned}$$

$$\text{Hence } \frac{d\tilde{W}_1}{dt} + R\tilde{W}_1 \leq 2x \leq 2, \text{ for large } t, \text{ since } \lim_{t \rightarrow \infty} \sup x(t) \leq 1.$$

Applying a theorem on differential inequalities, we obtain

$$0 \leq \tilde{W}_1(x, y) \leq \frac{2}{R} + \frac{\tilde{W}_1(x(0), y(0))}{e^{Rt}} \Rightarrow 0 \leq \tilde{W}_1 \leq \frac{2}{R} \text{ as } t \rightarrow \infty.$$

Thus, all solutions of system (4) enter into the region

$$\Gamma_1 = \left\{ (x, y) : 0 \leq \tilde{W}_1 < \frac{2}{R} + \epsilon \text{ for any } \epsilon > 0 \right\}.$$

This proves the theorem. \square

4. Extinction scenarios

This section deals with the conditions for which both the species of the underlying system (4) will be going to extinct in long run. Here we shall use the following fact (for large time t):

$$x(t) \leq 1.$$

Theorems 4.1 show the extinction of prey population and Theorem 4.2 deals with the extinction of predator population.

Theorem 4.1. *If $h > (g + 1)$, then $\lim_{t \rightarrow \infty} x(t) = 0$.*

Proof. We have

$$\begin{aligned} \frac{dx}{dt} &\leq x - \frac{hx}{g+x} \\ &= x \left(1 - \frac{h}{g+x}\right) \\ \frac{dx}{dt} &\leq \frac{x}{g+x} (g + 1 - h) \\ &\leq \frac{x}{g} (g + 1 - h) \\ &< 0, \text{ provided } h > (g + 1) \end{aligned}$$

Hence, $\limsup_{t \rightarrow \infty} x(t) = 0$ if $h > (g + 1)$. □

Theorem 4.2. *If $d > b$, then $\lim_{t \rightarrow \infty} y(t) = 0$.*

Proof. We have

$$\begin{aligned} \frac{dy}{dt} &\leq \frac{bxy}{x+y^\gamma} - dy \\ &= \frac{by}{1+\frac{y^\gamma}{x}} - dy \\ \frac{dy}{dt} &\leq y(b - d) \\ &< 0, \text{ provided } b < d \end{aligned}$$

Hence, $\limsup_{t \rightarrow \infty} y(t) = 0$ if $b < d$. □

Remark: So, when death rate is grater than capturing rate of prey by predator, evidently the predator population will be going to extinct.

5. Boundary equilibria and their stability

5.1. Existence of the boundary equilibrium points. The persistence of the system (4) is determined by the information collected through the computations of boundary equilibria and their stability. The system (4) has two boundary equilibrium points namely $E_0(0, 0)$ and $E_1(x_1, 0)$. The point $E_0(0, 0)$ always exist. Now, we show the existence of other boundary equilibrium points as follows:

5.1.1. Existence of $E_1(x_1, 0)$. Here, x_1 is the positive solution of the following equation:

$$x_1 = \frac{-(g - 1) \pm \sqrt{(g - 1)^2 - 4(h - g)}}{2}$$

Remarks: Case-1: When $g > 1$ and $h - g < 0$, then x_1 has one positive and one negative value.

Case-2: When $g < 1$ and $h - g < 0$, then x_1 has one positive and one negative value.

Case-3: When $g > 1$, $h - g > 0$ and $(g - 1)^2 > 4(h - g)$, then x_1 has two negative values.

Case-4: When $g < 1$, $h - g > 0$ and $(g - 1)^2 > 4(h - g)$, then x_1 has two positive values.

In the rest of the paper we have considered at least one positive values of x_1 .

5.1.2. Existence of $E^*(x^*, y^*)$. The Interior equilibrium point $E^*(x^*, y^*)$ of system (4) is given by

$$x^* = \frac{y^{*\gamma}(d + cy^*)}{b - d - cy^*}$$

$$\frac{y^{*\gamma}(d + cy^*)}{b - d - cy^*} + \frac{ay^*(b - d - cy^*)}{by^{*\gamma}} + \frac{h}{g + \frac{y^{*\gamma}(d + cy^*)}{y^* - d - cy^*}} = 1. \quad (5)$$

This interior equilibrium exists only when $y^* > \frac{d}{b-c}$ where $b > c$.

6. Stability Analysis

We now study the local stability of the prescribed equilibrium points. The variational matrix corresponding to system (4) is:

$$J(x, y) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

where

$$v_{11} = 1 - 2x - \frac{ay^{\gamma+1}}{(x+y^\gamma)^2} - \frac{gh}{(g+x)^2}, \quad v_{12} = -\frac{ax^2 + axy^\gamma(1-\gamma)}{(x+y^\gamma)^2},$$

$$v_{21} = \frac{by^{\gamma+1}}{(x+y^\gamma)^2}, \quad v_{22} = \frac{bx^2 + bxy^\gamma(1-\gamma)}{(x+y^\gamma)^2} - d - 2cy.$$

The variational matrix corresponding to E_0 is:

$$J(E_0) = \begin{bmatrix} 1 - \frac{h}{g} & 0 \\ 0 & -d \end{bmatrix}$$

All the eigen-values of J_{E_0} are: $1 - \frac{h}{g}$ and $-d$.

Theorem 6.1. *The trivial equilibrium E_0 is locally asymptotically stable (LAS) if $\frac{h}{g} > 1$.*

The variational matrix corresponding to E_1 is:

$$J(E_1) = \begin{bmatrix} 1 - 2x_1 - \frac{gh}{(g+x_1)^2} & -a \\ 0 & b - d \end{bmatrix}.$$

The eigen-values are: $1 - 2x_1 - \frac{gh}{(g+x_1)^2}$ and $b - d$.

Theorem 6.2. *The axial equilibrium point E_1 is locally asymptotically stable if and only if $2x_1 + \frac{gh}{(g+x_1)^2} > 1$ and $b < d$.*

The variational matrix corresponding to E^* is :

$$J(E^*) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

where

$$\begin{aligned} m_{11} &= x^* \left\{ -1 + \frac{ay^*}{(x^*+y^*\gamma)^2} + \frac{h}{(g+x^*)^2} \right\}, \\ m_{12} &= -x^* \left\{ \frac{ax^*+ay^*\gamma(1-\gamma)}{(x^*+y^*\gamma)^2} \right\}, \\ m_{21} &= \frac{by^{*\gamma+1}}{(x^*+y^*\gamma)^2}, \\ m_{22} &= -\frac{b\gamma x^*y^{*\gamma}}{(x^*+y^*\gamma)^2} - cy^* \end{aligned}$$

The characteristic equation is:

$$\lambda^2 + P_1\lambda + P_2 = 0. \tag{6}$$

where $P_1 = -(m_{11} + m_{22})$, $P_2 = m_{11}m_{22} - m_{12}m_{21}$.

Theorem 6.3. *The interior equilibrium point E^* is locally asymptotically stable if and only if $(m_{11} + m_{22}) < 0$ and $m_{11}m_{22} > m_{12}m_{21}$.*

7. Hopf Bifurcation

Characteristic equation of system (6) at $E^*(x^*, y^*)$ is given by

$$\lambda^2 + P_1(\gamma)\lambda + P_2(\gamma) = 0, \tag{7}$$

where

$$\begin{aligned} P_1(\gamma) &= -x^* \left\{ -1 + \frac{ay^*}{(x^*+y^*\gamma)^2} + \frac{h}{(g+x^*)^2} \right\} + \frac{b\gamma x^*y^{*\gamma}}{(x^*+y^*\gamma)^2} + cy^*, \\ P_2(\gamma) &= -x^* \left\{ -1 + \frac{ay^*}{(x^*+y^*\gamma)^2} + \frac{h}{(g+x^*)^2} \right\} \left\{ \frac{b\gamma x^*y^{*\gamma}}{(x^*+y^*\gamma)^2} + cy^* \right\} \\ &+ x^* \left\{ \frac{ax^*+ay^*\gamma(1-\gamma)}{(x^*+y^*\gamma)^2} \right\} \frac{by^{*\gamma+1}}{(x^*+y^*\gamma)^2}. \end{aligned}$$

In order to see the instability of system (4), let us consider γ as bifurcation parameter. For this purpose let us first state the following theorem:

Theorem 7.1. Hopf Bifurcation Theorem [20]

If $P_1(\gamma)$ and $P_2(\gamma)$ are the smooth functions of γ in an open interval about $\gamma^ \in \mathbb{R}$ such that the characteristic equation (7) has a pair of imaginary eigen values $\lambda = r_1(\gamma) \pm ir_2(\gamma)$ with $r_1(\gamma)$ and $r_2(\gamma) \in \mathbb{R}$ so that they become purely imaginary at $\gamma = \gamma^*$ and $\frac{dr_1}{d\gamma}|_{\gamma=\gamma^*} \neq 0$, then a Hopf-bifurcation occurs around E^* at $\gamma = \gamma^*$ (i.e, a stability change of $E^*(x^*, y^*)$ accompanied by the creation of a limit cycle around $\gamma = \gamma^*$).*

Theorem 7.2. *System (6) possesses a Hopf-bifurcation around E^* when γ passes through γ^* provided $P_1(\gamma^*) = 0$.*

Proof. At $\gamma = \gamma^*$, the characteristic equation (7) can be written as $\lambda^2 + P_2(\gamma) = 0$. The roots of this equation are $\lambda_1 = i\sqrt{P_2}$ and $\lambda_2 = -i\sqrt{P_2}$. Thus there exists a pair of purely complex eigen values. Also, P_1 and P_2 are smooth functions of γ . So, for γ in a neighborhood of γ^* , the roots have of the forms: $\lambda_1 = r_1(\gamma) + ir_2(\gamma)$ and $\lambda_2 = r_1(\gamma) - ir_2(\gamma)$, where $r_i(\gamma)$ are real functions of γ in an open neighbourhood of γ^* for $i = 1, 2$.

Next, let us verify the transversality condition:

$$\left. \frac{d}{d\gamma}(\operatorname{Re}\lambda_i(\gamma)) \right|_{\gamma=\gamma^*} \neq 0, i = 1, 2. \quad (8)$$

Substituting $\lambda(\gamma) = r_1(\gamma) + ir_2(\gamma)$ in the characteristic equation (7), we get

$$(r_1(\gamma) + ir_2(\gamma))^2 + P_1(\gamma)(r_1(\gamma) + ir_2(\gamma)) + P_2(\gamma) = 0 \quad (9)$$

Differentiating both sides with respect to γ , we get

$$2(r_1(\gamma) + ir_2(\gamma))(r_1(\gamma) + ir_2(\gamma)) + P_1(\gamma)(r_1(\gamma) + ir_2(\gamma)) + \dot{P}_1(\gamma)(r_1(\gamma) + ir_2(\gamma)) + \dot{P}_2(\gamma) = 0 \quad (10)$$

Equating real and imaginary parts from both sides, we get

$$2r_1\dot{r}_1 - 2r_2\dot{r}_2 + P_1\dot{r}_1 + \dot{P}_1r_1 + \dot{P}_2 = 0 \quad (11)$$

$$2r_1\dot{r}_2 + 2r_2\dot{r}_1 + P_1\dot{r}_2 + \dot{P}_1r_2 = 0 \quad (12)$$

Let us rewrite (11) as follows:

$$\begin{aligned} r_1(2r_1 + P_1) + r_2(-2r_2) + \dot{P}_1r_1 + \dot{P}_2 &= 0 \\ \Rightarrow r_1S_1 - r_2S_2 + S_3 &= 0 \end{aligned} \quad (13)$$

Also, rewrite (12) as follows:

$$\begin{aligned} r_1(2r_2) + r_2(2r_1 + P_1) + \dot{P}_1r_2 &= 0 \\ \Rightarrow r_1S_2 + r_2S_1 + S_4 &= 0 \end{aligned} \quad (14)$$

where $S_1 = (2r_1 + P_1) \neq 2r_1$, $S_2 = 2r_2$, $S_3 = \dot{P}_1r_1 + \dot{P}_2$ and $S_4 = \dot{P}_1r_2$. Multiplying (13) by S_1 and (14) by S_2 and then adding we have

$$(S_1^2 + S_2^2)r_1 + S_1S_3 + S_2S_4 = 0 \Rightarrow r_1 = -\frac{(S_1S_3 + S_2S_4)}{S_1^2 + S_2^2} \quad (15)$$

At $\gamma = \gamma^*$:

Case 1: $r_1 = 0$, $r_2 = \sqrt{P_2}$. Then $S_1 \neq 0$, $S_2 = 2\sqrt{P_2}$, $S_3 = \dot{P}_2$ and $S_4 = \dot{P}_1\sqrt{P_2}$.
 $\therefore S_1S_3 + S_2S_4 \neq 0$.

Case 2: $r_1 = 0$, $r_2 = -\sqrt{P_2}$. Then $S_1 \neq 0$, $S_2 = -2\sqrt{P_2}$, $S_3 = \dot{P}_2$ and $S_4 = -\dot{P}_1\sqrt{P_2}$.
 $\therefore S_1S_3 + S_2S_4 \neq 0$.

Hence it is proved by Theorem 7.2. □

Remarks: Similarly, we consider harvesting coefficient, intra-specific competition of predator as bifurcating parameters, statements and proofs are similar as in this theorem.

8. Numerical simulation

In this section, Extensive numerical simulations have been performed for various values of parameters to determine the the dynamics of the system. We present computer simulations of different solutions of the system (4) using MATLAB. This study provides stability analysis of each of the equilibrium points and occurrence of Hopf-bifurcation of the system.

First we take the parameters as $a = 1.7, \gamma = 0.35, h = 0.1, g = 0.4, b = 0.9, d = 0.1, c = 0.3$. Then conditions are satisfied, and hence $E^*(0.1901, 0.3259,)$ exists. Also the conditions of Theorem 6.3 are satisfied. Consequently, E^* is locally asymptotically stable. The stable behaviour of x, y with t and the phase portrait are presented in Figures 1.

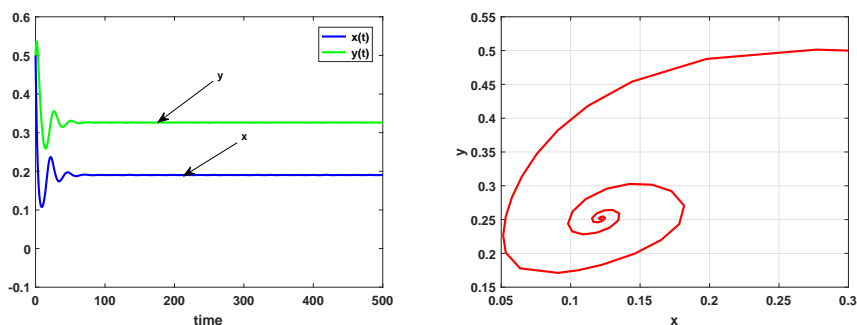


FIGURE 1. Local asymptotic stability of E^* , where $x^* = 0.1901, y^* = 0.3259$.

In this context it is mentioned that the Hassell- Varley constant, γ has an important role on the dynamics the underlying system. The system (4) undergoes a Hopf-bifurcation around E^* at $\gamma^* = 0.7416$. Fig. 2(First Fig.) show the stable behaviour of $E^*(x^*, y^*)$ in time of the underlying system respectively when the value of γ is less than critical value γ^* (taking $\gamma = 0.5 < \gamma^* = 0.7416$). Also Fig. 2(Second Fig.) depict the unstable behaviour of x^*, y^* in time of the

system (4) respectively, when the value γ is greater than γ^* (taking $\gamma = 0.8 > \gamma^* = 0.7416$), the other parameters remain same as before. The corresponding Hopf-bifurcation diagrams with respect to the parameter γ are presented in Fig. 3.

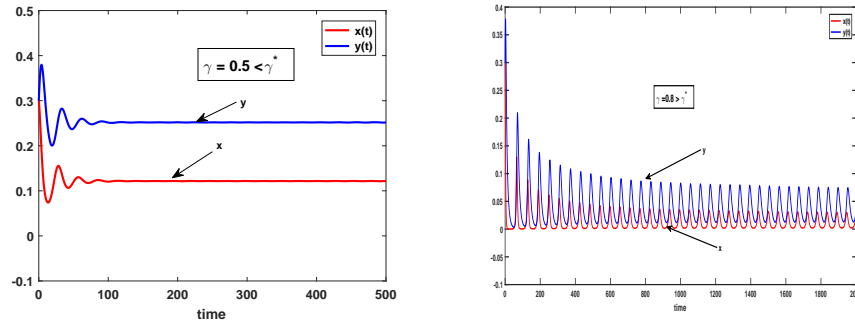


FIGURE 2. *Stable behaviour of E^* when $\gamma = 0.5 < \gamma^* = 0.7416$ and occurrence of oscillatory behaviour of E^* when $\gamma = 0.8 > \gamma^* = 0.7416$.*

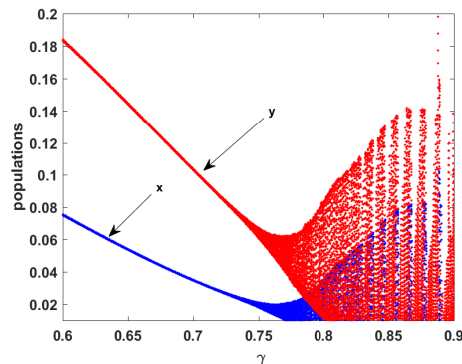


FIGURE 3. *Bifurcation diagram for the parameter γ with $\gamma^* = 0.7416$*

In this context it is mentioned that the biological parameter, h (harvesting coefficient) has an important role on the dynamics the underlying system. The system (4) undergoes a Hopf-bifurcation around E^* at $h^* = 0.2696$. Fig. 4 (First Fig.) show the stable behaviour of $E^*(x^*, y^*)$ in time of the underlying system respectively when the value of h is less than critical value h^* (taking $h =$

$0.2 < h^* = 0.2696$). Also Fig. 4(Second Fig.) depict the unstable behaviour of x^*, y^* in time of the system (4) respectively, when the value h is greater than h^* (taking $h = 0.3 > h^* = 0.2696$), the other parameters remain same as before. The corresponding Hopf-bifurcation diagrams with respect to the parameter h are presented in Fig. 5.

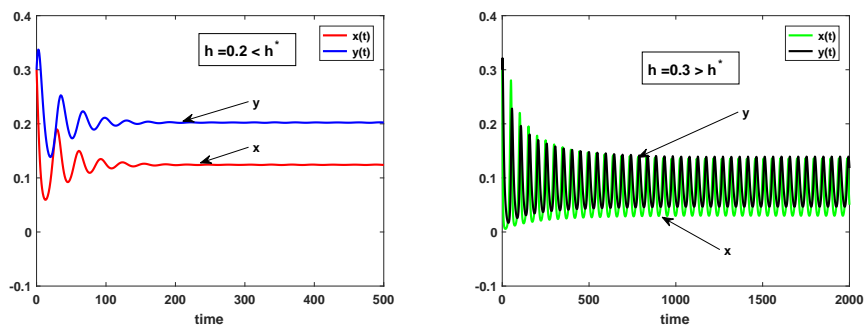


FIGURE 4. *Stable behaviour of E^* when $h = 0.2 < h^* = 0.2696$ and occurrence of oscillatory behaviour of E^* when $h = 0.3 > h^* = 0.2696$.*

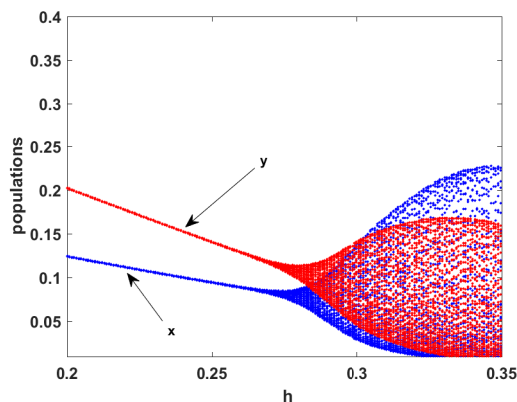


FIGURE 5. *Bifurcation diagram for the parameter h with $h^* = 0.2696$*

In this context also it is mentioned that the biological parameter, c (intra-species competition) has an important role on the dynamics the underlying system. The system (4) undergoes a Hopf-bifurcation around E^* at $c^* = 0.0246$.

Fig. 6(First Fig.) show the stable behaviour of $E^*(x^*, y^*)$ in time of the underlying system respectively when the value of c is greater than critical value c^* (taking $c = 0.06 > c^* = 0.0246$). Also Fig. 4(Second Fig.) depict the unstable behaviour of x^*, y^* in time of the system (4) respectively, when the value c is less than c^* (taking $c = 0.001 < c^* = 0.0246$), the other parameters remain same as before. The corresponding Hopf-bifurcation diagrams with respect to the parameter c are presented in Fig. 7.

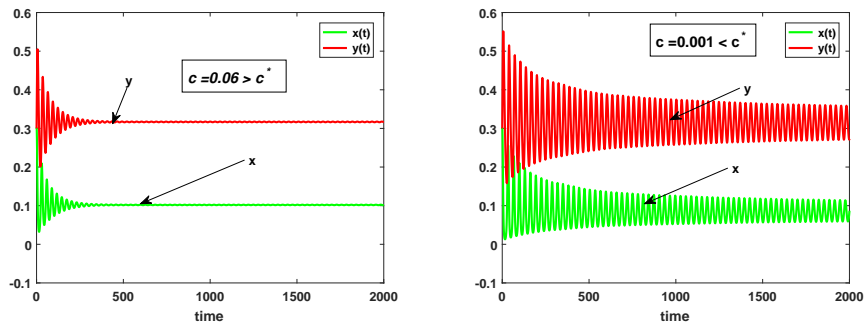


FIGURE 6. Stable behaviour of E^* when $c = 0.06 > c^* = 0.0246$ and occurrence of oscillatory behaviour of E^* when $c = 0.001 < c^* = 0.0246$.

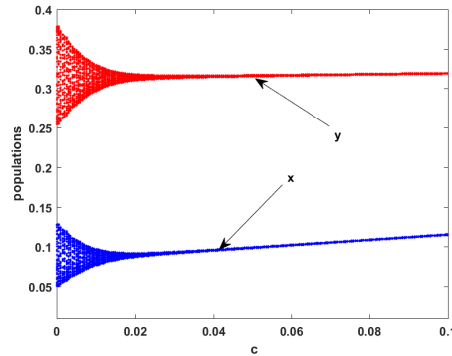


FIGURE 7. Bifurcation diagram for the parameter c with $c^* = 0.0246$

9. Conclusion

Due to the confinement of prey in the small arenas of laboratory system, it is often observed that coexistence of prey and predator population exist in the natural system but not in the laboratory system. This problem has been solved by some mathematicians as well as ecologists who have suggested that direct predation only is not enough for the study of prey- predator model and that the effect of harvesting should be considered to observe the coexistence of prey and predator's population. This could be one of the possible reasons to observe the coexistence.

In this paper, we have investigated the significant effect of harvesting on a prey- predator model with Holling type-II response function incorporated Hassell- Varley type predation model. This study helps us to state that, one of the possible ways to observe the coexistence of prey and predator is by considering the influence of harvesting on the proposed model. Positivity and boundedness of the system are shown in section 3. Extinction criteria of the populations are discussed in section 4. Also, stability behaviour of the equilibrium points are studied and validated by computer simulations.

Here we have analyzed all the boundary equilibrium points extensively. The predator free point will be stable only when $2x_1 + \frac{gh}{(g+x_1)^2} > 1$ and when predator's death rate is more than the interaction between prey and predator. The interior equilibrium points E^* also exist under certain conditions. Further we have studied the local stability behaviour of the interior equilibrium point E^* . Numerical simulations suggest the co-existence of two species for some hypothetical set of parameteric values. The important mathematical findings for the dynamical behaviour of the underlying model are also numerically verified using MATLAB. The most important equilibrium point i.e. as interior equilibrium point satisfying existence criteria are shown graphically. The Hopf-bifurcation condition has been derived in terms of γ as bifurcation parameter. Here it is observed that as γ increases the system exhibits oscillatory behaviour around coexistence equilibrium E^* . Also, occurrence of Hopf- bifurcations at interior equilibrium points under the different parameters such as h and c have been reported in Figures (4,5,6,7) and each of these bifurcation parameters has an important role on the dynamics of the system.

Finally, our model can be applicable in various fields of ecological as well as epidemiological systems. Further studies are required to analyze the dynamics of more realistic but complex systems such as considering different response functions and also applying time delays in different species.

Acknowledgements : The authors are grateful to the anonymous referees and the Editor in chief (Prof. Cheon Seoung Ryou) for their careful reading, valuable comments and helpful suggestion, which have helped them to improve the presentation of this work significantly.

REFERENCES

1. R. Arditi and L.R. Ginzburg, *Coupling in predator - prey dynamics: ratio dependence*, J. Theor. Biol. **139** (1989), 311-326.
2. S. Chakraborty, S. Pal and N. Bairagi, *Predator-Prey interaction with harvesting: mathematical study with biological ramifications*, Appl. Math. Model. **36** (2012), 4044-4059.
3. L. Chen, X. Song and Z. Lu, *Mathematical Models and Methods in Ecology*, Scient. and Tech. Publisher of Sichuan, Chengdu, 2003.
4. T. Das, R.N. Mukherjee and K.S. Chaudhuri, *Bioeconomic Harvesting of a Prey-Predator Fishery*, J. of Biol. Dyna. **3** (2009), 447- 462.
5. M. Fan and Y. Kuang, *Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional*, J. of Math. Anal. and Appl. **295** (2004), 15-39.
6. M.P. Hassell and G.C. Varley, *New inductive population model for insect parasites and its bearing on biological control*, Nature **223** (1969), 1133-1137.
7. C.S. Holling, *The components of predation as revealed by a study of small mammal predation of the European pine sawy*, Can. Entomol. **91** (1959), 293-320.
8. S.B. Hsu, T.W. Hwang and Y. Kuang, *Global analysis of the Michaelis-Menten type ratio-dependent predator-prey system*, J. of Math. Biol. **42** (2001), 489-506.
9. H.F. Huo and W.T. Li, *Stable periodic solution of the discrete periodic Leslie-Gower predator-prey model*, Math. and Comput. Model. **40** (2004), 261-269.
10. A.J. Lotka, *Elements of Physical Biology*, Williams and Wilkins Co. Inc., Baltimore, 1924.
11. Y. Li, S. Huang and T. Zhang, *Dynamics of a non-selective harvesting predator - prey model with Hassell-Varley type functional response and impulsive effects*, Math. Meth. Appl. Sci. 2015.
12. S. Liu and E. Beretta, *A stage-structured predator-prey model of Beddington-DeAngelis type*, SIAM J. on Appl. Maths. **66** (2006), 1101-1129.
13. Z.N. Ma, *Modelling, Mathematical and Study of Species Ecology*, Anhui Education Publishing Company, Hefei, China, 1996.
14. A. Mondal, A.K. Pal and G.P. Samanta, *On the dynamics of evolutionary Leslie-Gower predator-prey eco-epidemiological model with disease in predator*, Ecol. Genet. and Geono. **10** (2019), 100034.
15. A. Mondal, A.K. Pal and G.P. Samanta, *Analysis of a Delayed Eco-Epidemiological Pest-Plant Model with Infected Pest*, Biophysical Reviews and Letters **14** (2019), 141-170.
16. A. Mondal, A.K. Pal and G.P. Samanta, *Stability and Bifurcation Analysis of a Delayed Three Species Food Chain Model with Crowley-Martin Response Function*, Appl. and Appl. Maths.(AAM) **13** (2018), 709-749.
17. A. Mondal, A.K. Pal and G.P. Samanta, *Evolutionary Dynamics of a Single-Species Population Model with Multiple Delays in a Polluted Environment*, Discontinuity, Nonlinearity, and Complexity **9** (2020), 433-459.
18. A. Mondal, A.K. Pal and G.P. Samanta, *Rich dynamics of non-toxic phytoplankton, toxic phytoplankton and zooplankton system with multiple gestation delays*, Int. J. of Dyna. and Cont. **8** (2020), 112-131.
19. A. Mondal, A.K. Pal, R. Kar and A.K. Shaw, *Analysis of a complex four species food-web system: A mathematical model*, Math. in Eng., Sci. and Aeros. **12** (2021), 1-21.
20. J.D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1989.
21. R.K. Naji and N.A. Mustafa, *The dynamics of a eco- epidemiological model with nonlinear incidence rate*, J. of Appl. Math. 2012.
22. A.K. Pal, *Stability analysis of a delayed predator-prey model with nonlinear harvesting efforts using imprecise biological parameters*, Z. Naturforschung A **76** (2021), DOI:10.1515/ZNA-2021-0131.
23. S. Pathak, A. Maiti and G.P. Samanta, *Rich Dynamics of a food chain model with Hassell-Varley type functional response*, Appl. Math. and Comput. **208** (2009), 303-317.

24. F. Rao, S. Jiang, Y. Li and H. Liu, *Stochastic Analysis of a Hassell-Varley Type Predation Model*, Abstract and Applied Analysis, 2013.
25. C. Raymond, A.A. Hugo and M. Kung'aro, *Modelling dynamics of prey- predator fishery model with harvesting: a bioeconomic model*, J. of Appl. Maths. 2019.
26. V. Volterra, *Variazioni e uttauzionidelnumero d individual in species animals conviventi*, Mem. Acad. Lincei **2** (1926), 31-33.
27. X. Wang, L. Zanette and X. Zou, *Modelling the fear effect in predator-prey interactions*, J. Math. Biol. **73** (2016), 11-79.
28. D. Wang, *On a Non-Selective Harvesting Prey-Predator Model with Hassell-Varley type functional response*, Applied Mathematics and Computation **246** (2014), 678-695.
29. Rui Yuan, Weihua Jiang and Yong Wang, *Saddle-node-Hopf Bifurcation in a modified Leslie-Gower predator-prey model with time-delay and prey harvesting*, J. Math. Anal. Appl. **422** (2015), 1072-1090.
30. Z. Xiao, X. Xie and Y. Xue, *Stability and bifurcation in a Holling type II predator-prey model with Allee effect and time delay*, Adv Differ Equ **288** (2018).
31. P. Yang and Y. Wang, *Periodic Solutions of a Delayed Eco-Epidemiological Model with Infection-Age Structure and Holling Type II Functional Response*, Int. J. of Bif. and Chaos **30** (2020), 2050011.
32. G. Zeng, F. Wang and J.J. Nieto, *Complexity of a delayed predator-prey model with impulsive harvest and Holling Type II functional response*, Adv. in Compl. Syst. **11** (2008), 77-97.

Anindita Bhattacharyya received her Ph.D. in Applied Mathematics from Indian Institute of Technology Kharagpur in 2005 and her M.Sc. in 2000 from Rajabazar Science College, University of Calcutta. At present she is associated with the Department of Mathematics, Amity University Kolkata as an Associate Professor. Her research interest is in Mathematical Fluid Mechanics and Mathematical Biology.

Department of Mathematics, Amity University, Kolkata 700135.

e-mail: ambhattacharyya@kol.amity.edu

Ashok Mondal received his M.Sc. in Applied Mathematics from Calcutta University in 2014 and Ph.D. from Indian Institute of Engineering Science and Technology, Shibpur Howrah in 2021. Now he is working as an Assistant Professor of Mathematics at Regent Education and Research Foundation, Kolkata. His research interest is Mathematical Biology.

Department of Mathematics, Regent Education and Research Foundation, Barrackpore, West Bengal 700121, India.

e-mail: ashokm@regent.ac.in, ashoke.2012@yahoo.com

A.K. Pal received his M.Sc. in Applied Mathematics from Calcutta University in 2000 and Ph.D. from Bengal Engineering and Science University, Shibpur, Howrah in 2010. Now he is working as an Assistant Professor of Mathematics at Seth Anandram Jaipuria College, Kolkata. His research interest is Mathematical Biology.

Department of Mathematics, Seth Anandram Jaipuria College, Kolkata 700005, India.

e-mail: akpal_2002@yahoo.co.in

Nikhita Singh received her M.Sc. in Applied Mathematics from Amity University Kolkata in 2021. At present she is looking for a suitable Ph.D. position at different institutes.

Department of Mathematics, Amity University, Kolkata 700135.

e-mail: nikhitasingh98@gmail.com