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CONTINUITY OF THE FRACTIONAL PART FUNCTION AND DYNAMICS OF CIRCLE

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Abstract. In this paper, we obtain some subsets of real numbers ($\mathbb{R}$) on which a fractional part function is defined as a real-valued continuous function. This gives rise to the analysis of the continuous properties of the fractional part function as a real-valued function. The analysis of fractional part function is helpful in the study of the dynamics of circle.

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1. Introduction

Dynamics is the study of the motion of the body, or more generally evolution of a system with time (see, e.g., Brin and Stuck [2], Gadgil [3]). In dynamics of circle, motion takes place through the points of a circle. The movement of these points is represented by certain functions defined on the circles which are called circle maps. The study of dynamics of circle also includes the information about the continuity of the circle maps. A circle in the plane $\mathbb{R} \times \mathbb{R}$ ($\mathbb{R}^2$) is the boundary of a disc with centre at a point of the plane and some radius. While studying circles in $\mathbb{R} \times \mathbb{R}$ ($\mathbb{R}^2$) we have considered only the unit circle i.e., the circle with center at $(0, 0)$ and radius 1 which is denoted by S^1. A circle map is a function whose domain and co-domain both are S^1. Dynamics of the circle and dynamics of circle maps have been studied by many researchers, e.g., Gadgil [3], Sharma and Nagar [4], Zhang [5], and Birkhoff [6].

Every real number can be written as the sum of its integral part and fractional part. Using Archimedean property of real numbers (for a given a real number x there exists an integer n such that $n \geq x$) and well ordering property of the natural numbers (for each non-empty subset of natural numbers ($\mathbb{N}$) has the least element), there exists smallest integer n_x for arbitrary real number x ($\in \mathbb{R}$),

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such that $x \leq n_x$. This integer n_x is recognized as the integral value of x and is characterized by $[x]$. Formerly, $[x] \leq x < [x]+1$ proposed that, $0 \leq x-[x] < 1$. Here $x-[x]$ is called the fractional part of x and is denoted by r_x.

Every real number x can be written as, $x = [x] + r_x$ with $0 \leq r_x < 1$. This gives a function $r : \mathbb{R} \to [0,1)$ taking x to r_x. The function r is called the fractional part function. The fractional part function is helpful in some part of the study of dynamics of circle maps, in particular and functions in dynamical systems in general.

Now, we can deduce from the definition of S^1 and the definition of fractional part function that $S^1 = \{(\cos x, \sin x) : x \in [0, 2\pi)\} = \{(\cos 2\pi y, \sin 2\pi y) : y \in [0,1)\} = \{(\cos 2\pi x, \sin 2\pi x) : x \in \mathbb{R}\}$ (see, Lal *et al.* [1]). For $x \in \mathbb{R}$, $e^{ix} = \cos x + i \sin x$ is a periodic function with periodicity 2π. Euler's identity, $e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x)$, allows us to lift the circle to the real line. There is a covering mapping $\pi : \mathbb{R} \to C$ (the set of complex numbers), defined as $\pi(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$. As a point of $\mathbb{R} \times \mathbb{R}$, $\pi(x) = (\cos 2\pi x, \sin 2\pi x)$. Thus π is also a function from $\mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$.

In particular, $\pi : [0,1) \to S^1$ is one-one and onto (see Lal *et al.* [1]). A circle map is a continuous map $f : S^1 \to S^1$. For example, for a fixed ω, $0 \leq \omega < 2\pi$, define $f_\omega : S^1 \to S^1$ as $f_\omega(\cos x, \sin x) = (\cos(x+\omega), \sin(x+\omega))$, then f_ω is called a circle map. Similarly, for a fixed ω, $0 \leq \omega < 2\pi$, define $f^*_\omega : S^1 \to S^1$ as $f^*_\omega(\cos x, \sin x) = (\cos(x+2\pi\omega), \sin(x+2\pi\omega))$, then f^*_ω is called a circle map.

Lift of f is a continuous map $F : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions.

(i) There exists $k \in \mathbb{Z}$ such that $F(x+1) = F(x) + k$ for every $x \in \mathbb{R}$.
(ii) $\pi \circ F = f \circ \pi$.

Following from the introduction, for a fixed ω with $0 < \omega < 2\pi$, we have two circle maps f_ω and f^*_ω, defined as follows $f_\omega(\cos x, \sin x) = (\cos(x+\omega), \sin(x+\omega))$ and $f^*_\omega(\cos x, \sin x) = (\cos(x+2\pi\omega), \sin(x+2\pi\omega))$. $F : \mathbb{R} \to \mathbb{R}$, $F(x) = x+\omega/2\pi$, and $F^* : \mathbb{R} \to \mathbb{R}$, $F^*(x) = x+\omega$ are lifts of f_ω and f^*_ω respectively.

For detailed description of the circle maps and its properties, one can go through Lal et al. [1], Gadgil [3] and references therein.

The continuity of a real-valued function of a real variable is defined using ϵ-δ definition. Now the question arise 'what if the domain Y is a proper subset of $\mathbb{R}$'. If Y is an interval, the same ϵ-δ definition of continuity works excluding at the end point(s), where we talk of left-hand/right-hand continuity depending upon the end point. When the domain Y is not necessarily an interval, we need an exact definition of continuity. An appropriate way for that is to consider subspace topology. But that definition is very typical to utilize. There is another equivalent definition of continuity of a real-valued function for a real variable defined on proper subset Y of $\mathbb{R}$, in terms of ϵ-δ, which is easy to use as compared to subspace topology (see Munkres [7] and Kelly [8]).

The main focus of our study is to determine such subsets of $\mathbb{R}$ on which the fractional part function is continuous.

The layout of this article is as follows. In section 2, some basic definitions, properties and notations are given which help derive the fundamental results. Section 3 is devoted to the study of functional iteration and orbit of dynamical systems. In Section 4, some subsets of $\mathbb{R}$ on which the fractional part function is continuous have been derived. In the last section, a conclusion of the work is given.

2. Preliminaries

In this section, we present some notations and properties of subsets of $\mathbb{R}$ which are used in establishing the continuity of the fractional part function. Following this, a brief insight on the functional iteration and orbit of the dynamical system are given which are important for the study of the dynamics of the circle.

Throughout this article, $\mathbb{Z}$ denotes the set of integers. For further analysis of the subsets of $\mathbb{R}$, we present the following notations.

Let $Y \subset \mathbb{R}$, $0 < s < 1$ and $n \in \mathbb{Z}$.

$cl(Y)$ denotes the closure of Y.

A collection $\{Y_j : j \in j\}$ of subsets of $\mathbb{R}$ is called *separated* or pair wise separated if, for every $j, k \in j$, $j \neq k$, $cl(Y_j) \cap Y_k = \phi$.

For further analysis we present the following notations.

$A_s = \cup\{[n, n+s] : n \in \mathbb{Z}\}$.
$A_s^0 = \cup\{(n, n+s) : n \in \mathbb{Z}\}$.
$C_s = \cup\{(n+s, n+1) : n \in \mathbb{Z}\}$.
$C_s^* = \cup\{[n+s, n+1) : n \in \mathbb{Z}\}$.
$B_s = \{x \in \mathbb{R} : r(x) \leq s\}$.
$B_s^* = \{x \in \mathbb{R} : r(x) \geq s\}$.
$E_s = \{x \in \mathbb{R} : r(x) = s\} = B_s \cap B_s^*$.
$A_s^n = [n, n+s]$.
$V_s^n = [n, n+s)$.

Note. Also, we can define A_s and A_s^n, also for $s = 0$, as A_0 turns out to be $\mathbb{Z}$ and $A_0^n = \{n\}$, $n \in \mathbb{Z}$.

Let $Y \subset \mathbb{R}$ and $g : Y \to \mathbb{R}$, for $x \in Y$, $g : Y \to \mathbb{R}$ is said to be continuous at a point x if $g : (Y, \tau^*) \to \mathbb{R}$ is continuous at x, where τ^* is the induced topology of the usual topology of $\mathbb{R}$. $g : Y \to \mathbb{R}$ is continuous if $g : Y \to \mathbb{R}$ is continuous at every point of Y (see Munkers [7] and Kelley [8]).

Remark 2.1. $\mathbb{R} = \cup\{V_1^n : n \in \mathbb{Z}\} = \cup\{[n, n+1) : n \in \mathbb{Z}\}$.

Proof. Let $x \in \mathbb{R}$. $x = [x] + r(x)$ with $0 \leq r(x) < 1$. Therefore, $x \in [m, m+1)$, where $m = [x]$. Thus $\mathbb{R} = \cup\{[n, n+1) : n \in \mathbb{Z}\}$. □

Remark 2.2. (i) For every s, $0 \leq s < 1$, $A_s = \cup\{A_s^n : n \in \mathbb{Z}\}$.
(ii) $\mathbb{R} = \cup\{[n, n+s) : n \in \mathbb{Z}, 0 < s < 1\}$.
(iii) $\mathbb{R} = \cup\{A_s^n : n \in \mathbb{Z}, 0 < s < 1\}$.

Proof. (i) The proof of this part is directly followed from the definitions of A_s and A_s^n.
(ii) Let $x \in \mathbb{R}$. $x = [x] + r(x)$ with $0 \leq r(x) < 1$. Let t be such that $0 \leq r(x) < t < 1$. Therefore $x \in [m, m+t)$, where $m = [x]$. Thus $\mathbb{R} = \cup\{[n, n+s) : n \in \mathbb{Z}, 0 < s < 1\}$.
(iii) Using property (ii), $\mathbb{R} = \cup\{[n, n+s) : n \in \mathbb{Z}, 0 < s < 1\} \subset \cup\{A_s^n : n \in \mathbb{Z}, 0 < s < 1\}$. Therefore, $\mathbb{R} = \cup\{A_s^n : n \in \mathbb{Z}, 0 < s < 1\}$. □

Lemma 2.1. *For every s, $0 < s < 1$, $A_s = B_s$.*

Proof. Let $x \in A_s$. Then, $x \in [n, n+s]$ for some $n \in \mathbb{Z}$. Later, $n \leq x \leq n+s < n+1$ implies that $n = [x]$. Subsequently, $x = [x] + r(x)$ and $n + r(x) \leq n + s$ suggest that $r(x) \leq s$. Thus, $x \in B_s$. Conversely, let $x \in B_s$. Because $[x] \leq x = [x] + r(x) \leq [x] + s$ signify $r(x) \leq s$. Follows, $x \in [[x], [x]+s]$. Thus, $x \in A_s$. □

Lemma 2.2. *For each s, $0 < s < 1$, $E_s = \{n + s : n \in \mathbb{Z}\}$.*

Proof. Let $x \in E_s$. For $r(x) = s$ we have $x = [x] + s$. Therefore, $x \in \{n + s : n \in \mathbb{Z}\}$. Conversely, let $x \in \{n + s : n \in \mathbb{Z}\}$. Then $x = m + s$ for some $m \in \mathbb{Z}$. So $x - m = s$. Since $[x - m] = -m + [x]$ and $0 < s < 1$, therefore $m = [x]$. This implies that $r(x) = s$. Therefore, $x \in E_s$. □

Lemma 2.3. *Let $H \subset \mathbb{R}$. Let $h : H \to \mathbb{R}$. Let $x \in H$. If there exists a $\delta^* > 0$ such that for $y \in (x - \delta^*, x + \delta^*)$,*

(i) *$x - y = h(x) - h(y)$, or*
(ii) *$|h(x) - h(y)| \leq \lambda|x - y|$ for some fixed $\lambda > 0$.*

Then h is continuous at x.

Proof. For the case (i). Let $\epsilon > 0$. Let $\delta = \min\{\epsilon, \delta^*\}$. Let $|y - x| < \delta$. Then $y \in (x - \delta^*, x + \delta^*)$ as $\delta \leq \delta^*$. Therefore by the given condition (i), $|h(x) - h(y)| = |x - y| < \delta \leq \epsilon$.

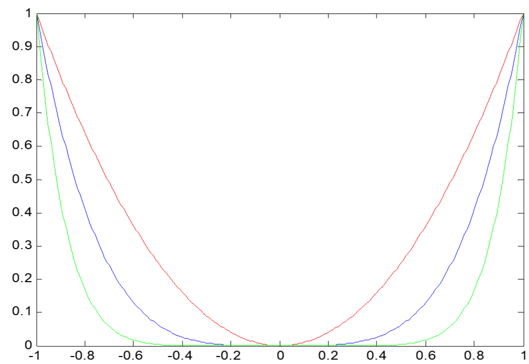
For the case (ii). Let $\epsilon > 0$. Let $\delta = \min\{\epsilon/\lambda, \delta^*\}$. Let $|y - x| < \delta$. Then $y \in (x - \delta^*, x + \delta^*)$ as $\delta \leq \delta^*$. Therefore by the given condition (ii), $|h(x) - h(y)| \leq \lambda|x - y| < \delta\lambda \leq \epsilon$. □

3. Functional Iteration and Orbit and Dynamical Systems

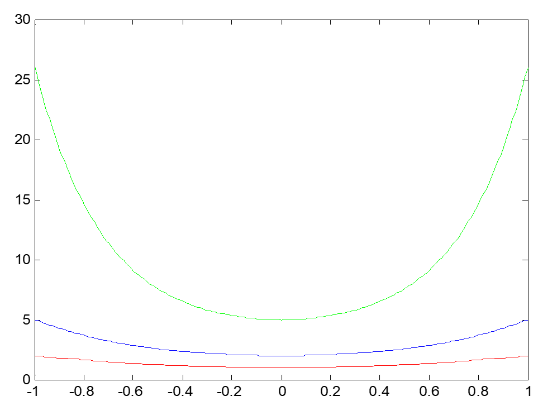
Dynamics study the pattern in the map under repeated iterations. In the study of circle maps and their lifts, the study of fractional iteration is considered useful e.g. Gadgil [3] and Zhang [5].

The framework of functional iteration can be understood from the following concept. If $f : \mathbb{R} \to \mathbb{R}$, then we can write, $f^0(x) = x$ and $f^1(x) = f(x)$. After applying functional iteration we have $f^2(x) = (f \circ f)(x) = f(f(x))$. Furthermore $f^3(x) = (f \circ f^2)(x) = f(f^2(x)) = f(f(f(x)))$. In general $f^n(x) = (f \circ f^{n-1})(x) = f(f^{n-1}(x))$, for $n \geq 3$. Further, $f^n(x)$ is called the nth iteration of f for $n \geq 0$. Following are some examples of functional iterations.

Example 3.1. Let $f(x) = x^2$. Then $f^2(x) = (f \circ f)(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4$. $f^3(x) = (f \circ f^2)(x) = f(f^2(x)) = f(x^4) = (x^4)^2 = x^8$. In generalized form we have, $f^n(x) = (x^2)^n$. The following figure presents the graph of the three functions $f(x)$, $f^2(x)$, $f^3(x)$ for $f(x) = x^2$.



Example 3.2. Let us consider another example of iterated functions.
Let $f(x) = x^2 + 1$. Again $f^2(x) = (f \circ f)(x) = f(f(x)) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$. By using iteration of functions $f^3(x) = (f \circ f^2)(x) = f(f^2(x)) = f(x^4 + 2x^2 + 2) = (x^4 + 2x^2 + 2)^2 + 1 = x^8 + 4x^6 + 8x^4 + 8x^2 + 5$. In this case it is not easy to generalize. Here we can see the comparison of the iterations of the function $f(x)$.



3.1. Orbit and Dynamical Systems. Let $x_0 \in \mathbb{R}$. We define the orbit of x_0 under f to be the sequence of points $x_0, x_1, x_2, x_3, \ldots, x_n, \ldots$ such that $x_{n+1} = f(x_n)$, for $n \geq 0$. That is the orbit of x_0 under f is the sequence of iteration $x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots, f^n(x_0), \ldots$. Here x_0 is called the seed of the orbit. The changing values in an orbit represent a dynamical system after repeated iterations.

Example 3.3. Let $x = \pi/3$. Consider $f(x) = \cos x$. Then $f(\pi/3) = \cos(\pi/3) = 1/2$. Using iteration of the functions, $f^2(x) = f(f(x)) = f(\cos x) = \cos(\cos(x))$ and $f^2(\pi/3) = \cos(\cos(\pi/3)) = \cos(1/2) = 0.8776$. Furthermore, $f^3(x) = f(f^2(x)) = f(\cos(\cos(x))) = \cos(\cos(\cos(x)))$ and $f^3(\pi/3) = \cos(\cos(\cos(\pi/3))) = \cos(0.8776) = 0.6390$.

The following is the table for $\cos x$ for $x = 0.5$ for first 30 iteration.

Sr. No	Values	Sr. No	Values	Sr. No	Values
1	0.8776	11	0.7418	21	0.7391
2	0.639	12	0.7372	22	0.739
3	0.8027	13	0.7403	23	0.7391
4	0.6948	14	0.7382	24	0.7391
5	0.7682	15	0.7396	25	0.7391
6	0.7192	16	0.7387	26	0.7391
7	0.7524	17	0.7393	27	0.7391
8	0.7301	18	0.7389	28	0.7391
9	0.7451	19	0.7392	29	0.7391
10	0.735	20	0.739	30	0.7391

As we can follow from the table that the value of the iteration for the function $\cos x$ remains static after 20 iterations.

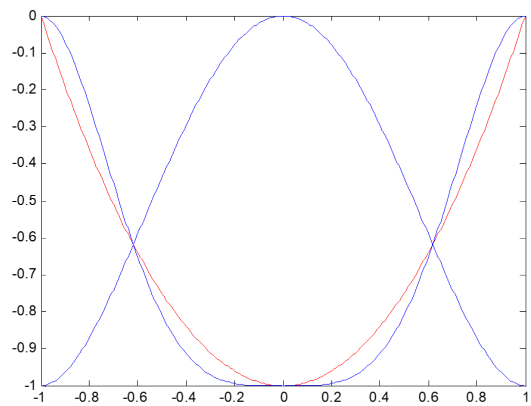
Example 3.4. Let $f(x) = x^2 - 1$. Use $x_0 = 1/2$. Therefore, $f(x_0) = f(1/2) = (1/2)^2 - 1 = 1/4 - 1 = -3/4$. Through iteration process on functions $f^2(x_0) = (f \circ f)(x_0) = f(f(x_0)) = f(-3/4) = (-3/4)^2 - 1 = 9/16 - 1 = -7/16$. Similarly we have, $f^3(x_0) = (f \circ f^2)(x_0) = f(f^2(x_0)) = f(-7/16) = (-7/16)^2 - 1 = 49/256 - 1 = -207/256$.

Sr. No	Values	Sr. No	Values
1	−0.7500000000	11	−0.9996000000
2	−0.4375000000	12	−0.0007594800
3	−0.8086000000	13	−1.0000000000
4	−0.3462000000	14	−0.0000011536
5	−0.8802000000	15	−1.0000000000
6	−0.2253000000	16	0.0000000000
7	−0.9492000000	17	−1.0000000000
8	−0.0990000000	18	0.0000000000
9	−0.9902000000	19	−1.0000000000
10	−0.0195000000	20	0.0000000000

Also for the value of $x = 0.4$ and 0.9 the value of the iteration after some iterations interpolates between -1 and 0.

Let $f(x) = x^2 - 1$. Using iteration of functions $f^2(x) = (f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$. In similar way $f^3(x) = (f \circ f)(x) = f(f^2(x)) = f(x^4 - 2x^2) = (x^4 - 2x^2)^2 - 1 = x^8 - 4x^6 + 4x^4 - 1$.

The following figure shows the comparison for $f(x) = x^2 - 1$, $f^2(x)$ and $f^3(x)$.



Remark 3.1. The sequences of iterated images $f^n(x)$ when n is an even goes to zero and when n is an odd the sequence goes to -1, where $f^{n+1}(x) = (f^n(x))^2 - 1$ for large values of n.

Proof. Consider the function $f(x) = x^2 - 1$.
Case (i) Let $0 \leq x < 1$. If $x = 0$ then $f(x) = -1$. Again $f^2(x) = f(f(x)) = f(-1) = (-1)^2 - 1 = 1 - 1 = 0$. In similar way $f^3(x) = f(f^2(x)) = f(0) = -1$ and so on.
Case (ii) Consider $0 < x < 1$. Subcase (i) Let $f(x) = x^2 - 1 < 0$. With iteration of functions $f^2(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2 = x^2(x^2 - 2) = -x^2(2 - x^2)$. Again $f^3(x) = f(f^2(x)) = f(-x^2(2 - x^2)) = x^4(2 - x^2)^2 - 1 = y^2 - 1$ where $y = x^2(2 - x^2)$. Similarly, $f^4(x) = f(f^3(x)) = f(y^2 - 1) = (y^2 - 1)^2 - 1 = y^4 - 2y^2 = -y^2(2 - y^2) = -x^4(2 - x^2)^2(2 - x^4(2 - x^2)^2)$. Furthermore, $f^5(x) = f(f^4(x)) = f(-y^2(2 - y^2)) = (-y^2(2 - y^2))^2 - 1 = y^4(2 - y^2)^2 - 1 = z^2 - 1$ where $z = y^2(2 - y^2)$. □

Remark 3.2. If $f(x) = (x)^{1/2}$ then $f^n(x) = x^{(1/2)^n}$ and hence $f^{n+1}(x) = x^{(1/2)^{n+1}}$.

Proof. Let $f(x) = (x)^{1/2}$. Using iteration process on functions $f^2(x) = f(f(x)) = f((x)^{1/2}) = x^{(1/2)^2}$. Then $f^{n+1}(x) = f(f^n(x)) = (f^n(x))^{1/2}$. We claim that $f^n(x) = x^{(1/2)^n}$. By induction method for $n = 1$, $f(x) = (x)^{1/2}$. Let us suppose the result is true for n, i.e. $f^n(x) = x^{(1/2)^n}$. Furthermore $f^{n+1}(x) = f(f^n(x)) = (x^{(1/2)^n})^{1/2} = x^{(1/2)^{n+1}}$. □

Remark 3.3. If $x < 1$ then, $x^{(1/2)^n} < 1$.

Proof. For $n = 1$, if $0 < x < 1$ then $x^{1/2} < 1$. Suppose the result is true for n, i.e. $x^{(1/2)^n} < 1$. Now $x^{(1/2)^{n+1}} = (x^{(1/2)^n})^{1/2} < 1$. This proves for $n + 1$. □

Remark 3.4. If $x > 1$ then, $x^{1/2} < x$.

Proof. If $t, y > 0$. If $t^2 < y^2$ if and only if $t < y$. If $t < y$ if and only if $t^{1/2} < y^{1/2}$. Whenever $1 < x$ at that instant $x < x^2$ i.e. $(x^{1/2})^2 < x^2$. It follows that $x^{1/2} < x$. □

Remark 3.5. If $1 < x$ then, $x^{(1/2)^n} \leq x$.

Proof. By induction if $n = 1$, $x^{1/2} < x$. Suppose that it is true for n i.e. $x^{(1/2)^n} \leq x$. Now $x^{(1/2)^{n+1}} = (x^{(1/2)^n})^{1/2} < x^{1/2} < x$. This proves for $n + 1$. □

Remark 3.6. We claim that $1 < x^{(1/2)^n}$.

Proof. By induction if $n = 1$, $1 < x^{1/2}$. Suppose it is true for n i.e. $1 < x^{(1/2)^n}$. Now $1 < (x^{(1/2)^n})^{1/2} = x^{(1/2)^{n+1}}$.This proves for $n + 1$. □

Remark 3.7. If $0 < x < 1$, then $\lim x^{(1/2)^n} = 1$.

Proof. Let $t = \lim x^{(1/2)^n}$ exists. Therefore, $t = x^{\lim(1/2)^n} = x^0 = 1$. When $0 < x < 1$. Let $y = 1/x$, then $y^{(1/2)^n} = 1/x^{(1/2)^n}$. If $y > 1$. Therefore, $\lim y^{(1/2)^n} = 1$. Therefore, $1 = 1/\lim x^{(1/2)^n}$. Hence, $\lim x^{(1/2)^n} = 1$.

The following is a table for $f(x) = x^{(1/2)}$, $x = 0.1$ up to 30 iterations in which one can see that up to 23 iteration approaches to 1 and in the 24 and onward iteration becomes exactly 1.

Sr. No	Values	Sr. No	Values	Sr. No	Values
1	0.3162000000	11	0.7499000000	21	0.9997000000
2	0.5623000000	12	0.8660000000	22	0.9999000000
3	0.7499000000	13	0.9306000000	23	0.9999000000
4	0.8660000000	14	0.9647000000	24	1.0000000000
5	0.9306000000	15	0.9822000000	25	1.0000000000
6	0.9647000000	16	0.9910000000	26	1.0000000000
7	0.9822000000	17	0.9955000000	27	1.0000000000
8	0.9910000000	18	0.9978000000	28	1.0000000000
9	0.3162000000	19	0.9989000000	29	1.0000000000
10	0.5623000000	20	0.9994000000	30	1.0000000000

4. Results

In this section, we present the fundamental results in which the subsets of $\mathbb{R}$ are derived where the fractional part function is continuous.

Lemma 4.1. *Let $\{Y_j : j \in J\}$ be a collection of subsets of $\mathbb{R}$ such that $\cup\{Y_j : j \in J\}$ is closed. Then $\{Y_j : j \in J\}$ are pair wise separated iff $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed.*

Proof. If $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed then clearly $\{Y_j : j \in J\}$ are pair wise separated. Now suppose that $\{Y_j : j \in J\}$ are pair wise separated. Let $k \in j$. Let $x \in cl(Y_k)$. Suppose $x \notin Y_k$. For $j \in J$,

$j \neq k$, since $cl(Y_k) \cap Y_j = \phi$, $x \notin Y_j$. Thus $x \notin \cup\{Y_j : j \in J\}$. Therefore $x \in B = \mathbb{R} - \cup\{Y_j : j \in J\}$. Since B is open and $x \in cl(Y_k)$, $B \cap Y_k \neq \phi$. But $B \cap Y_k = \phi$ as $B \subset \mathbb{R} - Y_k$. □

Corollary 4.2. *Let $\{B_j : j \in J\}$ and $\{Y_j : j \in J\}$ be such that $\mathbb{R} - \cup\{Y_j : j \in J\} = \cup\{B_j : j \in J\}$. If $\cup\{B_j : j \in J\}$ is open then $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed iff $\{Y_j : j \in J\}$ are pair wise separated.*

Proof. By the given condition $\cup\{Y_j : j \in J\}$ is closed. Therefore the result follows by Lemma 4.1.

Lemma 4.3. *Let $\mathbb{R} = \cup\{Y_j : j \in J\}$ be such that $Y_j \cap Y_k = \phi$ for every $j, k \in j$, $j \neq k$. If for each $j \in J$, $Y_j = H_j \cup K_j$ with $H_j \cap K_j = \phi$, then $\mathbb{R} - \cup\{H_j : j \in J\} = \cup\{K_j : j \in J\}$.*

Proof. Let $j, k \in J$, $k \neq j$. Since $Y_j \cap Y_k = \phi$, $Y_j \cap H_k = \phi$. Thus $Y_j \subset \mathbb{R} - H_k$. So $Y_j - H_k = Y_j \cap (\mathbb{R} - H_k) = Y_j$. This implies that $\cap\{Y_j - H_k : k \in j, k \neq j\} = Y_j$. Therefore, $\cap\{Y_j - H_k : k \in j\} = Y_j - H_j$. Now $\mathbb{R} - \cup\{H_j : j \in J\} = (\cup\{Y_j : j \in J\}) - (\cup\{H_k : k \in J\}) = \cup\{(Y_j - (\cup\{H_k : k \in J\}) : j \in J\} = \cup\{\cap(Y_j - H_k : k \in J) : j \in J\} = \cup\{Y_j - H_j : j \in J\} = \cup\{K_j : j \in J\}$, as $Y_j - H_j = K_j$. □

Lemma 4.4. *For every s, $0 < s < 1$, $\mathbb{R} - A_s = C_s = \cup\{(n+s, n+1); n \in \mathbb{Z}\}$.*

Proof. By Remark 2.1, $\mathbb{R} = \cup\{[n, n+1) : n \in \mathbb{Z}\}$. Let $j = \mathbb{Z}$. Let, for $n \in \mathbb{Z}$, $Y_n = [n, n+1)$. Then $Y_n \cap Y_k = \phi$ for every $n, k \in \mathbb{Z}$, $n \neq k$. Let $H_n = [n, n+s]$ and $K_n = (n+s, n+1)$. Then $Y_n = H_n \cup K_n$ and $H_n \cap K_n = \phi$. Therefore by Lemma 4.3, $\mathbb{R} - A_s = C_s = \cup\{(n+s, n+1) : n \in \mathbb{Z}\}$. □

Remark 4.1. For every s, $0 < s < 1$, $B_s^* = \cup\{[n+s, n+1) : n \in \mathbb{Z}\}$.

Proof. $B_s^* = \{x : r(x) > s\} \cup \{x : r(x) = s\}$. Therefore, by definition of B_s, $B_s^* = (\mathbb{R} - B_s) \cup E_s = (\mathbb{R} - A_s) \cup E_s$ as $B_s = A_s$ by Lemma 2.1. Since, by Lemma 4.4, $\mathbb{R} - A_s = C_s = \cup\{(n+s, n+1); n \in \mathbb{Z}\}$ and $E_s = \{n+s : n \in \mathbb{Z}\}$, $B_s^* = \cup\{[n+s, n+1); n \in \mathbb{Z}\}$. □

Remark 4.2. Let $0 < s < 1$. For every $n \in \mathbb{Z}$, n is a limit point of B_s^*, so B_s^* is not closed.

Proof. By Remark 4.1, $B_s^* = \cup\{[n+s, n+1) : n \in \mathbb{Z}\}$. As $n \in \mathbb{Z}$ if and only if $n - 1 \in \mathbb{Z}$ at this point $B_s^* = \cup\{[n-1+s, n) : n \in \mathbb{Z}\}$. For $n \in \mathbb{Z}$, n is a limit point of B_s^*, so B_s^* is not closed. □

Remark 4.3. $\mathbb{R} - A_s^0 = C_s^* = \cup\{[n+s, n+1); n \in \mathbb{Z}\}$, where $A_s^0 = \cup\{(n, n+s) : n \in \mathbb{Z}\}$.

Proof. $\mathbb{R} = \cup\{[n, n+1) : n \in \mathbb{Z}\}$. Let $J = \mathbb{Z}$. Let for $n \in \mathbb{Z}$, $Y_n = [n, n+1)$. Then $Y_n \cap Y_k = \phi$ for every $n, k \in \mathbb{Z}$, $n \neq k$. Let $H_n = (n, n+s)$ and $K_n = [n+s, n+1)$. Then $Y_n = H_n \cup K_n$ and $H_n \cap K_n = \phi$. Therefore, by Lemma 4.4, $\mathbb{R} - A_s^0 = C_s^* = \cup\{[n+s, n+1); n \in \mathbb{Z}\}$. □

Remark 4.4. The Lemma 4.4 can also be proved using the function r i.e. using Lemma 2.1. We give the proof below.

Lemma 4.5. *(i)* $\mathbb{R} - A_s = C_s = \cup\{(n+s, n+1); n \in \mathbb{Z}\}$. *(ii)* A_s *is closed.*

Proof. (i) Since by Lemma 2.1, $A_s = B_s$, we prove that $\mathbb{R} - B_s = \cup\{(n+s, n+1) : n \in \mathbb{Z}\}$. As $\mathbb{R} - B_s = \{x \in \mathbb{R}; r(x) > s\}$. Let $x \in \mathbb{R} - B_s$. Therefore, $r(x) > s$. Already, $x = [x] + r(x)$. We claim $x \in ([x] + s, [x] + 1)$. As $r(x) > s$ therefore $[x] + r(x) > [x] + s$. Again $r(x) < 1$ then we have $[x] + r(x) < [x] + 1$. Therefore, $[x] + s < x < [x] + 1$. Hence, $x \in ([x] + s, [x] + 1)$. For the converse, suppose that $x \in (n + s, n + 1)$ for some $n \in \mathbb{Z}$. Since $n + s < x < n + 1$, $n = [x]$. So $r(x) = x - [x] > s$. Therefore, $x \in \mathbb{R} - B_s$.
(ii) Since $\cup\{(n + s, n + 1); n \in \mathbb{Z}\}$ is open, by (i), $\mathbb{R} - A_s$ is open. Therefore, A_s is closed. □

The following is the direct proof that A_s is closed. We need the following remark for that.

Remark 4.5. Let $a, x, b \in \mathbb{R}$ such that $a < x < b$. Take $\delta = \min\{x - a, b - x\}$ then $a \le x - \delta$ and $x + \delta \le b$. Therefore, $(x - \delta, x + \delta) \subset (a, b)$.

Proof. As $\delta = \min\{x - a, b - x\}$ therefore, $\delta \le x - a$ and $\delta \le b - x$. Consequently, $a \le x - \delta$ and $x + \delta \le b$. Hence, $(x - \delta, x + \delta) \subset (a, b)$.

Proof of A_s is closed. Suppose $x \notin A_s$. This suggest that, $x \notin [[x], [x] + s]$. Accordingly, $r(x) > s$ as $x = [x] + r(x)$. As follows, $[x] + s < x < [x] + 1$. Let $\delta = \min\{r(x) - s, 1 - r(x)\}$. Because $x - ([x] + s) = r(x) - s$, and $[x] + 1 - x = 1 - r(x)$, then by Remark 4.5 we have $(x - \delta, x + \delta) \subset ([x] + s, [x] + 1)$. Consequently, we have, $(x - \delta, x + \delta) \cap [[x], [x] + s] = \phi$. Let $n \in \mathbb{Z}$ and $n \neq [x]$. Then either $n < [x]$ or $[x] + 1 \le n$. Suppose $n < [x]$. Through Remark 4.5, $s \le r(x) - \delta$. Hence, $n + s < x - \delta$. Now consider $[x] + 1 \le n$. Applying Remark 4.5, $x + \delta \le [x] + 1 \le n$. Along these lines we have $(x - \delta, x + \delta) \cap [n, n + s] = \phi$ for every $n \in \mathbb{Z}$. Thus $(x - \delta, x + \delta) \cap A_s = \phi$. Therefore x is not a limit point of A_s. Hence A_s is closed. □

As mentioned in the introduction, we need to have a working definition of the continuity of a real-valued function of a real variable is defined using ϵ-δ, when the domain Y of the function is a proper subset of $\mathbb{R}$. If Y is an interval, the same ϵ -δ definition of continuity works except at the end point(s). When the domain Y is not necessarily an interval, we have the following definition which works even when Y is an interval. But first we have the theoretical definition. Then we have an equivalent definition of continuity of a real-valued function of a real variable in terms of ϵ-δ, which we use later.

Remark 4.6. Let $Y \subset \mathbb{R}$ and $g : Y \to \mathbb{R}$. Suppose $x \in Y$. (i) If g is continuous at x if and only if for given $\epsilon > 0$, there exists $\delta > 0$ such that for $y \in (x - \delta, x + \delta) \cap Y$ then $|g(x) - g(y)| < \epsilon$. (ii) If there exists $\delta > 0$ such that for $y \in (x - \delta, x + \delta) \cap Y$ and $|g(x) - g(y)| \le |x - y|$, then g is continuous at x.

Proof. (i) It follows as $(x-\delta, x+\delta)\cap Y$ is open in the induced topology on Y. (ii) For given $\epsilon > 0$, if we take $\delta^* = \min\{\epsilon, \delta\}$, then g is continuous at x. □

Remark 4.7. Let $H \subset Y \subset \mathbb{R}$. Let $g : Y \to \mathbb{R}$. Let $x \in H$. (i) If $g : H \to \mathbb{R}$ is not continuous at x, then $g : Y \to \mathbb{R}$ is not continuous at x. (ii) The converse of (i) is not true. That is, if $g : Y \to \mathbb{R}$ is not continuous at x, then $g : H \to \mathbb{R}$ may be continuous at x.

Proof. (i) $g : H \to \mathbb{R}$ is not continuous at x, so by Remark 4.6, there exists some $\epsilon > 0$ such that whatever $\delta > 0$ we take there exists $y \in (x-\delta, x+\delta)\cap H$ such that $|g(x) - g(y)| \geq \epsilon$. Since $H \subset Y$, $y \in (x-\delta, x+\delta)\cap Y$. Therefore, in view of Remark 4.6, $g : Y \to \mathbb{R}$ is not at x. (ii) Take $H = \mathbb{Z}$ and $Y = \mathbb{R}$. Every $g : \mathbb{Z} \to \mathbb{R}$ is continuous. But every $g : \mathbb{R} \to \mathbb{R}$ is not continuous. □

Remark 4.8. Let $x, y \in \mathbb{R}$. If $[x] = [y]$ then $x - y = r(x) - r(y)$. Prove that $r(x) < r(y)$ if and only if $x < y$.

Proof. As of now $x = [x]+r(x)$ and $y = [y]+r(y)$. Accordingly, $x-y = r(x)-r(y)$ as $[x] = [y]$. Immediately it follows that $r(x) < r(y)$ if and only if $x < y$. □

Proposition 4.6. *Let $A_s = \cup\{[n, n+s]; n \in \mathbb{Z}\}$. $r : A_s \to \mathbb{R}$ is continuous.*

Proof. Let $x \in A_s$. First we find $\delta > 0$ such that for $y \in (x-\delta, x+\delta)\cap A_s$ and $[y] = [x]$. Suppose $x \in [m, m+s]$ for some $m \in \mathbb{Z}$. Case (i) Let $x \in (m, m+s)$ and $\delta = \min\{x-m, m+s-x\}$. Applying Remark 4.5 we get $(x-\delta, x+\delta) \subset (m, m+s)$. Let $y \in (x-\delta, x+\delta)\cap A_s$. Then we have $y \in (m, m+s)$. Consequently $[y] = [x]$ as $s < 1$. Case (ii) Suppose $x = m+s$. Take $\delta = 1/2(\min\{s, (1-s)\})$. Through Remark 4.5 we obtain $(x-\delta, x+\delta) \subset (m, m+1)$. Let $y \in (x-\delta, x+\delta)\cap A_s$. Then we have $y \in (m, m+1)$. Hence $[y] = m = [x]$ as $0 < s < 1$. Case (iii) Let $x = m$. Take $\delta = 1/2(\min\{s, (1-s)\})$. Using Remark 4.5 we get $(x-\delta, x+\delta) \subset (m-1, m+s)$. Let $y \in (x-\delta, x+\delta)\cap A_s$. Then $y \in [m, m+s)$ as $(m-1, m+s)\cap A_s \subset [m, m+s)$. Hence, $[y] = [x]$. Thus in every case, for $y \in (x-\delta, x+\delta)\cap A_s$ we obtain $[y] = [x]$. By Remark 4.8, $x - y = r(x) - r(y)$. Now, by Remark 4.6(ii), r is continuous at x. □

Remark 4.9. By Proposition 4.6, for every s with $0 < s < 1$, r is continuous on A_s. The collection $\{A_s : s \in \mathbb{R}, 0 < s < 1\}$ is a totally ordered subset of the p.o. set $(\mathbb{R}, \subset)$ because for $s, t \in \mathbb{R}$, $0 < s$, $t < 1$, $A_s \subset A_t$, or $A_t \subset A_s$ depending upon $s \leq t$, or $t \leq s$. Let $x \in \mathbb{R}$. $x = [x] + r(x)$. Since $0 \leq r(x) < 1$, $x \in A_t$ for every t such that $r(x) < t$. Therefore $\cup\{A_s : s \in \mathbb{R}, 0 < s < 1\} = \mathbb{R}$. By definition $A_s^0 = \cup\{(n, n+s) : n \in \mathbb{Z}\}$. It can be seen that $\cup\{A_s^0 : s \in \mathbb{R}, 0 < s < 1\} = \mathbb{R}-\mathbb{Z}$.

Remark 4.10. We have seen above that, for $s = 0$, $A_s = \mathbb{Z}$, i.e. $A_0 = \mathbb{Z}$. It can be seen (below) that r is continuous also on A_0.

Remark 4.11. (i) r is continuous on $\mathbb{R}-\mathbb{Z}$. (ii) r is continuous on $\mathbb{Z}$. (iii) r is continuous on H, if $H \subset \mathbb{R}-\mathbb{Z}$ or $\mathbb{Z}$.

Proof. (i) Let $x \in \mathbb{R} - \mathbb{Z}$. Since $[x] < x < [x] + 1$, by Remark 4.5 there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset ([x], [x] + 1)$. Therefore, for $y \in (x - \delta, x + \delta)$, $[y] = [x]$. By Remark 4.8, $x - y = r(x) - r(y)$. Therefore r is continuous at x. Therefore r is continuous on $\mathbb{R} - \mathbb{Z}$. (ii) Let $m \in \mathbb{Z}$. Let $\epsilon > 0$. For $\delta < 1$, $(m-\delta, m+\delta) \cap \mathbb{Z} = \{m\}$, therefore $r(y) - r(m) = 0$ for every $y \in (m-\delta, m+\delta) \cap \mathbb{Z}$. (iii) Restriction of a continuous function is continuous. □

5. Conclusion

In this article, we have discussed the circle maps, functional iterations, orbit and dynamical systems and their relations following some examples. Further, we have demonstrated the relationship between circle maps and fractional part functions. Following this, some particular subsets of $\mathbb{R}$ have been presented in which the fractional part function is continuous. The work in this paper is new from the point of analysis.

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