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ON THE NUMBER OF FUZZY SUBGROUPS OF $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ †

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Abstract. In this paper we are concerned with the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ of rank three with order $p^{m+n+\ell}$. We obtain a recurrence relation for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$. In order to show that using this recurrence relation, one can find explicit formulas for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ consecutively, we give explicit formulas for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ where $(n, \ell) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2)$.

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1. Introduction

Let G be a group with a multiplicative binary operation, and let $\mu : G \to [0, 1]$ be a fuzzy subset of G. Then μ is *fuzzy subgroup* of G if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. The set $\{\mu(x) \mid x \in G\}$ is called the *image* of μ and is denoted by $\mu(G)$. For all $a \in \mu(G)$, the set $\mu_a = \{x \in G \mid \mu(x) \geq a\}$ is called a *level subset* of μ. It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G (see [3]).

Let G be a group. A *chain of subgroups* of G is a set of subgroups of G linearly ordered by set inclusion. A chain of subgroups of G is called *rooted* if it contains G. Otherwise, it is called *unrooted*.

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For any fuzzy subgroups μ and ν of a given group G, μ and ν are *equivalent*, written as $\mu \sim \nu$, if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ for all $x, y \in G$. It follows that $\mu \sim \nu$ if and only if μ and ν have the same set of level subgroups (see [8]). Hence there exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G and the collection of rooted chains of subgroups of G (see [8]).

There is another equivalence relation on the set of fuzzy subgroups of a given group G in [9]. The number of fuzzy subgroups of a group G under the equivalence relation in [9] can be obtained from the number of fuzzy subgroups of G under the equivalence relation in [8] by using Proposition 2.1(for details, see [4]).

S.L. Ngcibi et al. [9] gave an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where m is a positive integer and $n = 1, 2, 3$. Their method could be used to find the numbers with $n = 4, 5, \ldots$ consecutively. But it did not give an explicit formula for that number for an arbitrarily given positive integer n. The first author [4] gave an explicit formula for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ of rank two with order p^{m+n}. I.K. Appiach et al. [2] found the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ of rank three where m is a positive integer, $n = 1$ and $\ell = 1$.

In this paper we are concerned with the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ of rank three with order $p^{m+n+\ell}$. We obtain a recurrence relation for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$(see Theorem 4.4). In order to show that using this recurrence relation, one can find explicit formulas for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ consecutively, we give explicit formulas for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ where $(n, \ell) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2)$.

This paper is organized as follows. In Section 2 we present some notations and results. In Section 3 we give some known results about a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ of rank two, which are necessary in subsequent sections. In Section 4 we obtain a recurrence relation for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$. In Sections 5 and 6, using the recurrence relation obtained in Section 4 we give explicit formulas for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ where $(n, \ell) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2)$.

2. Preliminaries

Let G be a group with a multiplicative binary operation and identity e. A *chain of subgroups* of G is a set of subgroups of G linearly ordered by set inclusion.

A chain of subgroups of G is called *rooted* (more precisely G-*rooted*) if it contains G. Otherwise, it is called *unrooted*.

Given a group G, let $\mathcal{C}(G)$, $\mathcal{D}(G)$ and $\mathcal{F}(G)$ be the collection of chains of subgroups of G, of unrooted chains of subgroups of G and of G-rooted chains of subgroups of G, respectively. Let $C(G) := |\mathcal{C}(G)|$, $D(G) := |\mathcal{D}(G)|$ and $F(G) := |\mathcal{F}(G)|$.

The following is useful for enumerating chains of subgroups of a given group.

Proposition 2.1. [4] *Let G be a finite group. Then $F(G) = D(G) + 1$ and $C(G) = F(G) + D(G) = 2F(G) - 1$.*

Given a group G, let $\mathcal{S}(G)$ and $\mathcal{T}(G)$ be the set of subgroups of G and the set of proper subgroups of G, respectively. Let $S(G) := |\mathcal{S}(G)|$ and $T(G) := |\mathcal{T}(G)|$. Then $S(G) = T(G) + 1$.

For undefined group theoretical terminologies we refer the reader to [6, 7]. For a general theory of solving a recurrence relation using a generating function we refer the reader to [1, 10].

3. The chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$

Throughout this section we assume that

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \left\langle a, b \mid a^{p^m} = e = b^{p^n}, bab^{-1} = a \right\rangle,$$

where m and n are non-negative integers and p is a prime number. Without loss of generality, we may assume that $m \geq n$. As a set

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{a^i b^j \mid i = 0, 1, \ldots, p^m - 1, j = 0, 1, \ldots, p^n - 1\}.$$

Let $a_{m,n} := |\mathcal{F}(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})|$, that is, the number of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

Lemma 3.1. [4] *Let m and n be non-negative integers such that $m \geq n$.*

(1) *If $m > n$, then*

$$a_{m,n} = 2a_{m-1,n} + 2pa_{m,n-1} - 2pa_{m-1,n-1}. \tag{1}$$

(2) *If $m = n$, then*

$$a_{m,m} = 2(1 + p)a_{m,m-1} - 2pa_{m-1,m-1}. \tag{2}$$

Proposition 3.2. [4] *Let $a_{m,n}$ be the number of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where m and n are positive integers such that $m \geq n$. Then the following holds.*

(1)

$$a_{m,1} = 2^m[mp+2].$$

(2)

$$a_{m,2} = 2^m\left[\frac{1}{2}p^2m^2 + \left(2p + \frac{3}{2}p^2\right)m - 2p^2 + 4 + 2p\right].$$

(3)

$$a_{m,3} = 2^m\left[\frac{1}{6}p^3m^3 + \left(p^2 + \frac{3}{2}p^3\right)m^2 + \left(4p + 5p^2 + \frac{1}{3}p^3\right)m + 8 - 8p^3 + 8p\right].$$

(4)

$$a_{m,4} = 2^m\left[\frac{1}{24}p^4m^4 + \left(\frac{3}{4}p^4 + \frac{1}{3}p^3\right)m^3 + \left(4p^3 + 2p^2 + \frac{59}{24}p^4\right)m^2 \right.$$
$$\left. + \left(\frac{23}{3}p^3 + 8p + 14p^2 - \frac{29}{4}p^4\right)m + 16 + 24p + 12p^2 - 12p^3 - 24p^4\right].$$

(5)

$$a_{m,5} = 2^m\left[\frac{1}{120}p^5m^5 + \left(\frac{1}{12}p^4 + \frac{1}{4}p^5\right)m^4 + \left(\frac{2}{3}p^3 + \frac{11}{6}p^4 + \frac{47}{24}p^5\right)m^3 \right.$$
$$+ \left(10p^3 + \frac{1}{4}p^5 + 4p^2 + \frac{119}{12}p^4\right)m^2 + \left(16p + \frac{1}{6}p^4 + \frac{100}{3}p^3 + 36p^2 - \frac{517}{15}p^5\right)m$$
$$\left. + 32 + 64p + 56p^2 - 64p^5 - 56p^4\right].$$

4. The chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$

Throughout this section we assume that

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell} = \left\langle a, b, c \mid a^{p^m} = b^{p^n} = c^{p^\ell} = e, [a,b] = [a,c] = [b,c] = e \right\rangle,$$

where m, n and ℓ are non-negative integers such that $m \geq n \geq \ell$ and p is a prime number.

Lemma 4.1. [5] *There exist* (p^2+p+1) *index* p^2 *subgroups of* $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ *containing the subgroup* $\langle a^p, b^p, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$ *as follows:*

(1) $\left\langle a^i b^j c, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$ *and* $j = 0, 1, \ldots, p-1$,
(2) $\left\langle a^p, b^k c, c^p\right\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $k = 1, 2, \ldots, p-1$,
(3) $\langle a^p, b^p, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$,
(4) $\left\langle a^i b, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$,
(5) $\langle a^p, b, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$,
(6) $\langle a, b^p, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$.

Lemma 4.2. [5] *The group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ has (p^2+p+1) index p subgroups as follows:*

(1) $\langle a, b, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$,
(2) $\left\langle a^i b, b^p, c\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$; $i = 1, 2, \ldots, p-1$,
(3) $\left\langle a^i b, b^j c, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$ *and* $j = 1, 2, \ldots, p-1$,
(4) $\langle a^p, b, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$,
(5) $\left\langle a^i c, b, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$,
(6) $\langle a, b^p, c\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$,
(7) $\left\langle a, b^i c, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$.

Lemma 4.3. [5] *If K is an index p^2 subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ and K contains the subgroup $\langle a^p, b^p, c^p\rangle$, then there exist $(p+1)$ index p subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing K.*

Let $a_{m,n,\ell} := F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell})$ be the number of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$.

Theorem 4.4. (1) *If $m > n > \ell$, then*

$$a_{m,n,\ell} = 2a_{m-1,n,\ell} + 2p(a_{m,n-1,\ell} - a_{m-1,n-1,\ell}) + 2p^2(a_{m,n,\ell-1} - a_{m-1,n,\ell-1}) - 2p^3(a_{m,n-1,\ell-1} - a_{m-1,n-1,\ell-1}). \quad (3)$$

(2) *If $m = n > \ell$, then*

$$a_{n,n,\ell} = (2p+2)a_{n,n-1,\ell} - 2pa_{n-1,n-1,\ell} + 2p^2 a_{n,n,\ell-1} - 2p^2 a_{n,n-1,\ell-1} - 2p^3 a_{n,n-1,\ell-1} + 2p^3 a_{n-1,n-1,\ell-1}. \quad (4)$$

(3) *If $m > n = \ell$, then*

$$a_{m,\ell,\ell} = 2a_{m-1,\ell,\ell} + (2p^2+2p)a_{m,\ell,\ell-1} - (2p^2+2p)a_{m-1,\ell,\ell-1} - 2p^3 a_{m,\ell-1,\ell-1} + 2p^3 a_{m-1,\ell-1,\ell-1}. \quad (5)$$

(4) *If $m = n = \ell$, then*

$$a_{\ell,\ell,\ell} = (2p^2+2p+2)a_{\ell,\ell,\ell-1} - (2p^3+2p^2+2p)a_{\ell,\ell-1,\ell-1} + 2p^3 a_{\ell-1,\ell-1,\ell-1}. \quad (6)$$

Proof. We only give the proof when $m > n > \ell$. The remaining can be proved similarly.

For notational brevity we set $b_{m,n,\ell} := 2a_{m,n,\ell} - 1$, which is equal to the number of chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$. By Lemma 4.2,

$$\mathcal{D}(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}) = \mathcal{C}(\langle a, b, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}})$$

$$\bigcup_{i=1}^{p-1}\mathcal{C}(\left\langle a^ib,b^p,c\right\rangle\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}\times\mathbb{Z}_{p^\ell})\bigcup_{1\le i,j\le p-1}\mathcal{C}(\left\langle a^ib,b^jc,c^p\right\rangle\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^{\ell-1}})$$
$$\bigcup\mathcal{C}(\langle a^p,b,c\rangle\cong\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell})\bigcup_{i=1}^{p-1}\mathcal{C}(\left\langle a^ic,b,c^p\right\rangle\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^{\ell-1}})$$
$$\bigcup\mathcal{C}(\langle a,b^p,c\rangle\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}\times\mathbb{Z}_{p^\ell})\bigcup_{i=1}^{p-1}\mathcal{C}(\left\langle a,b^ic,c^p\right\rangle\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^{\ell-1}}).$$

Using the inclusion-exclusion principle and Lemmas 4.1 and 4.3, we have

$$\begin{aligned}
a_{m,n,\ell}-1={}&p^2b_{m,n,\ell-1}+pb_{m,n-1,\ell}+b_{m-1,n,\ell}\\
&-\left[p^2\binom{p+1}{2}b_{m,n-1,\ell-1}+p\binom{p+1}{2}b_{m-1,n,\ell-1}+\binom{p+1}{2}b_{m-1,n-1,\ell}\right]\\
&+\Bigg[p^2\binom{p+1}{3}b_{m,n-1,\ell-1}+p\binom{p+1}{3}b_{m-1,n,\ell-1}+\binom{p+1}{3}b_{m-1,n-1,\ell}\\
&\qquad+\left[\binom{p^2+p+1}{3}-(p^2+p+1)\binom{p+1}{3}\right]b_{m-1,n-1,\ell-1}\Bigg]\\
&-\Bigg[p^2\binom{p+1}{4}b_{m,n-1,\ell-1}+p\binom{p+1}{4}b_{m-1,n,\ell-1}+\binom{p+1}{4}b_{m-1,n-1,\ell}\\
&\qquad+\left[\binom{p^2+p+1}{4}-(p^2+p+1)\binom{p+1}{4}\right]b_{m-1,n-1,\ell-1}\Bigg]\\
&+\cdots+(-1)^{p+2}\Bigg[p^2\binom{p+1}{p+1}b_{m,n-1,\ell-1}\\
&\qquad+p\binom{p+1}{p+1}b_{m-1,n,\ell-1}+\binom{p+1}{p+1}b_{m-1,n-1,\ell}\\
&\qquad+\left[\binom{p^2+p+1}{p+1}-(p^2+p+1)\binom{p+1}{p+1}\right]b_{m-1,n-1,\ell-1}\Bigg]\\
&+(-1)^{p+3}\binom{p^2+p+1}{p+2}b_{m-1,n-1,\ell-1}+(-1)^{p+4}\binom{p^2+p+1}{p+3}b_{m-1,n-1,\ell-1}\\
&+\cdots+(-1)^{p^2+p+2}\binom{p^2+p+1}{p^2+p+1}b_{m-1,n-1,\ell-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
a_{m,n,\ell}-1={}&p^2b_{m,n,\ell-1}+pb_{m,n-1,\ell}+b_{m-1,n,\ell}-p^3b_{m,n-1,\ell-1}\\
&-p^2b_{m-1,n,\ell-1}-pb_{m-1,n-1,\ell}+p^3b_{m-1,n-1,\ell-1},
\end{aligned}$$

and so

$$\begin{aligned}
a_{m,n,\ell}={}&2p^2a_{m,n,\ell-1}+2pa_{m,n-1,\ell}+2a_{m-1,n,\ell}-2p^3a_{m,n-1,\ell-1}\\
&-2p^2a_{m-1,n,\ell-1}-2pa_{m-1,n-1,\ell}+2p^3a_{m-1,n-1,\ell-1}.
\end{aligned}$$

$\square$

5. A general method

By Lemma 4.4 and inductive process, one can see that

$$a_{m,n,\ell} = 2a_{m-1,n,\ell} + 2^m f(m,n,\ell) \tag{7}$$

where $f(m,n,\ell) = a_0 + a_1 m + \cdots + a_{n+\ell-1} m^{n+\ell-1}$ for some $a_i \in \mathbb{R}$ with $a_{n+\ell-1} \neq 0$ (in this stage, we do not need the degree of $f(m,n,\ell)$). In fact, if $m > n > \ell$, then

$$\begin{aligned} 2^m f(m,n,\ell) = {} & 2p(a_{m,n-1,\ell} - a_{m-1,n-1,\ell}) + \\ & 2p^2(a_{m,n,\ell-1} - a_{m-1,n,\ell-1}) - 2p^3(a_{m,n-1,\ell-1} - a_{m-1,n-1,\ell-1}); \end{aligned} \tag{8}$$

if $m > n = \ell$, then

$$\begin{aligned} 2^m f(m,\ell,\ell) = {} & (2p^2+2p)a_{m,\ell,\ell-1} - (2p^2+2p)a_{m-1,\ell,\ell-1} \\ & - 2p^3 a_{m,\ell-1,\ell-1} + 2p^3 a_{m-1,\ell-1,\ell-1}. \end{aligned} \tag{9}$$

Now, we follows a general method of solving the linear non-homogenous recurrence relation (7). Let $a^{(p)}_{m,n,1} := m(b_0 + b_1 m + \cdots + b_{n+\ell-1} m^{n+\ell-1}) 2^m$ be a particular solution of Eq. (7). For a brevity, we set $g(m,n,\ell) := b_0 + b_1 m + \cdots + b_{n+\ell-1} m^{n+\ell-1}$. Now taking $a^{(p)}_{m,n,1}$ in Eq. (7), we have

$$mg(m,n,\ell)2^m = 2^m(m-1)g(m-1,n,\ell) + 2^m f(m,n,\ell),$$

and so

$$mg(m,n,\ell) = (m-1)g(m-1,n,\ell) + f(m,n,\ell).$$

Let $c_m := mg(m,n,\ell)$. Then $c_m - c_{m-1} = f(m,n,\ell)$. Since

$$\begin{aligned} c_1 - c_0 &= f(1,n,\ell), \\ c_2 - c_1 &= f(2,n,\ell), \\ &\vdots \\ c_m - c_{m-1} &= f(m,n,\ell), \end{aligned}$$

we have $c_m = \sum_{t=1}^m f(t,n,\ell)$, and so $g(m,n,\ell) = \frac{1}{m}\sum_{t=1}^m f(t,n,\ell)$. Thus

$$a^{(p)}_{m,n,\ell} = 2^m \sum_{t=1}^m f(t,n,\ell).$$

Hence the general solution of Eq. (7) is

$$a_{m,n,\ell} = A \cdot 2^m + 2^m \sum_{t=1}^m f(t,n,\ell).$$

Since $a_{n,n,\ell} = A\cdot 2^n + 2^n \sum_{t=1}^{n} f(t,n,\ell)$, we have $A = 2^{-n}a_{n,n,\ell} - \sum_{t=1}^{n} f(t,n,\ell)$, and so

$$\begin{aligned} a_{m,n,\ell} &= 2^{m-n}a_{n,n,\ell} + 2^m\left(\sum_{t=1}^{m} f(t,n,\ell) - \sum_{t=1}^{n} f(t,n,\ell)\right) \\ &= 2^{m-n}a_{n,n,\ell} + 2^m \sum_{t=n+1}^{m} f(t,n,\ell). \end{aligned} \tag{10}$$

By definition of $f(m,n+1,\ell)$, we know that

$$\begin{aligned} f(m,n+1,\ell) = \frac{1}{2^m}\big[&2p(a_{m,n,\ell} - a_{m-1,n,\ell}) + 2p^2(a_{m,n+1,\ell-1} - a_{m-1,n+1,\ell-1}) \\ &-2p^3(a_{m,n,\ell-1} - a_{m-1,n,\ell-1}\big]. \end{aligned}$$

Since we know by Eq. (10) that

$$\begin{aligned} a_{m,n,\ell} - a_{m-1,n,\ell} &= 2^{m-n-1}a_{n,n,\ell} + 2^{m-1}\left(\sum_{t=n+1}^{m} 2f(t,n,\ell) - \sum_{t=n+1}^{m-1} f(t,n,\ell)\right) \\ &= 2^{m-n-1}a_{n,n,\ell} + 2^{m-1}\left(\sum_{t=n+1}^{m} f(t,n,\ell) + f(m,n,\ell)\right), \end{aligned}$$

$$\begin{aligned} a_{m,n+1,\ell-1} - a_{m-1,n+1,\ell-1} &= 2^{m-n-2}a_{n+1,n+1,\ell-1} \\ &\quad + 2^{m-1}\left(\sum_{t=n+2}^{m} f(t,n+1,\ell-1) + f(m,n+1,\ell-1)\right) \end{aligned}$$

and

$$\begin{aligned} &a_{m,n,\ell-1} - a_{m-1,n,\ell-1} \\ &\quad = 2^{m-n-1}a_{n,n,\ell-1} + 2^{m-1}\left(\sum_{t=n+1}^{m} f(t,n,\ell-1) + f(m,n,\ell-1)\right), \end{aligned}$$

we have

$$\begin{aligned} f(m,n+1,\ell) &= p\left[2^{-n}a_{n,n,\ell} + \sum_{t=n+1}^{m} f(t,n,\ell) + f(m,n,\ell)\right] \\ &\quad + p^2\left[2^{-n-1}a_{n+1,n+1,\ell-1} + \sum_{t=n+2}^{m} f(t,n+1,\ell-1) + f(m,n+1,\ell-1)\right] \\ &\quad - p^3\left[2^{-n}a_{n,n,\ell-1} + \sum_{t=n+1}^{m} f(t,n,\ell-1) + f(m,n,\ell-1)\right]. \end{aligned} \tag{11}$$

We know by Eq. (4) that

$$
\begin{aligned}
a_{n+1,n+1,\ell} &= 2a_{n+1,n,\ell} + 2p(a_{n+1,n,\ell} - a_{n,n,\ell}) \\
&\quad + 2p^2(a_{n+1,n+1,\ell-1} - a_{n+1,n,\ell-1}) - 2p^3(a_{n+1,n,\ell-1} - a_{n,n,\ell-1}). \quad (12)
\end{aligned}
$$

Since

$$
\begin{aligned}
a_{n+1,n,\ell} &= 2a_{n,n,\ell} + 2^{n+1}f(n+1,n,\ell) \text{ and} \\
a_{n+1,n,\ell-1} &= 2a_{n,n,\ell-1} + 2^{n+1}f(n+1,n,\ell-1) \text{ by Eq. (7),}
\end{aligned}
$$

we have

$$
\begin{aligned}
2a_{n+1,n,\ell} + 2p(a_{n+1,n,\ell} - a_{n,n,\ell}) &= (2+2p)a_{n+1,n,\ell} - 2pa_{n,n,\ell} \\
&= (2+2p)(2a_{n,n,\ell} + 2^{n+1}f(n+1,n,\ell)) - 2pa_{n,n,\ell} \\
&= 2(2+p)a_{n,n,\ell} + 2^{n+2}(1+p)f(n+1,n,\ell)
\end{aligned}
$$

and

$$
\begin{aligned}
&2p^2(a_{n+1,n+1,\ell-1} - a_{n+1,n,\ell-1}) - 2p^3(a_{n+1,n,\ell-1} - a_{n,n,\ell-1}) \\
&= 2p^2a_{n+1,n+1,\ell-1} + 2p^3a_{n,n,\ell-1} - 2(p^2+p^3)a_{n+1,n,\ell-1} \\
&= 2p^2a_{n+1,n+1,\ell-1} + 2p^3a_{n,n,\ell-1} - 2(p^2+p^3)(2a_{n,n,\ell-1} + 2^{n+1}f(n+1,n,\ell-1)) \\
&= 2p^2a_{n+1,n+1,\ell-1} - (4p^2+2p^3)a_{n,n,\ell-1} - 2^{n+2}(p^2+p^3)f(n+1,n,\ell-1)).
\end{aligned}
$$

Taking these to Eq. (12), we have

$$
\begin{aligned}
a_{n+1,n+1,\ell} &= 2(2+p)a_{n,n,\ell} + 2^{n+2}(1+p)f(n+1,n,\ell) \\
&+ 2p^2a_{n+1,n+1,\ell-1} - (4p^2+2p^3)a_{n,n,\ell-1} - 2^{n+2}(p^2+p^3)f(n+1,n,\ell-1). \quad (13)
\end{aligned}
$$

In the following section, by using Eqs. (6), (8), (9), (10), (11) and (13), we give an explicit formulas for $a_{m,n,\ell}$ where $n = 1,2,3,4,5$ and $\ell = 1,2$.

6. Computations

For brevity, we write $a_{m,n,0}$ by $a_{m,n}$ and $f(m,n,0)$ by $f(m,n)$, respectively.

Eq. (8) with $\ell = 0$ gives that

$$2^m f(m,n,0) = 2p(a_{m,n-1,0} - a_{m-1,n-1,0}).$$

That is,

$$2^m f(m,n) = 2p(a_{m,n-1} - a_{m-1,n-1}). \quad (14)$$

Using Proposition 3.2 and Eq. (14), one can obtain the following.

$$\begin{aligned} f(m,1) =& p,\\ f(m,2) =& p(2+p)+p^2m,\\ f(m,3) =& 4p+4p^2-p^3+\left(2p^2+\frac{5}{2}p^3\right)m+\frac{1}{2}p^3m^2,\\ f(m,4) =& 8p+12p^2+4p^3-9p^4+\left(4p^2+7p^3+\frac{17}{6}p^4\right)m+\left(p^3+2p^4\right)m^2+\\ & \frac{1}{6}p^4m^3,\\ f(m,5) =& 16p+32p^2+24p^3-8p^4-33p^5+\left(8p^2+\frac{44}{3}p^4+18p^3-\frac{53}{12}p^5\right)m+\\ & \left(2p^3+5p^4+\frac{107}{24}p^5\right)m^2+\left(\frac{1}{3}p^4+\frac{11}{12}p^5\right)m^3+\frac{1}{24}p^5m^4. \end{aligned} \tag{15}$$

1. $a_{m,n,1}$ *where* $n=1,2,3,4,5$

Eq. (9) with $\ell=1$ gives

$$\begin{aligned} &2^m f(m,1,1)\\ &=(2p^2+2p)a_{m,1}-(2p^2+2p)a_{m-1,1}-2p^3a_{m,0}+2p^3a_{m-1,0}\\ &=(2p^2+2p)2^m(mp+2)-(2p^2+2p)2^{m-1}((m-1)p+2)-2p^32^m+2p^32^{m-1}\\ &=2^m(mp^3+mp^2+3p^2+2p). \end{aligned}$$

Thus

$$f(m,1,1)=2p+3p^2+(p^2+p^3)m \quad \text{by Eq. (9).} \tag{16}$$

Eq. (6) with $\ell=1$ gives

$$\begin{aligned} a_{1,1,1} &= (2p^2+2p+2)a_{1,1,0}-(2p^3+2p^2+2p)a_{1,0,0}+2p^3a_{0,0,0}\\ &=(2p^2+2p+2)(2p+4)-2(2p^3+2p^2+2p)+2p^3\\ &=2p^3+8p^2+8p+8. \end{aligned}$$

Thus

$$a_{1,1,1}=2p^3+8p^2+8p+8.$$

$$\begin{aligned} &a_{m,1,1}\\ &=2^{m-1}a_{1,1,1}+2^m\sum_{t=2}^{m}f(t,1,1)\\ &=2^m\left[p^3+4p^2+4p+4+(2p+3p^2)(m-1)\ +(p^2+p^3)\left(\frac{m(m+1)}{2}-\frac{1\cdot 2}{2}\right)\right]\\ &=2^m\left[\frac{p^3+p^2}{2}m^2+\frac{p^3+7p^2+4p}{2}m+2p+4\right]. \end{aligned}$$

Eq. (11) with $(n, \ell) = (1, 1)$ gives

$$\begin{aligned}
&f(m,2,1)\\
&= p\left[2^{-1}a_{1,1,1} + \sum_{t=2}^{m} f(t,1,1) + f(m,1,1)\right] + p^2\left[2^{-2}a_{2,2} + \sum_{t=3}^{m} f(t,2) + f(m,2)\right]\\
&\quad - p^3\left[2^{-1}a_{1,1} + \sum_{t=2}^{m} f(t,1) + f(m,1)\right]\\
&= \left(\frac{1}{2}p^3 + p^4\right)m^2 + \left(2p^2 + \frac{13}{2}p^3 + 3p^4\right)m + 8p^2 + 4p + 5p^3 - 2p^4.
\end{aligned}$$

Eq. (13) with $(n, \ell) = (1, 1)$ gives

$$\begin{aligned}
a_{2,2,1} &= 2(2+p)a_{1,1,1} + 2^3(1+p)f(2,1,1) + 2p^2a_{2,2} - (4p^2 + 2p^3)a_{1,1}\\
&\quad - 2^3(p^2 + p^3)f(2,1)\\
&= 32p^4 + 104p^3 + 120p^2 + 64p + 32.
\end{aligned}$$

Eq. (10) with $(n, \ell) = (2, 1)$ gives

$$\begin{aligned}
a_{m,2,1} &= 2^{m-2}a_{2,2,1} + 2^m\left(\sum_{t=1}^{m} f(t,2,1) - \sum_{t=1}^{2} f(t,2,1)\right)\\
&= 2^{m-2}a_{2,2,1} + 2^m\sum_{t=3}^{m} f(t,2,1)\\
&= 2^m\left[\left(\frac{1}{6}p^3 + \frac{1}{3}p^4\right)m^3 + \left(\frac{7}{2}p^3 + 2p^4 + p^2\right)m^2\right.\\
&\quad \left. + \left(9p^2 + 4p - \frac{1}{3}p^4 + \frac{25}{3}p^3\right)m - 2p^4 - 6p^3 + 8p^2 + 8p + 8\right].
\end{aligned}$$

$$a_{3,3} = 88p^3 + 192p^2 + 160p + 64.$$

Eq. (11) with $(n, \ell) = (2, 1)$ gives

$$\begin{aligned}
&f(m,3,1)\\
&= p\left[2^{-2}a_{2,2,1} + \sum_{t=3}^{m} f(t,2,1) + f(m,2,1)\right]\\
&\quad + p^2\left[2^{-3}a_{3,3} + \sum_{t=4}^{m} f(t,3) + f(m,3)\right] - p^3\left[2^{-2}a_{2,2} + \sum_{t=3}^{m} f(t,2) + f(m,2)\right]\\
&= \left(\frac{1}{2}p^5 + \frac{1}{6}p^4\right)m^3 + \left(5p^4 + p^3 + \frac{9}{2}p^5\right)m^2\\
&\quad + \left(4p^2 + \frac{119}{6}p^4 + 15p^3 + 3p^5\right)m + 20p^2 + 24p^3 + 8p - p^4 - 12p^5.
\end{aligned}$$

Eq. (13) with $(n, \ell) = (2, 1)$ gives

$$\begin{aligned} a_{3,3,1} &= 2(2+p)a_{2,2,1} + 2^4(1+p)f(3,2,1) + 2p^2 a_{3,3} - (4p^2+2p^3)a_{2,2} \\ &\quad - 2^4(p^2+p^3)f(3,2) \\ &= 408p^5 + 1248p^4 + 1504p^3 + 960p^2 + 384p + 128. \end{aligned}$$

Eq. (10) with $(n, \ell) = (3, 1)$ gives

$$\begin{aligned} a_{m,3,1} &= 2^{m-3}a_{3,3,1} + 2^m \sum_{t=4}^{m} f(t,3,1) \\ &= 2^m \left[\left(\frac{1}{8}p^5 + \frac{1}{24}p^4 \right) m^4 + \left(\frac{1}{3}p^3 + \frac{7}{4}p^4 + \frac{7}{4}p^5 \right) m^3 \right. \\ &\quad + \left(8p^3 + \frac{31}{8}p^5 + \frac{299}{24}p^4 + 2p^2 \right) m^2 + \left(22p^2 + \frac{95}{3}p^3 + 8p + \frac{39}{4}p^4 - \frac{39}{4}p^5 \right) m \\ &\quad \left. -36p^4 + 12p^3 + 36p^2 + 24p + 16 - 12p^5 \right]. \end{aligned}$$

$$a_{4,4} = 2608\,p^4 + 4736\,p^3 + 3136\,p^2 + 896\,p + 256.$$

Eq. (11) with $(n, \ell) = (3, 1)$ gives

$$\begin{aligned} & f(m,4,1) \\ &= p \left[2^{-3}a_{3,3,1} + \sum_{t=4}^{m} f(t,3,1) + f(m,3,1) \right] \\ &\quad + p^2 \left[2^{-4}a_{4,4} + \sum_{t=5}^{m} f(t,4) + f(m,4) \right] - p^3 \left[2^{-3}a_{3,3} + \sum_{t=4}^{m} f(t,3) + f(m,3) \right] \\ &= \left(\frac{1}{24}p^5 + \frac{1}{6}p^6 \right) m^4 + \left(\frac{1}{3}p^4 + \frac{9}{4}p^5 + 3p^6 \right) m^3 \\ &\quad + \left(\frac{515}{24}p^5 + 2p^3 + \frac{65}{6}p^6 + 11p^4 \right) m^2 + \left(8p^2 + \frac{182}{3}p^4 - 14p^6 + 34p^3 + \frac{149}{4}p^5 \right) m \\ &\quad + 48p^2 + 80p^3 + 70p^6 + 16p + 144p^4 + 143p^5. \end{aligned}$$

Eq. (13) with $(n, \ell) = (3, 1)$ gives

$$\begin{aligned} a_{4,4,1} &= 2(2+p)a_{3,3,1} + 2^5(1+p)f(4,3,1) + 2p^2 a_{4,4} - (4p^2+2p^3)a_{3,3} \\ &\quad - 2^5(p^2+p^3)f(4,3) \\ &= 8640p^6 + 20672p^5 + 21280p^4 + 13184p^3 + 6272p^2 + 2048p + 512. \end{aligned}$$

Eq. (10) with $(n, \ell) = (4, 1)$ gives

$$\begin{aligned}
a_{m,4,1} &= 2^{m-4} a_{4,4,1} + 2^m \sum_{t=5}^{m} f(t,4,1) \\
&= 2^m \left[\left(\frac{1}{30} p^6 + \frac{1}{120} p^5 \right) m^5 + \left(\frac{1}{12} p^4 + \frac{7}{12} p^5 + \frac{5}{6} p^6 \right) m^4 \right. \\
&\quad + \left(\frac{31}{6} p^6 + \frac{2}{3} p^3 + \frac{23}{6} p^4 + \frac{199}{24} p^5 \right) m^3 \\
&\quad + \left(4p^2 + \frac{359}{12} p^5 + 18p^3 - \frac{5}{6} p^6 + \frac{431}{12} p^4 \right) m^2 \\
&\quad + \left(\frac{1057}{6} p^4 + 52p^2 + 16p + \frac{826}{5} p^5 + \frac{292}{3} p^3 + \frac{324}{5} p^6 \right) m \\
&\quad \left. + 32 + 120p^2 + 104p^3 - 216p^4 + 64p - 536p^5 - 284p^6 \right].
\end{aligned}$$

$$a_{5,5} = 6304p^5 + 15168p^4 + 16000p^3 + 10752p^2 + 4608p + 1024.$$

Eq. (11) with $(n, \ell) = (4, 1)$ gives

$$\begin{aligned}
&f(m,5,1) \\
&= p \left[2^{-4} a_{4,4,1} + \sum_{t=5}^{m} f(t,4,1) + f(m,4,1) \right] \\
&\quad + p^2 \left[2^{-5} a_{5,5} + \sum_{t=6}^{m} f(t,5) + f(m,5) \right] - p^3 \left[2^{-4} a_{4,4} + \sum_{t=5}^{m} f(t,4) + f(m,4) \right] \\
&= \left(\frac{1}{120} p^6 + \frac{1}{24} p^7 \right) m^5 + \left(\frac{1}{12} p^5 + \frac{5}{4} p^7 + \frac{17}{24} p^6 \right) m^4 \\
&\quad + \left(\frac{2}{3} p^4 + \frac{81}{8} p^7 + \frac{99}{8} p^6 + \frac{29}{6} p^5 \right) m^3 \\
&\quad + \left(4p^3 + \frac{683}{12} p^5 + \frac{1471}{24} p^6 + \frac{41}{4} p^7 + 24p^4 \right) m^2 \\
&\quad + \left(\frac{502}{3} p^4 + \frac{12157}{60} p^6 + 76p^3 + 16p^2 + \frac{1621}{6} p^5 + \frac{49}{3} p^7 \right) m \\
&\quad + 112p^2 + 232p^3 - 641p^6 + 32p + 240p^4 - 168p^5 - 396p^7.
\end{aligned}$$

Eq. (13) with $(n, \ell) = (4, 1)$ gives

$$\begin{aligned}
&a_{5,5,1} \\
&= 2(2+p) a_{4,4,1} + 2^6 (1+p) f(5,4,1) + 2p^2 a_{5,5} - (4p^2 + 2p^3) a_{4,4} - 2^6 (p^2 + p^3) f(5,4) \\
&= 2048 + 10240p + 36864p^2 + 94720p^3 + 184192p^4 + 249856p^5 + 200448p^6 + 67808p^7.
\end{aligned}$$

Eq. (10) with $(n, \ell) = (5, 1)$ gives

$$\begin{aligned}
a_{m,5,1} &= 2^{m-5} a_{5,5,1} + 2^m \sum_{t=6}^{m} f(t,5,1) \\
&= 2^m \left[\left(\frac{1}{720}p^6 + \frac{1}{144}p^7 \right) m^6 + \left(\frac{1}{60}p^5 + \frac{7}{48}p^6 + \frac{13}{48}p^7 \right) m^5 \right. \\
&\quad + \left(\frac{457}{144}p^7 + \frac{497}{144}p^6 + \frac{5}{4}p^5 + \frac{1}{6}p^4 \right) m^4 \\
&\quad + \left(\frac{4}{3}p^3 + \frac{25}{3}p^4 + \frac{1289}{48}p^6 + \frac{257}{12}p^5 + \frac{427}{48}p^7 \right) m^3 \\
&\quad + \left(40p^3 + \frac{48617}{360}p^6 + \frac{575}{6}p^4 + \frac{1139}{72}p^7 + 8p^2 + \frac{659}{4}p^5 \right) m^2 \\
&\quad + \left(120p^2 + 32p + \frac{983}{3}p^4 - \frac{703}{30}p^5 + \frac{812}{3}p^3 - \frac{1059}{2}p^6 - \frac{2317}{6}p^7 \right) m \\
&\quad \left. + 64 + 352p^2 + 440p^3 + 576p^4 + 160p + 296p^5 - 396p^7 - 456p^6 \right].
\end{aligned}$$

2. $a_{m,n,2}$ *where* $n = 1, 2, 3, 4, 5$ Eq. (6) with $\ell = 2$ gives

$$\begin{aligned}
a_{2,2,2} &= (2p^2 + 2p + 2) a_{2,2,1} - (2p^3 + 2p^2 + 2p) a_{2,1,1} + 2p^3 a_{1,1,1} \\
&= 44p^6 + 192p^5 + 384p^4 + 440p^3 + 352p^2 + 160p + 64.
\end{aligned}$$

Eq. (9) with $\ell = 2$ gives

$$\begin{aligned}
2^m f(m,2,2) &= (2p^2 + 2p) a_{m,2,1} - (2p^2 + 2p) a_{m-1,2,1} - 2p^3 a_{m,1,1} + 2p^3 a_{m-1,1,1} \\
&= 2^m \left[\left(\frac{1}{3}p^6 + \frac{1}{2}p^5 + \frac{1}{6}p^4 \right) m^3 + \left(p^3 + 5p^4 + \frac{13}{2}p^5 + \frac{5}{2}p^6 \right) m^2 \right. \\
&\quad \left. + \left(4p^2 + 13p^5 + \frac{143}{6}p^4 + 15p^3 + \frac{7}{6}p^6 \right) m + 20p^2 + 8p - 8p^5 + 11p^4 - 4p^6 + 24p^3 \right].
\end{aligned}$$

Thus

$$\begin{aligned}
f(m,2,2) &= \left(\frac{1}{3}p^6 + \frac{1}{2}p^5 + \frac{1}{6}p^4 \right) m^3 + \left(p^3 + 5p^4 + \frac{13}{2}p^5 + \frac{5}{2}p^6 \right) m^2 \\
&\quad + \left(4p^2 + 13p^5 + \frac{143}{6}p^4 + 15p^3 + \frac{7}{6}p^6 \right) m + 20p^2 + 8p - 8p^5 + 11p^4 - 4p^6 + 24p^3.
\end{aligned}$$

By Eq. (10) with $(n, \ell) = (2, 2)$, we have

$$\begin{aligned}
a_{m,2,2} &= 2^{m-2} a_{2,2,2} + 2^m \sum_{t=3}^{m} f(t,2,2) \\
&= 2^m \left[\left(\frac{1}{12}p^6 + \frac{1}{24}p^4 + \frac{1}{8}p^5 \right) m^4 + \left(p^6 + \frac{7}{4}p^4 + \frac{1}{3}p^3 + \frac{29}{12}p^5 \right) m^3 \right. \\
&\quad + \left(\frac{23}{12}p^6 + 8p^3 + 2p^2 + \frac{79}{8}p^5 + \frac{347}{24}p^4 \right) m^2 \\
&\quad + \left(22p^2 - \frac{5}{12}p^5 - 3p^6 + 8p + \frac{95}{3}p^3 + \frac{95}{4}p^4 \right) m \\
&\quad \left. + 16 - 12p^5 + 36p^2 + 12p^3 + 24p - 24p^4 \right].
\end{aligned}$$

Eq. (11) with $(n, \ell) = (2, 2)$ gives

$$\begin{aligned}
&f(m,3,2)\\
&= p\left[2^{-2}a_{2,2,2} + \sum_{t=3}^{m} f(t,2,2) + f(m,2,2)\right]\\
&+ p^2\left[2^{-3}a_{3,3,1} + \sum_{t=4}^{m} f(t,3,1) + f(m,3,1)\right] - p^3\left[2^{-2}a_{2,2,1} + \sum_{t=3}^{m} f(t,2,1) + f(m,2,1)\right]\\
&= \left(\frac{5}{24}p^7 + \frac{1}{6}p^6 + \frac{1}{24}p^5\right)m^4 + \left(\frac{13}{4}p^7 + \frac{14}{3}p^6 + \frac{9}{4}p^5 + \frac{1}{3}p^4\right)m^3\\
&+ \left(11p^4 + \frac{659}{24}p^5 + 2p^3 + \frac{179}{6}p^6 + \frac{235}{24}p^7\right)m^2\\
&+ \left(8p^2 + 34p^3 + \frac{333}{4}p^5 + \frac{206}{3}p^4 - \frac{45}{4}p^7 + \frac{82}{3}p^6\right)m\\
&+ 48p^2 + 80p^3 + 16p - 24p^7 + 7p^5 + 80p^4 - 56p^6.
\end{aligned}$$

Eq. (13) with $(n, \ell) = (2, 2)$ gives

$$\begin{aligned}
&a_{3,3,2}\\
&= 2(2+p)a_{2,2,2} + 2^4(1+p)f(3,2,2) + 2p^2a_{3,3,1} - (4p^2+2p^3)a_{2,2,1} - 2^4(p^2+p^3)f(3,2,1)\\
&= 256 + 896p + 2496p^2 + 4608p^3 + 6800p^4 + 6960p^5 + 4144p^6 + 1080p^7.
\end{aligned}$$

Eq. (10) with $(n, \ell) = (3, 2)$ gives

$$\begin{aligned}
a_{m,3,2} &= 2^{m-3}a_{3,3,2} + 2^m \sum_{t=4}^{m} f(t,3,2)\\
&= 2^m\left[\left(\frac{1}{120}p^5 + \frac{1}{24}p^7 + \frac{1}{30}p^6\right)m^5 + \left(\frac{11}{12}p^7 + \frac{5}{4}p^6 + \frac{7}{12}p^5 + \frac{1}{12}p^4\right)m^4\right.\\
&\quad + \left(\frac{247}{24}p^5 + \frac{23}{6}p^4 + \frac{119}{24}p^7 + \frac{37}{3}p^6 + \frac{2}{3}p^3\right)m^3\\
&\quad + \left(\frac{479}{12}p^4 + \frac{671}{12}p^5 + 18p^3 + \frac{119}{4}p^6 + \frac{1}{12}p^7 + 4p^2\right)m^2\\
&\quad + \left(16p + \frac{292}{3}p^3 + \frac{697}{6}p^4 + 52p^2 + \frac{266}{5}p^5 - \frac{1121}{30}p^6 - 28p^7\right)m\\
&\quad \left. +32 + 120p^2 + 104p^3 - 80p^6 + 64p - 120p^5 + 32p^4\right].
\end{aligned}$$

Eq. (11) with $(n, \ell) = (3, 2)$ gives

$$\begin{aligned}
&f(m,4,2)\\
&= p\left[2^{-3}a_{3,3,2} + \sum_{t=4}^{m} f(t,3,2) + f(m,3,2)\right]\\
&+ p^2\left[2^{-4}a_{4,4,1} + \sum_{t=5}^{m} f(t,4,1) + f(m,4,1)\right] - p^3\left[2^{-3}a_{3,3,1} + \sum_{t=4}^{m} f(t,3,1) + f(m,3,1)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{24}p^7 + \frac{3}{40}p^8 + \frac{1}{120}p^6\right)m^5 + \left(2p^7 + \frac{17}{24}p^6 + \frac{1}{12}p^5 + 2p^8\right)m^4 \\
&+ \left(\frac{205}{8}p^7 + \frac{131}{8}p^6 + \frac{29}{6}p^5 + \frac{113}{8}p^8 + \frac{2}{3}p^4\right)m^3 \\
&+ \left(4p^3 + \frac{2911}{24}p^6 + \frac{827}{12}p^5 + \frac{23}{2}p^8 + \frac{187}{2}p^7 + 24p^4\right)m^2 \\
&+ \left(76p^3 + 16p^2 + \frac{19597}{60}p^6 + \frac{1741}{6}p^5 + \frac{977}{6}p^7 + \frac{183}{10}p^8 + \frac{550}{3}p^4\right)m \\
&+ 112p^2 + 232p^3 + 240p^5 + 32p + 320p^4 - 214p^8 - 221p^6 - 492p^7.
\end{aligned}
$$

Eq. (13) with $(n,\ell)=(3,2)$ gives

$$
\begin{aligned}
a_{4,4,2} &= 2(2+p)a_{3,3,2} + 2^5(1+p)f(4,3,2) + 2p^2a_{4,4,1} - (4p^2+2p^3)a_{3,3,1} \\
&\quad - 2^5(p^2+p^3)f(4,3,1) \\
&= 1024 + 4608p + 15360p^2 + 35968p^3 + 68544p^4 + 103072p^5 + 114240p^6 \\
&\quad + 80160p^7 + 26464p^8.
\end{aligned}
$$

Eq. (10) with $(n,\ell)=(4,2)$ gives

$$
\begin{aligned}
a_{m,4,2} &= 2^{m-4}a_{4,4,2} + 2^m\sum_{t=5}^{m} f(t,4,2) \\
&= 2^m\left[\left(\frac{1}{80}p^8 + \frac{1}{720}p^6 + \frac{1}{144}p^7\right)m^6 + \left(\frac{1}{60}p^5 + \frac{101}{240}p^7 + \frac{7}{48}p^6 + \frac{7}{16}p^8\right)m^5\right. \\
&+ \left(\frac{1069}{144}p^7 + \frac{73}{16}p^8 + \frac{641}{144}p^6 + \frac{5}{4}p^5 + \frac{1}{6}p^4\right)m^4 \\
&+ \left(\frac{2143}{48}p^7 + \frac{185}{16}p^8 + \frac{4}{3}p^3 + \frac{25}{3}p^4 + \frac{2345}{48}p^6 + \frac{305}{12}p^5\right)m^3 \\
&+ \left(\frac{623}{6}p^4 + 40p^3 + \frac{723}{4}p^5 + 8p^2 + \frac{82097}{360}p^6 + \frac{737}{40}p^8 + \frac{9689}{72}p^7\right)m^2 \\
&+ \left(32p + \frac{812}{3}p^3 - 203p^8 + \frac{1247}{3}p^4 + \frac{11897}{30}p^5 - \frac{5926}{15}p^7 + 120p^2 - \frac{75}{2}p^6\right)m \\
&\left. +64 + 352p^2 + 440p^3 - 780p^7 + 160p - 236p^8 + 384p^4 - 780p^6\right].
\end{aligned}
$$

Eq. (11) with $(n,\ell)=(4,2)$ gives

$$
\begin{aligned}
&f(m,5,2) \\
&= p\left[2^{-4}a_{4,4,2} + \sum_{t=5}^{m} f(t,4,2) + f(m,4,2)\right] \\
&+ p^2\left[2^{-5}a_{5,5,1} + \sum_{t=6}^{m} f(t,5,1) + f(m,5,1)\right] - p^3\left[2^{-4}a_{4,4,1} + \sum_{t=5}^{m} f(t,4,1) + f(m,4,1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{720}p^7 + \frac{1}{120}p^8 + \frac{7}{360}p^9\right) m^6 \\
&+ \left(\frac{1}{60}p^6 + \frac{41}{240}p^7 + \frac{73}{120}p^8 + \frac{19}{24}p^9\right) m^5 \\
&+ \left(\frac{923}{144}p^7 + \frac{3}{2}p^6 + \frac{1}{6}p^5 + \frac{719}{72}p^9 + \frac{311}{24}p^8\right) m^4 \\
&+ \left(\frac{1397}{16}p^7 + \frac{2375}{24}p^8 + \frac{463}{12}p^6 + \frac{31}{3}p^5 + \frac{4}{3}p^4 + \frac{877}{24}p^9\right) m^3 \\
&+ \left(\frac{699}{2}p^6 + \frac{8279}{180}p^9 + 52p^4 + \frac{23584}{45}p^7 + 8p^3 + \frac{1007}{6}p^5 + \frac{11191}{30}p^8\right) m^2 \\
&+ \left(\frac{17941}{60}p^7 + \frac{1400}{3}p^4 - \frac{22847}{30}p^8 + \frac{5252}{5}p^6 + 168p^3 + \frac{2657}{3}p^5 + 32p^2 - \frac{1816}{3}p^9\right) m \\
&+ 256p^2 + 624p^3 + 1208p^5 + 64p - 1976p^8 - 801p^7 + 1056p^4 + 872p^6 - 1028p^9.
\end{aligned}
$$

Eq. (13) with $(n, \ell) = (4, 2)$ gives

$$
\begin{aligned}
&a_{5,5,2} \\
&= 2(2+p)a_{4,4,2} + 2^6(1+p)f(5,4,2) + 2p^2 a_{5,5,1} - (4p^2 + 2p^3)a_{4,4,1} - 2^6(p^2+p^3)f(5,4,1) \\
&= 4096 + 22528p + 87040p^2 + 242688p^3 + 552448p^4 + 1032064p^5 \\
&\quad + 1557824p^6 + 1708608p^7 + 1150208p^8 + 341824p^9.
\end{aligned}
$$

Eq. (10) with $(n, \ell) = (5, 2)$ gives

$$
\begin{aligned}
&a_{m,5,2} = 2^{m-5} a_{5,5,2} + 2^m \sum_{t=6}^{m} f(t,5,2) \\
&= 2^m \Bigg[\left(\frac{1}{5040}p^7 + \frac{1}{840}p^8 + \frac{1}{360}p^9\right) m^7 + \left(\frac{17}{120}p^9 + \frac{19}{180}p^8 + \frac{1}{360}p^6 + \frac{7}{240}p^7\right) m^6 \\
&+ \left(\frac{1}{30}p^5 + \frac{173}{72}p^9 + \frac{197}{144}p^7 + \frac{37}{120}p^6 + \frac{29}{10}p^8\right) m^5 \\
&+ \left(\frac{749}{72}p^6 + \frac{1205}{48}p^7 + \frac{347}{24}p^9 + \frac{1133}{36}p^8 + \frac{1}{3}p^4 + \frac{8}{3}p^5\right) m^4 \\
&+ \left(\frac{3271}{24}p^6 + \frac{9922}{45}p^7 + \frac{8}{3}p^3 + \frac{21377}{120}p^8 + 18p^4 + \frac{6647}{180}p^9 + \frac{367}{6}p^5\right) m^3 \\
&+ \left(16p^2 + \frac{127727}{180}p^6 + \frac{779}{3}p^4 + \frac{13001}{30}p^7 - \frac{1353}{5}p^9 + \frac{1588}{3}p^5 - \frac{7631}{45}p^8 + 88p^3\right) m^2 \\
&+ \left(1298p^4 + 64p + 272p^2 + \frac{8394}{5}p^5 - \frac{160653}{70}p^8 - \frac{7901}{14}p^7 - \frac{3970}{3}p^9 + \frac{7277}{5}p^6 + \frac{2128}{3}p^3\right) m \\
&+128 + 960p^2 + 1504p^3 + 472p^9 + 384p - 2616p^7 + 1824p^4 - 1084p^8 - 880p^6 + 1208p^5 \Bigg].
\end{aligned}
$$

7. Conclusion

We have found a recurrence relation for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$. Using this recurrence relation, we gave some explicit formulas for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ where m is a positive integer, and $(n, \ell) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2)$.

Starting with small numbers n and ℓ, one get explicit formulas sequentially by using the recurrence relation founded in Section 4. But our result does not provide an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ where m, n and ℓ are arbitrary positive numbers. We leave this as an open question.

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