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KUMMER-TYPE CONGRUENCES FOR THE HIGHER ORDER EULER NUMBERS AND POLYNOMIALS†

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ABSTRACT. In this paper, by using the multiple fermionic p-adic integrals, we obtain Kummer-type congruences for the higher order Euler numbers and polynomials.

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1. Introduction

Euler numbers, denoted by E_m for $m \geq 0$, count the number of odd alternating permutations of a set with an even number of elements. They are related to the Bernoulli numbers. The odd-indexed Euler numbers are all zero since its generating function is even (see [1, 2, 3, 7, 29]). The Euler numbers E_m satisfy the following recurrence relation (cf. [29, (1.2)])

$$E_0 = 1, \quad (E+1)^m + (E-1)^m = 0, \quad m \geq 1. \tag{1}$$

From this, by the induction we can also conclude that the odd-indexed Euler numbers are all zero and all the Euler numbers $E_0, E_2, \ldots$ are integers.

Let ℓ be a positive integer. Recently, Liu [18] introduced the higher order Euler numbers and gave some applications related to them. It is known [17, 18] that the higher order Euler numbers are defined by the following generating function

$$e^{E^{(\ell)}t} \equiv \sum_{m=0}^{\infty} \frac{(E^{(\ell)}t)^m}{m!} \equiv \sum_{m=0}^{\infty} E_m^{(\ell)} \frac{t^m}{m!} = \left(\frac{2}{e^t + e^{-t}}\right)^{\ell}, \tag{2}$$

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where the symbol $\equiv$ is used to denote symbolic or umbral equivalences understand as $(E^{(\ell)})^m \equiv E_m^{(\ell)}$. From the multinomial theorem, we have

$$\sum_{m=0}^{\infty} E_m^{(\ell)} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{\substack{j_1+\cdots+j_\ell=m \\ j_1,\ldots,j_\ell \geq 0}} \binom{m}{j_1,\ldots,j_\ell} E_{j_1} \cdots E_{j_\ell} \right) \frac{t^m}{m!}. \tag{3}$$

By (3), we see that the higher order Euler numbers are linked with the ordinary Euler numbers by the following identity

$$E_m^{(\ell)} = \sum_{\substack{j_1+\cdots+j_\ell=m \\ j_1,\ldots,j_\ell \geq 0}} \binom{m}{j_1,\ldots,j_\ell} E_{j_1} \cdots E_{j_\ell}, \quad m \geq 0. \tag{4}$$

It is seen from (1) and (4) that the higher order Euler numbers $E_m^{(\ell)}$ are integers. These numbers satisfy the following recurrence formula

$$\sum_{j=0}^{\ell} \binom{\ell}{j} (E^{(\ell)} + 2j - \ell)^m = \begin{cases} 2^\ell, & m = 0, \\ 0, & m \geq 1, \end{cases} \tag{5}$$

in which we understand that the expression on the left is expanded in powers of $E^{(\ell)}$, and each terms $(E^{(\ell)})^m$ is replaced by $E_m^{(\ell)}$. The higher order Euler polynomials $E_m^{(\ell)}(x)$ satisfy the following generating function

$$e^{E^{(\ell)}(x)t} \equiv \sum_{m=0}^{\infty} \frac{(E^{(\ell)}(x)t)^m}{m!} \equiv \sum_{m=0}^{\infty} E_m^{(\ell)}(x) \frac{t^m}{m!} = \left(\frac{2}{e^t+1} \right)^\ell e^{xt}, \tag{6}$$

in which, the symbol $\equiv$ is used to denote symbolic or umbral equivalences. It has been appeared in [5, (3.15)], [17, (8)] and [22, (78)]. Moreover, the relation $E_m^{(\ell)} = 2^m E_m^{(\ell)}\left(\frac{\ell}{2}\right)$ follows by setting $x = \frac{\ell}{2}$ in (6), replacing t by $2t$ and then comparing with (2). From (6), it is easy to verify that $E_m^{(\ell)}(x+y) = \sum_{k=0}^{m} \binom{m}{k} E_k^{(\ell)}(x) y^{m-k}$. Note that we have $E_m^{(0)}(x) = x^m$.

It is also easy to see that $(\mathrm{d}/\mathrm{d}x) E_m^{(\ell)}(x) = m E_{m-1}^{(\ell)}(x)$ for $m > 0$. From (2) and (6), we have the following the identity

$$\left(\frac{2}{e^t+1} \right)^\ell e^{xt} = \left(\frac{2}{e^{t/2}+e^{-t/2}} \right)^\ell e^{(x-\ell/2)t}. \tag{7}$$

It implies the Taylor expansion of $E_m^{(\ell)}(x)$ around $x = \ell/2$ (cf. [24]):

$$E_m^{(\ell)}(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{E_k^{(\ell)}}{2^k} \left(x - \frac{\ell}{2} \right)^{m-k}, \tag{8}$$

which holds for all nonnegative integers m and all real x. Clearly, the classical Euler polynomials and numbers are given by

$$E_m(x) := E_m^{(1)}(x) \quad \text{and} \quad E_m := E_m^{(1)} = 2^m E_m\left(\frac{1}{2}\right), \tag{9}$$

respectively (cf. [29]). From the generating function (6) we have $E_m(0) = 0$ if m is even. Therefore, $E_m \neq E_m(0)$; in fact

$$E_m(0) = -E_m(1) = \frac{2}{m+1}(1 - 2^{m+1})B_{m+1}, \quad m \geq 0, \tag{10}$$

here we recall that the Bernoulli numbers B_m are defined by the generating function

$$e^{Bt} \equiv \sum_{m=0}^{\infty} \frac{(Bt)^m}{m!} \equiv \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{t}{e^t - 1}. \tag{11}$$

We also mention that the Bernoulli polynomials $B_m(x)$ are defined by $B_m(x) = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} B_k$.

Recently, the higher order Euler numbers and polynomials have been investigated by many experts from different viewpoints such as number theory, mathematical analysis and statistics (see [2, 11, 25, 26, 28]). In [4], Chen obtained many interesting congruences related to Euler polynomials $E_n(x)$ by using the results of Eie and Ong [6]. Recently, the congruences for higher order Euler numbers have been further investigated by Liu [17, 18].

The main aim of this paper is to prove Kummer-type congruences for the higher order Euler numbers and polynomials by using the multiple fermionic p-adic integrals.

2. Higher order Euler numbers, polynomials and multiple Hurwitz-Euler eta functions

In this section, we shall introduce the higher order Euler numbers and polynomials, the multiple Hurwitz-Euler eta functions and analyze their elementary properties and relations.

For $q \geq 1$, we write

$$\begin{aligned} \left(\frac{2e^t}{e^{2t}+1}\right)^{\ell} \left(1 - (-e^{2t})^q\right)^{\ell} &= (2e^t)^{\ell} \left(\frac{1 - (-e^{2t})^q}{1 - (-e^{2t})}\right)^{\ell} \\ &= 2^{\ell} \sum_{j_1,\ldots,j_\ell=0}^{q-1} (-1)^{j_1+\cdots+j_\ell} e^{(2(j_1+\cdots+j_\ell)+\ell)t}. \end{aligned} \tag{12}$$

On the other hand, by using the binomial theorem and (2), we have

$$\begin{aligned} \left(\frac{2e^t}{e^{2t}+1}\right)^{\ell} \left(1 - (-e^{2t})^q\right)^{\ell} &= e^{E^{(\ell)}t} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{(q+1)j} e^{(2qj)t} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{(q+1)j} e^{(E^{(\ell)}+2qj)t}. \end{aligned} \tag{13}$$

Comparing the coefficients of t^m in the Taylor expansion around 0 for the right-hand sides of (12) and (13), we get the following proposition.

Proposition 2.1. *Let ℓ and q be positive integers. For any non-negative integer m, we have*

$$\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^{(q+1)j}\left(E^{(\ell)}+2qj\right)^m = 2^\ell \sum_{j_1,\ldots,j_\ell=0}^{q-1}(-1)^{j_1+\cdots+j_\ell}(2(j_1+\cdots+j_\ell)+\ell)^m$$

with the usual convention of replacing $(E^{(\ell)})^i$ by $E_i^{(\ell)}$.

Remark 2.1. Letting $\ell = 1$ in Proposition 2.1, we have

$$E_m + (-1)^{q+1}\sum_{j=0}^{m}\binom{m}{j}(2q)^{m-j}E_j = 2\sum_{j=0}^{q-1}(-1)^j(2j+1)^m. \tag{14}$$

This identity is due to Maïga [21, Proposition 2.3].

Lemma 2.2. *Let q be an odd integer with $q \geq 1$. Then for any non-negative integer m, we have*

$$E_m^{(\ell)} \equiv \sum_{j_1,\ldots,j_\ell=0}^{q-1}(-1)^{j_1+\cdots+j_\ell}(2(j_1+\cdots+j_\ell)+\ell)^m \pmod{q}.$$

Proof. For $m \geq 0$ we have

$$\left(E^{(\ell)}+2qj\right)^m = \sum_{k=0}^{m}\binom{m}{k}E_{m-k}^{(\ell)}(2qj)^k.$$

For an odd integer $q \geq 1$, the left hand side of Proposition 2.1 implies

$$\sum_{j=0}^{\ell}\binom{\ell}{j}\sum_{k=0}^{m}\binom{m}{k}E_{m-k}^{(\ell)}(2qj)^k \equiv 2^\ell E_m^{(\ell)} \pmod{q}, \tag{15}$$

since $\sum_{j=0}^{\ell}\binom{\ell}{j} = 2^\ell$. Therefore, by Proposition 2.1 and (15) we obtain the assertion. □

Letting $\ell = 1$ in the above lemma, we immediately get the following result.

Corollary 2.3 ([8, Lemma 2.5])**.** *Let q be an odd integer with $q \geq 1$. Then for any non-negative integer m, we have*

$$E_m \equiv \sum_{j=0}^{q-1}(-1)^j(2j+1)^m \pmod{q}.$$

Theorem 2.4. *Let m be a positive integer and p an odd prime. We have*

$$E_{(p-1)+2m}^{(\ell)} \equiv E_{2m}^{(\ell)} \pmod{p}.$$

Proof. By Lemma 2.2, we have

$$E_{2m}^{(\ell)} \equiv \sum_{j_1,\ldots,j_\ell=0}^{p-1}(-1)^{j_1+\cdots+j_\ell}(2(j_1+\cdots+j_\ell)+\ell)^{2m} \pmod{p}$$

and

$$E^{(\ell)}_{(p-1)+2m} \equiv \sum_{j_1,\dots,j_\ell=0}^{p-1} (-1)^{j_1+\cdots+j_\ell}(2(j_1+\cdots+j_\ell)+\ell)^{(p-1)+2m} \pmod p.$$

Then by Fermat's Little Theorem we get

$$E^{(\ell)}_{(p-1)+2m} \equiv \sum_{j_1,\dots,j_\ell=0}^{p-1} (-1)^{j_1+\cdots+j_\ell}(2(j_1+\cdots+j_\ell)+\ell)^{2m} \equiv E^{(\ell)}_{2m} \pmod p,$$

which completes the proof of Theorem 2.4. □

Putting $\ell = 1$ in Theorem 2.4 we immediately get the following result.

Corollary 2.5 ([8, Theorem 3.2])**.** *Let m be a positive integer and p an odd prime. We have*

$$E_{(p-1)+2m} \equiv E_{2m} \pmod p.$$

The following is the definition for multiple Hurwitz-Euler eta functions.

Definition 2.6 ([5, p. 314, (3.3)])**.** For $x > 0$ and $\ell \geq 1$, the multiple Hurwitz-Euler eta function $\eta_\ell(s,x)$ is defined by

$$\eta_\ell(s,x) = \sum_{k_1,\dots,k_\ell=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_\ell}}{(k_1+\cdots+k_\ell+x)^s}, \quad \mathrm{Re}(s) > 0. \tag{16}$$

Here $u^s = e^{s\log u}$ and $\log u = \log|u| + i\arg u$ with $-\pi < \arg u < \pi$ for any complex number u not on the nonpositive real axis.

In the case of $\ell = 1$, it reduces to the Hurwitz-Euler eta function

$$\eta(s,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^s}, \quad \mathrm{Re}(s) > 0. \tag{17}$$

Further setting $x = 1$ in the above equation, we recover the Dirichlet eta function (or the alternating Riemann zeta function)

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}, \quad \mathrm{Re}(s) > 0. \tag{18}$$

The analytic continuation and special values of $\eta_\ell(s,x)$ are implied by the following contour integral representation of $\eta_\ell(s,x)$.

Theorem 2.7 ([5, Theorem 4])**.** *The multiple Hurwitz-Euler eta function $\eta_\ell(s,x)$ is expressed as a contour integral*

$$\eta_\ell(s,x) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-t)^{s-1}e^{-xt}}{(1+e^{-t})^\ell}\,dt,$$

where $0 < c < \pi$ and C is the path from $+\infty$ to c along the real axis, going along the circle around 0 of radius c counter-clockwise to c, and then going back *to $+\infty$. This expression gives us the analytic continuation of η_ℓ to the whole complex s-plane, and also for a positive integer m we find that*

$$\eta_\ell(-m,x)=\frac{(-1)^m}{2^\ell}E_m^{(\ell)}(\ell-x).$$

In particular, for the Hurwitz-Euler eta function $\eta(s,x)$, we have $\eta(-m,x)=(-1)^m E_m(1-x)/2$.

Let p be an odd prime number. We get rid of the terms $1/(k_1+\cdots+k_\ell+x)^s$ with $k_1+\cdots+k_\ell$ divisible by p in (16) by defining

$$\tilde{\eta}_\ell(s,x)=\sum_{\substack{k_1,\ldots,k_\ell=0\\ p\nmid(k_1+\cdots+k_\ell)}}^{\infty}\frac{(-1)^{k_1+\cdots+k_\ell}}{(k_1+\cdots+k_\ell+x)^s}, \tag{19}$$

for $\mathrm{Re}(s)>0$ and $x>0$. From (19), we have

$$\begin{aligned}\tilde{\eta}_\ell(s,x)&=\eta_\ell(s,x)-\sum_{\substack{k_1,\ldots,k_\ell=0\\ p\mid(k_1+\cdots+k_\ell)}}^{\infty}\frac{(-1)^{k_1+\cdots+k_\ell}}{(k_1+\cdots+k_\ell+x)^s}\\ &=\eta_\ell(s,x)-\sum_{\substack{j_1,\ldots,j_\ell=0\\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1}\sum_{k_1',\ldots,k_\ell'=0}^{\infty}\frac{(-1)^{j_1+pk_1'+\cdots+j_\ell+pk_\ell'}}{(j_1+pk_1'+\cdots+j_\ell+pk_\ell'+x)^s}\\ &=\eta_\ell(s,x)-p^{-s}\sum_{\substack{j_1,\ldots,j_\ell=0\\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1}(-1)^{j_1+\cdots+j_\ell}\eta_\ell\left(s,\frac{j_1+\cdots+j_\ell+x}{p}\right).\end{aligned} \tag{20}$$

Since

$$E_m^{(\ell)}(x)=(-1)^m E_m^{(\ell)}(\ell-x),\quad m\geq 0,$$

from Theorem 2.7 and (20) we have

$$\begin{aligned}&\frac{1}{2^\ell}\left(E_m^{(\ell)}(x)-p^m\sum_{\substack{j_1,\ldots,j_\ell=0\\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1}(-1)^{j_1+\cdots+j_\ell}E_m^{(\ell)}\left(\frac{j_1+\cdots+j_\ell+x}{p}\right)\right)\\ &\quad=\eta_\ell(-m,x)-p^m\sum_{\substack{j_1,\ldots,j_\ell=0\\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1}(-1)^{j_1+\cdots+j_\ell}\eta_\ell\left(-m,\frac{j_1+\cdots+j_\ell+x}{p}\right)\\ &\quad=\tilde{\eta}_\ell(-m,x).\end{aligned} \tag{21}$$

Thus we get the following proposition.

Proposition 2.8. *Let $m \geq 0$ and $x > 0$. Then*

$$\tilde{\eta}_\ell(-m,x) = \frac{1}{2^\ell}\left(E_m^{(\ell)}(x) - p^m \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j_1+\cdots+j_\ell} E_m^{(\ell)}\left(\frac{j_1+\cdots+j_\ell+x}{p}\right)\right).$$

3. Kummer-type congruences for $E_m^{(\ell)}$ and $E_m^{(\ell)}(x)$

In this section, let p be a fixed odd prime number, let $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively, let $|\cdot|_p$ be the p-adic valuation on $\mathbb{Q}$ with $|p|_p = p^{-1}$. As usual, the extended valuation on $\mathbb{C}_p$ is also denoted by the same symbol $|\cdot|_p$.

Setting

$$z + p^N\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x - z|_p \leq p^{-N}\},$$

where $z \in \mathbb{Z}$ lies in $0 \leq z < p^N$. For any positive integers N, we define

$$\mu_{-1}(z + p^N\mathbb{Z}_p) = (-1)^z, \tag{22}$$

which is known as be fermionic p-adic measures on $\mathbb{Z}_p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly (or strictly) differentiable function on $\mathbb{Z}_p$. Using the fermionic p-adic measure, we define the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) = \lim_{N\to\infty} \sum_{z=0}^{p^N-1} f(z)(-1)^z, \tag{23}$$

for $f \in UD(\mathbb{Z}_p)$. The fermionic p-adic integral (23) were independently found by Katz [9, p. 486] (in Katz's notation, the $\mu^{(2)}$-measure), Shiratani and Yamamoto [27], Osipov [23], Lang [16] (in Lang's notation, the $E_{1,2}$-measure), T. Kim [10] from very different viewpoints. Let E be the translation with $(Ef)(z) = f(z+1)$. The formula (23) reduces to

$$\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) = 2f(0) - \int_{\mathbb{Z}_p} (Ef)(z)d\mu_{-1}(z). \tag{24}$$

Let

$$\int_{\mathbb{Z}_p^\ell} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{\ell \text{ times}}. \tag{25}$$

The multiple fermionic p-adic integrals considered here are defined as the iterated integrals. At the kth iteration with $1 \leq k \leq \ell$, for each fixed vector $(z_{k+1}, \ldots, z_\ell) \in \mathbb{Z}_p^{\ell-k}$, we integrate

$$\int_{\mathbb{Z}_p} F_k(z_k, z_{k+1}, \ldots, z_\ell)d\mu_{-1}(z_k), \tag{26}$$

for $F_k(z_k, z_{k+1}, \dots, z_\ell) \in UD(\mathbb{Z}_p)$. Under these conditions, we use the notation (cf. [30, (2.29)])

$$\int_{\mathbb{Z}_p^\ell} f(\overline{z}) d\mu_{-1}(\overline{z}), \quad \text{where } \overline{z} = (z_1, \dots, z_\ell), \tag{27}$$

to denote the multivariate fermionic p-adic integral

$$\int_{\mathbb{Z}_p^\ell} f(z_1, \dots, z_\ell) d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_\ell). \tag{28}$$

Also, for any compact open subset O of $\mathbb{Z}_p^\ell$, the integral of on O is defined by

$$\int_O f(\overline{z}) d\mu_{-1}(\overline{z}) = \int_{\mathbb{Z}_p^\ell} f(\overline{z}) \cdot (\text{characteristic function of } O) d\mu_{-1}(\overline{z})$$

(cf. [13, Chap. II]). Setting

$$D = \left\{ t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}} \right\}.$$

For a fixed $t \in D$, $e^{(z_1+\cdots+z_\ell)t}$ is an analytic function for $\overline{z} = (z_1, \dots, z_\ell)$. Applying (28) to the function

$$f(\overline{z}) = e^{(z_1+\cdots+z_\ell)t}$$

we see that the generating function of higher order Euler polynomials can be represented by the fermionic p-adic integral on $\mathbb{Z}_p$, that is, for $t \in D$ and $x \in \mathbb{Z}_p$ we have

$$\int_{\mathbb{Z}_p^\ell} e^{(x+z_1+\cdots+z_\ell)t} d\mu_{-1}(\overline{z}) = \left(\frac{2}{e^t+1} \right)^\ell e^{xt} = \sum_{m=0}^\infty E_m^{(\ell)}(x) \frac{t^m}{m!} \tag{29}$$

(cf. [10]). By substituting the Taylor expansion of $e^{(x+z_1+\cdots+z_\ell)t}$ in the above equation, we see that

$$\sum_{m=0}^\infty \int_{\mathbb{Z}_p^\ell} (x+z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) \frac{t^m}{m!} = \sum_{m=0}^\infty E_m^{(\ell)}(x) \frac{t^m}{m!}. \tag{30}$$

Moreover, by comparing coefficients of $\frac{t^m}{m!}$ on the both sides in (30), for integers $m \geq 0$, we obtain

$$\int_{\mathbb{Z}_p^\ell} (x+z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) = E_m^{(\ell)}(x), \tag{31}$$

which is similar with those in [11, 28]. Differentiating both sides of (31) with respect to x, we get

$$\frac{d}{dx} E_m^{(\ell)}(x) = m E_{m-1}^{(\ell)}(x) \quad \text{and} \quad \deg E_m^{(\ell)}(x) = m.$$

From (29) and (31), we have the following lemma.

Lemma 3.1. (1) *For integers $m \geq 0$ and $n \in \mathbb{N}$,*

$$\int_{\mathbb{Z}_p^\ell} (x + n(z_1 + \cdots + z_\ell))^m d\mu_{-1}(\overline{z}) = n^m E_m^{(\ell)}\left(\frac{x}{n}\right).$$

(2) *For integers $m \geq 0$,*

$$\sum_{j=0}^{\ell} \binom{\ell}{j} \int_{\mathbb{Z}_p^\ell} (j + x + z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) = 2^\ell x^m,$$

which is equivalent to

$$E_m^{(\ell)}(x) + E_m^{(\ell)}(x+1) + \cdots + E_m^{(\ell)}(x+\ell) = 2^\ell x^m.$$

In particular, we have $E_m(x) + E_m(x+1) = 2x^m$.

Proof. Part (1) follows immediately from (31). To see Part (2), note that by (29) we have

$$(e^t + 1)^\ell \int_{\mathbb{Z}_p^\ell} e^{(x+z_1+\cdots+z_\ell)t} d\mu_{-1}(\overline{z}) = 2^\ell e^{xt}.$$

The result follows by equating the coefficients of t in the above equation. □

From (8) and Lemma 3.1(1) with $n = 2, x = \ell$, we get

$$E_m^{(\ell)} = 2^m E_m^{(\ell)}\left(\frac{\ell}{2}\right) \tag{32}$$

(see [20, Proposition 10]). By changing $t \to 2t$ and setting $x = \frac{\ell}{2}$ in (29), we obtain the following multiple fermionic p-adic integral representation for the generating function of the higher order Euler numbers.

Proposition 3.2. *Let $t \in D$. We have*

$$\int_{\mathbb{Z}_p^\ell} e^{(2(z_1+\cdots+z_\ell)+\ell)t} d\mu_{-1}(\overline{z}) = \left(\frac{1}{\cosh t}\right)^\ell.$$

In particular, for integers $m \geq 0$, we have

$$\int_{\mathbb{Z}_p^\ell} (2(z_1 + \cdots + z_\ell) + \ell)^m d\mu_{-1}(\overline{z}) = E_m^{(\ell)}.$$

Remark 3.1. From (31) and Proposition 3.2, we have (see (8) above)

$$\begin{aligned}
E_m^{(\ell)}(x) &= 2^{-m} \int_{\mathbb{Z}_p^\ell} (2x + 2(z_1 + \cdots + z_\ell))^m d\mu_{-1}(\overline{z}) \\
&= 2^{-m} \sum_{k=0}^{m} \binom{m}{k} (2x - \ell)^{m-k} \int_{\mathbb{Z}_p^\ell} (2(z_1 + \cdots + z_\ell) + \ell)^k d\mu_{-1}(\overline{z}) \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{1}{2^k} \left(x - \frac{\ell}{2}\right)^{m-k} E_k^{(\ell)},
\end{aligned}$$

which can be seen as an extension of the Taylor expansion for $E_m(x)$ around $x=1/2$:

$$E_m(x)=\sum_{k=0}^{m}\binom{m}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{m-k}$$

(see [22, p. 25, (32)]).

Proposition 3.3 ([12, Theorem 2.2(3)]). *Let q be an odd positive integer. For $f\in UD(\mathbb{Z}_p)$, we have*

$$\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z)=\sum_{j=0}^{q-1}(-1)^j\int_{\mathbb{Z}_p} f(j+qz)d\mu_{-1}(z).$$

Proof. Although it is known, we would like to provide a detail proof here for the completeness. From (23), we obtain

$$\begin{aligned}\sum_{j=0}^{q-1}(-1)^j\int_{\mathbb{Z}_p} f(j+qz)d\mu_{-1}(z)&=\sum_{j=0}^{q-1}(-1)^j\lim_{N\to\infty}\sum_{z=0}^{p^N-1} f(j+qz)(-1)^z\\&=\lim_{N\to\infty}\sum_{z=0}^{qp^N-1} f(z)(-1)^z\\&=\sum_{j=0}^{q-1}(-1)^j\lim_{N\to\infty}\sum_{z=0}^{p^N-1} f(jp^N+z)(-1)^z,\end{aligned}$$

since p is an odd prime and q is an odd positive integer. Therefore, due to the uniform convergence, we can put the limit into the sum and get

$$\begin{aligned}\lim_{N\to\infty}\sum_{z=0}^{p^N-1} f(jp^N+z)(-1)^z&=\lim_{N\to\infty}\sum_{z=0}^{p^N-1}\lim_{M\to\infty} f(jp^M+z)(-1)^z\\&=\lim_{N\to\infty}\sum_{z=0}^{p^N-1} f(z)(-1)^z=\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z)\end{aligned}$$

for any integer j. This completes our proof. □

From (31) and Proposition 3.3, we obtain the following corollary.

Corollary 3.4 (Multiple Raabe's theorem). *For an odd integer q and $m\geq 0$, we have*

$$E_m^{(\ell)}(qx)=q^m\sum_{j_1,\ldots,j_\ell=0}^{q-1}(-1)^{j_1+\cdots+j_\ell}E_m^{(\ell)}\left(x+\frac{j_1+\cdot+j_\ell}{q}\right).$$

Proposition 3.5. *For integers $m\geq 1$ and $\ell\geq 1$, we have*

$$E_m^{(\ell)}\equiv 0 \pmod{\ell}.$$

Remark 3.2. A different proof of Proposition 3.5 has been given in [18, Lemma 1].

Proof of Proposition 3.5. From Proposition 3.2 with $m=1$, we have

$$\begin{aligned}
E_1^{(\ell)} &= \int_{\mathbb{Z}_p^\ell} (2(z_1+\cdots+z_\ell)+\ell)d\mu_{-1}(\overline{z}) \\
&= \int_{\mathbb{Z}_p^\ell} (2z_1+1)d\mu_{-1}(\overline{z})+\cdots+\int_{\mathbb{Z}_p^\ell} (2z_\ell+1)d\mu_{-1}(\overline{z}) \\
&= E_1\cdot\int_{\mathbb{Z}_p^{\ell-1}} d\mu_{-1}(z_2,\dots,z_\ell)+\cdots+E_1\cdot\int_{\mathbb{Z}_p^{\ell-1}} d\mu_{-1}(z_1,\dots,z_{\ell-1}) \\
&= \ell E_1 = 0,
\end{aligned}$$

since $E_1=0$. On the other hand, for $m\geq 1$, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p^\ell} z_1(2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) &= \int_{\mathbb{Z}_p^\ell} z_2(2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) \\
&= \cdots \\
&= \int_{\mathbb{Z}_p^\ell} z_\ell(2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}).
\end{aligned} \tag{33}$$

From Proposition 3.2 and (33), we have

$$\begin{aligned}
E_{m+1}^{(\ell)} &= 2\int_{\mathbb{Z}_p^\ell} (z_1+\cdots+z_\ell)(2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) \\
&\quad + \ell\int_{\mathbb{Z}_p^\ell} (2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) \\
&= 2\ell\int_{\mathbb{Z}_p^\ell} z_1(2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) \\
&\quad + \ell\int_{\mathbb{Z}_p^\ell} (2(z_1+\cdots+z_\ell)+\ell)^m d\mu_{-1}(\overline{z}) \\
&\equiv 0 \pmod{\ell},
\end{aligned} \tag{34}$$

where $m\geq 0$. This completes the proof. □

Proposition 3.6. *For integers $m,n\geq 1$ and $\ell\geq 1$, we have*

$$E_m^{(\ell+n)} \equiv E_m^{(n)} \pmod{\ell}.$$

Remark 3.3. For a different proof of Proposition 3.6, see [18, Lemma 2].

Proof of Proposition 3.6. For $\overline{z}=(z_1,\dots,z_{\ell+n})\in\mathbb{Z}_p^{\ell+n}$, by Proposition 3.2 we have

$$\begin{aligned}
&E_m^{(\ell+n)}\\
&=\int_{\mathbb{Z}_p^{\ell+n}}(2(z_1+\cdots+z_{\ell+n})+\ell+n)^m d\mu_{-1}(\overline{z})\\
&=\int_{\mathbb{Z}_p^{\ell+n}}((2(z_1+\cdots+z_\ell)+\ell)+(2(z_{\ell+1}+\cdots+z_{\ell+n})+n))^m d\mu_{-1}(\overline{z})\\
&=\sum_{i=1}^m\binom{m}{i}\int_{\mathbb{Z}_p^{\ell+n}}(2(z_1+\cdots+z_\ell)+\ell)^i(2(z_{\ell+1}+\cdots+z_{\ell+n})+n)^{m-i}d\mu_{-1}(\overline{z})\\
&\quad+\int_{\mathbb{Z}_p^{\ell+n}}(2(z_{\ell+1}+\cdots+z_{\ell+n})+n)^m d\mu_{-1}(\overline{z})\\
&=\sum_{i=1}^m\binom{m}{i}\int_{\mathbb{Z}_p^{\ell}}(2(z_1+\cdots+z_\ell)+\ell)^i d\mu_{-1}(z_1,\dots,z_\ell)\\
&\quad\times\int_{\mathbb{Z}_p^{n}}(2(z_{\ell+1}+\cdots+z_{\ell+n})+n)^{m-i}d\mu_{-1}(z_{\ell+1},\dots,z_{\ell+n})\\
&\quad+\int_{\mathbb{Z}_p^{\ell}}d\mu_{-1}(z_1,\dots,z_\ell)\int_{\mathbb{Z}_p^{n}}(2(z_{\ell+1}+\cdots+z_{\ell+n})+n)^m d\mu_{-1}(z_{\ell+1},\dots,z_{\ell+n})\\
&\equiv E_m^{(n)}\pmod{\ell},
\end{aligned}\tag{35}$$

since

$$E_i^{(\ell)}=\int_{\mathbb{Z}_p^{\ell}}(2(z_1+\cdots+z_\ell)+\ell)^i d\mu_{-1}(z_1,\dots,z_\ell)\equiv 0\pmod{\ell},\quad i\geq 1$$

(see Proposition 3.5 above) and

$$E_0^{(\ell)}=\int_{\mathbb{Z}_p^{\ell}}d\mu_{-1}(z_1,\dots,z_\ell)=\left(\int_{\mathbb{Z}_p}d\mu_{-1}(z)\right)^{\ell}=(E_0)^\ell=1.$$

This completes the proof. □

Let $\mathbb{Z}_p^\times$ be the group of p-adic units. Here we consider the function $f(\overline{z})=e^{(z_1+\cdots+z_\ell)t}$ on the domains

$$(\mathbb{Z}_p^\ell)^\times=\{\overline{z}=(z_1,\dots,z_\ell)\in\mathbb{Z}_p^\ell\mid z_1+\cdots+z_\ell\in\mathbb{Z}_p^\times\},$$

and

$$p(\mathbb{Z}_p^\ell)=\{\overline{z}=(z_1,\dots,z_\ell)\in\mathbb{Z}_p^\ell\mid z_1+\cdots+z_\ell\in p\mathbb{Z}_p\}.$$

It is easy to see that

$$\int_{(\mathbb{Z}_p^\ell)^\times}(z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) = \int_{\mathbb{Z}_p^\ell}(z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) - \int_{p(\mathbb{Z}_p^\ell)}(z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) \tag{36}$$

(cf. [13]). In the following, we will show that the expression

$$\int_{(\mathbb{Z}_p^\ell)^\times}(z_1+\cdots+z_\ell)^m d\mu_{-1}(\overline{z}) \tag{37}$$

can be interpolated p-adically. To our purpose, we deal with the second integral on the right-hand side of (36). For $|t|_p < p^{-\frac{1}{p-1}}$, by (23) and (28), we have

$$\begin{aligned}
&\int_{p(\mathbb{Z}_p^\ell)} e^{(z_1+\cdots+z_\ell)t} d\mu_{-1}(\overline{z}) \\
&= \lim_{N\to\infty} \sum_{\substack{z_1,\ldots,z_\ell=0 \\ z_1+\cdots+z_\ell\equiv 0 \pmod p}}^{p^N-1} e^{(z_1+\cdots+z_\ell)t}(-1)^{z_1+\cdots+z_\ell} \\
&= \lim_{N\to\infty} \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1} \sum_{z_1',\ldots,z_\ell'=0}^{p^{N-1}-1} e^{((j_1+pz_1')+\cdots+(j_\ell+pz_\ell'))t} \\
&\quad \times (-1)^{(j_1+pz_1')+\cdots+(j_\ell+pz_\ell')} \\
&= \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1} e^{(j_1+\cdots+j_\ell)t}(-1)^{j_1+\cdots+j_\ell} \\
&\quad \times \lim_{N\to\infty}\left(\frac{1+e^{p^N t}}{1+e^{pt}}\right)^\ell .
\end{aligned} \tag{38}$$

Since $e^{p^N t} \to 1$ as $N\to\infty$, we find that

$$\int_{p(\mathbb{Z}_p^\ell)} e^{(z_1+\cdots+z_\ell)t} d\mu_{-1}(\overline{z}) = \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell\equiv 0 \pmod p}}^{p-1} (-1)^{j_1+\cdots+j_\ell} e^{(j_1+\cdots+j_\ell)t}\left(\frac{2}{1+e^{pt}}\right)^\ell . \tag{39}$$

By comparing the coefficients of $t^m (m \geq 0)$ in the above equation, we have

$$\int_{p(\mathbb{Z}_p^\ell)} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) = p^m \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell \equiv 0 \pmod{p}}}^{p-1} (-1)^{j_1+\cdots+j_\ell} E_m^{(\ell)}\left(\frac{j_1+\cdots+j_\ell}{p}\right). \tag{40}$$

Therefore, we obtain the following result.

Lemma 3.7. *For every nonnegative integers $m \geq 0$ and $\ell \geq 1$, we have*

$$\int_{p(\mathbb{Z}_p^\ell)} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) = p^m \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell \equiv 0 \pmod{p}}}^{p-1} (-1)^{j_1+\cdots+j_\ell} E_m^{(\ell)}\left(\frac{j_1+\cdots+j_\ell}{p}\right).$$

By (31), (36) and Lemma 3.7, we have the following result.

Lemma 3.8. *For every nonnegative integers $m \geq 0$ and $\ell \geq 1$, we have*

$$\int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) = E_m^{(\ell)}(0) - p^m \sum_{\substack{j_1,\ldots,j_\ell=0 \\ j_1+\cdots+j_\ell \equiv 0 \pmod{p}}}^{p-1} (-1)^{j_1+\cdots+j_\ell} E_m^{(\ell)}\left(\frac{j_1+\cdots+j_\ell}{p}\right).$$

For $z_1 + \cdots + z_\ell \in \mathbb{Z}_p^\times$ and $m \equiv n \pmod{p^N(p-1)}$, we have

$$\int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) \equiv \int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^n d\mu_{-1}(\overline{z}) \pmod{p^{N+1}}$$

(see [13, the corollary at the end of §5]). So by Lemma 3.8, we have the following result.

Theorem 3.9 (Kummer-type congruences)**.** *Let $m \equiv n \pmod{p^N(p-1)}$ with $p - 1 \nmid m$. We have*

$$E_m^{(\ell)}(0) - p^m \sum_{\alpha \in J_0} (-1)^{p\alpha} E_m^{(\ell)}(\alpha) \equiv E_n^{(\ell)}(0) - p^n \sum_{\alpha \in J_0} (-1)^{p\alpha} E_n^{(\ell)}(\alpha) \pmod{p^{N+1}},$$

where

$$J_0 = \left\{ \frac{1}{p}\overline{j} \;\middle|\; \begin{array}{l} \overline{j} = j_1 + \cdots + j_\ell \equiv 0 \pmod{p} \\ \text{for some } j_1, \ldots, j_\ell \text{ with } 0 \leq j_1, \ldots, j_\ell \leq p-1 \end{array} \right\}$$

and in $\sum_{\alpha \in J_0}$ we sum over $\alpha = \frac{1}{p}\overline{j}$ as many times as $\overline{j}$ being expressed in the form $\overline{j} = j_1 + \cdots + j_\ell$ by various j_i's.

Letting $\ell = 1$ in the above theorem, we immediately get:

Corollary 3.10. *If* $m \equiv n \pmod{p^N(p-1)}$ *with* $p-1 \nmid m$, *then*

$$(1-p^m)E_m(0) \equiv (1-p^n)E_n(0) \pmod{p^{N+1}}.$$

By (10), Corollary 3.10 and the congruence

$$2(1-2^{m+1}) \equiv 2(1-2^{n+1}) \pmod{p^{N+1}}$$

for $m = n \pmod{p^N(p-1)}$, we recover the following well-known Kummer congruence for Bernoulli numbers (see [4, 6, 13]).

Corollary 3.11 (The Kummer congruence for Bernoulli numbers)**.** *If* $m = n \pmod{p^N(p-1)}$ *with* $p-1 \nmid m$, *then*

$$(1-p^m)\frac{B_{m+1}}{m+1} \equiv (1-p^n)\frac{B_{n+1}}{n+1} \pmod{p^{N+1}}.$$

As a corollary, if $p-1 \nmid m$ and $m \equiv n \pmod{p^N(p-1)}$ and $m, n \geq N+2$, then

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p^{N+1}}. \tag{41}$$

This kind of congruence was first found by Kummer [15] around 1850s, but applying it to get the p-adic interpolation of the Riemann zeta function was discovered lately by Kubota and Leoplodt [14] in 1964.

Theorem 3.12. *Let* α, *etc., be defined as above. The function*

$$-m \longmapsto E_m^{(\ell)}(0) - p^m \sum_{\alpha \in J_0} (-1)^{p\alpha} E_m^{(\ell)}(\alpha) \tag{42}$$

admits a continuation from the set $\{0, -1, -2, \ldots\}$ *to* $\mathbb{Z}_p$ *as a* p*-adic continuous function* $\eta^*_{\ell,p} : \mathbb{Z}_p \to \mathbb{Q}_p$. *It has the integral representation*

$$\eta^*_{\ell,p}(s) = \int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^{-s} d\mu_{-1}(\overline{z}). \tag{43}$$

Proof. Let $z_1 + \cdots + z_\ell \in \mathbb{Z}_p^\times, (p, a) \neq 1$ and let $m \equiv m' \pmod{p^N(p-1)}$ with $(p-1, m) = 1$. It is easy to see that $(z_1 + \cdots + z_\ell)^m \equiv (z_1 + \cdots + z_\ell)^{m'} \pmod{p^{N+1}}$. Therefore, we have (using the corollary at the end of §5 in [13])

$$\int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z}) \equiv \int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^{m'} d\mu_{-1}(\overline{z}) \pmod{p^{N+1}},$$

which allows us to extend the function

$$f(m) = \int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^m d\mu_{-1}(\overline{z})$$

from $\{0, -1, -2, \ldots\}$ to $\mathbb{Z}_p$ by the continuation. We denote this function by $\eta^*_{\ell,p}(s)$ and it has the integral representation

$$\eta^*_{\ell,p}(s) = \int_{(\mathbb{Z}_p^\ell)^\times} (z_1 + \cdots + z_\ell)^{-s} d\mu_{-1}(\overline{z}). \tag{44}$$

Finally, the special values (42) follows from Proposition 2.8 and the proof of Lemma 3.8. □

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