J. Appl. Math. & Informatics Vol. **40***(2022), No. 5 - 6, pp. 983 - 993*
https://doi.org/10.14317/jami.2022.983

ON CLENSHAW-CURTIS SPECTRAL COLLOCATION METHOD FOR VOLTERRA INTEGRAL EQUATIONS†

HUANG CHAOLAN, FANG CHUNHUA*, WANG JIANYU, WAN ZHENGSU

Abstract. The main purpose of this paper is to solve the second kind Volterra integral equations by Clenshaw-Curtis spectral collocation method. First of all, we can transform the integral interval from $[-1,x]$ to $[-1,1]$ through a simple linear transformation, and discretize the integral term in the equation by Clenshaw-Curtis quadrature formula to obtain the collocation equations. Then we provide a rigorous error analysis for the proposed method. At last, several numerical example are used to verify the results of theoretical analysis.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Clenshaw-Curtis spectral collocation method, second kind Volterra integral equations, convergence analysis.

1. Introduction

Volterra integral equations (VIEs) abound in many mathematical problems in engineering and physics, They can be found in, for example, acoustic scattering [1, 2], electromagnetic scattering [3, 4], circuit simulation [5], mutual impedance between conductors [6]. However, for most integral equations, it is difficult to obtain the analytical solution, the numerical approximation method which is easy to implement, with high precision and converge fastly is usually used to obtain the numerical solution.

This paper is concerned with the second kind Volterra integral equations

$$u(x)+\int_{-1}^{x}k(x,s)u(s)ds=f(t),x\in[-1,1]. \tag{1}$$

Received November 11, 2021. Revised March 14, 2022. Accepted April 20, 2022.
*Corresponding author.
†This work was supported partly by Hunan Provincial Natural Science Foundation of China(No.2022JJ30276),the National Science Foundation of China(No.11701171), and the Science and Technology Program of Hunan Province(No.2019TP1014).

Where $u(x)$ is an unknown function, and the given function $f(x)$ and kernel function $k(x,s)$ is smooth enough.

In recent years, there has been tremendous interest in developing calculation of solving Volterra integral equations, such as discontinuous Galerkin method [7], Filon-type method [8, 9], collocation method [10, 11], collocation boundary value method [12, 13], collocation method on uniformmesh [14], collocation method on graded mesh [15], spectral collocation methods [16], and so on. In particular, in reference [16], the Legendre spectral collocation method is used to solve equation (1), and the convergence of this method is proved theoretically. However,the calculation of Gauss-Legendre points and weights involves the calculation of eigenvalues and eigenvectors of the matrix. While Clenshaw-Curtis points can be obtained by FFT quickly, and it's expression is explicit. In addition to saving the calculation time, the calculatine accuracy is similar to that of Gauss quadrature formula [17]. So, Clenshaw-Curtis collocation method is more popular in engineering calculation. In this paper, Clenshaw-Curtis collocation method is used to solve the equation (1). First, the integral term in the equation (1) was approximated by Clenshaw-Curtis quadrature rule, and expand the unknown function u(x) by Legrange interpolation polynomials at the Clenshaw-Curtis points to obtain the discrete format of the integral equation. Then we provide a rigorous error analysis for the proposed method in theory. At last, several numerical examples are used to verify the results of theoretical analysis.

2. Clenshaw-Curtis collocation method

Set the collocation points as the set of $(n+1)$ Clenshaw-Curtis points$\{x_i = \cos(\frac{i\pi}{n})\}_{i=0}^{n}$. Assume that the equation (1) holds at x_i

$$u(x_i) + \int_{-1}^{x_i} k(x_i,s)u(s)ds = f(x_i), 0 \le i \le n. \tag{2}$$

The key of obtaining high order accuracy is to compute the integral term in function (2) accurately. However, when x_i are small values, there is little information for $u(s)$. To overcome this difficulty, we transform the integral interval from $[-1, x_i]$ to $[-1, 1]$ by a simple linear transformation:

$$s = \frac{x_i+1}{2}\theta + \frac{x_i-1}{2}, \theta \in [-1,1]. \tag{3}$$

then (2) becomes

$$u(x_i) + \frac{x_i+1}{2}\int_{-1}^{1} k(x_i, s(x_i,\theta))u(s(x_i,\theta))d\theta = f(x_i), 0 \le i \le n. \tag{4}$$

Now, we will introduce Clenshaw-Curtis quadrature rule

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} f(x_i)\omega_i. \tag{5}$$

where $\omega_i = \int_{-1}^{1} L_i(x)dx$, $L_i(x)$ is the $i-th$ Lagrange interpolation basis function, $\{x_i\}_{i=0}^n$ and $\{\omega_i\}_{i=0}^n$ represent Clenshaw-Curtis points and integral weight respectively. Here we can use Waldvogel's method [18] to calculate. The algorithm is based on discrete Fourier transforms. The asymptotic complexity is $O(nlogn)$.The matlab codes are given as follows:

```
function [wcc]=Clenshaw_Curtis(n)
N=[1:2:n-1]';p=length(N);m=n-p;k=[0:m-1]';
v0=[2./N./(N-2);1/N(end);zeros(m,1)];
v2=-v0(1:end-1)-v0(end:-1:2);wf2=ifft(v2);
% Clenshaw_Curtis
g0=-ones(n,1);g0(1+p)=g0(1+p)+n;
g0(m+1)=g0(1+m)+n;
g=g0/(n^2-1+mod(n,2));
wcc=ifft(v2+g);
wcc(n+1)=wcc(1);
end
```

Using Clenshaw-Curtis quadrature rule (5) to approximate the integral term in equation (4), we have

$$u(x_i) + \frac{x_i+1}{2}\sum_{j=0}^{n} k(x_i, s(x_i,\theta_j))u(s(x_i,\theta_j))\omega_j \approx f(x_i), 0 \le i \le n. \tag{6}$$

In general, $\{s(x_i,\theta_j)\}$ are not coincide with the collocation points $\{x_i\}_{i=0}^n$. Here we assume that $\{x_i\}_{i=0}^n$ and $\{\theta_i\}_{i=0}^n$ are the same set of Clenshaw-Curtis points on $[-1,1]$. i.e. $x_i = \theta_i$, for $0 \le i \le n$. We expand $u(x)$ using Lagrange interpolation polynomial, i.e.

$$u(x) \approx \sum_{p=0}^{n} u(x_p)L_p(x) := I_n(u), \tag{7}$$

where $L_p(x)$ is the $p-th$ Lagrange interpolation basis function based on Clenshaw-Curtis points $\{x_i\}_{i=0}^n$, and then we have

$$u(s(x_i,\theta)) \approx \sum_{p=0}^{n} u(x_p)L_p(s(x_i,\theta)). \tag{8}$$

Let u_i are the approximate values of $u(x_i)$, we get the discrete format of (1):

$$u_i + \frac{x_i+1}{2}\sum_{j=0}^{n} \omega_j k(x_i, s(x_i,\theta_j))\left(\sum_{p=0}^{n} u_p L_p(s(x_i,\theta_j))\right) = f(x_i), 0 \le i \le n. \tag{9}$$

$$u_i + \frac{x_i+1}{2}\sum_{j=0}^{n} u_j \sum_{p=0}^{n} \omega_p k(x_i, s(x_i,\theta_p))L_j(s\,(x_i,\theta_p)) = f(x_i), 0 \le i \le n. \tag{10}$$

Denoting $U_n = (u_0, u_1, \ldots, u_n)^T$ and $F_n = (f(x_0), f(x_1), \ldots, f(x_n))^T$, we can write the equations (10) in matrix form:

$$U_n + AU_n = F_n. \tag{11}$$

where the entries of the matrix A is given by

$$a_{i,j} = \frac{x_i + 1}{2} \sum_{p=0}^{n} \omega_p k(x_i, s(x_i, \theta_p)) L_j(s(x_i, \theta_p))$$

3. Convergence analysis

In this section, a convergence analysis for the numerical schemes (10) will be provided. To this end ,we begin with introducing some useful Lemmas.

Lemma 3.1 (Estimates for the interpolation error[19]). *Assume that $u \in H^m(I)$ and denote $I_n(u)$ is interpolation polynomial associated with the $(n+1)$-points Clenshaw-Curtis points $\{x_i\}_{i=0}^n$, for $m \geq 1$, the following estimates hold*

$$\|u - I_n(u)\|_{H^l(I)} \leq Cn^{l-m}\|u\|_{H^m(I)}, 0 \leq l \leq 1.$$

Lemma 3.2 (Sobolev Inequality[19]). *Let $(a, b) \subset R$ be a bounded interval of the real line. For each function $u \in H^1(a, b)$, the following inequality hold*

$$\|u\|_{L^\infty(a,b)} \leq (\frac{1}{a+b} + 2)^{1/2}\|u\|^{1/2}_{L^2(a,b)}\|u\|^{1/2}_{H^1(a,b)}.$$

Lemma 3.3 (Gronwall inequality[16, 20]). *If a non-negative integrable function $E(t)$ satisfies*

$$E(t) \leq C_1 \int_{-1}^{1} E(s)ds + G(t), -1 \leq t \leq 1.$$

where $G(t)$ is an integrable function, then

$$\|E\|_{L_p(I)} \leq C\|G\|_{L_p(I)}, p \geq 1.$$

Lemma 3.4 ([21, 22]). *Let $F_i(x), i = 0, 1, \ldots, n$ be the Lagrange interpolation polynomials associated with $n+1$ point Clenshaw-Curtis points $\{x_i\}_{i=0}^n$, then*

$$\|I_n\|_{L^\infty(-1,1)} := \max_{x\in[-1,1]} \sum_{i=0}^{n} |F_i(x)| = O(logn).$$

Now, we begin to analyze the convergence of numerical schemes (9).

Theorem 3.5 (Integration error from Clenshaw-Curtis quadrature). *Let $u(x) \in H^m(I), m \geq 1$, and $\sum_{i=0}^n u(x)\omega_i$ is the approximation of $\int_{-1}^1 u(x)dx$ under the Clenshaw-Curtis quadrature rule. Then there exists a constant C independent of n such that:*

$$\left| \int_{-1}^{1} u(x)dx - \sum_{i=0}^{n} u(x_i)\omega_i \right| \leq Cn^{l-m}\|u\|_{H^m(I)}. \tag{12}$$

Proof. We konw that

$$\int_{-1}^{1} I_n(x)dx = \int_{-1}^{1} \sum_{i=0}^{n} u(x_i)L_i(x)dx = \sum_{i=0}^{n} u(x_i)\int_{-1}^{1} L_i(x)dx = \sum_{i=0}^{n} u(x_i)\omega_i$$

so we have

$$\begin{aligned}\left|\int_{-1}^{1} u(x)dx - \sum_{i=0}^{n} u(x_i)\omega_i\right| &= \left|\int_{-1}^{1} u(x)dx - \int_{-1}^{1} I_n(x)dx\right| \\ &\leq \int_{-1}^{1} |u(x) - I_n(x)|dx \\ &\leq 2|u(x) - I_n(x)|\end{aligned}$$

using Lemmas 1, we get

$$\left|\int_{-1}^{1} u(x)dx - \sum_{i=0}^{n} u(x_i)\omega_i\right| \leq Cn^{l-m}\|u\|_{H^m(I)}.$$

□

Theorem 3.6. *Assume that* $u(x) \in H^m(I)$ *is the exact solution of Volterra integral equation* (1) *, and* $u_n(x) := \sum_{i=0}^{n} u_i L_i(x)$ *is the approximate solution achived by using Clenshaw-Curtis collocation method from* (10)*,and*

$$u_n(x) := \sum_{i=0}^{n} u_i L_i(x). \tag{13}$$

then for $m \geq 1$*,*

$$\|u - u_n\|_{L^\infty(I)} \leq Cn^{l-m}(logn)\|u\|_{H^m(I)} + Cn^{1/2-m}\|u\|_{H^m(I)}. \tag{14}$$

provided that n *is sufficiently large, where* C *is a constant independent of* n *.*

Proof. Adding to the notation(13), then equation 9 can be written as

$$u_i + \frac{x_i+1}{2}\sum_{j=0}^{n} \omega_j k(x_i, s(x_i,\theta_j))u_n\left(s(x_i,\theta_j)\right) = f(x_i), 0 \leq i \leq n. \tag{15}$$

adding one integral to both sides of (15) and change the order, we have

$$u_i + \frac{x_i+1}{2}\int_{-1}^{1} k(x_i, s(x_i,\theta))u_n(s(x_i,\theta))d\theta = f(x_i) - Y_1(x_i), \tag{16}$$

where

$$Y_1(x) := \frac{x+1}{2}\left(\int_{-1}^{1} k(x, s(x,\theta))u(s(x,\theta))d\theta - \sum_{j=0}^{n} \omega_j k(x, s(x,\theta_j))u_n(s(x,\theta_j))\right). \tag{17}$$

Through the inverse process of (3), equation (16) can be transformed into

$$u_i + \int_{-1}^{x_i} k(x_i, s)u_n(s)ds = f(x_i) + Y_1(x_i), \tag{18}$$

multiplying $L_i(x)$ on the both sides of (18) and summing up from 0 to n yield

$$\sum_{i=0}^{n} u_i L_i(x) + \sum_{i=0}^{n} \int_{-1}^{x_i} k(x_i, s)u_n(s)dsL_i(x) = \sum_{i=0}^{n} f(x_i)L_i(x) + \sum_{i=0}^{n} Y_1(x_i)L_i(x)$$

that is

$$u_n(x) + I_n\left(\int_{-1}^{x} k(x, s)u_n(s)ds\right) = I_n(f) + I_n(Y_1), \tag{19}$$

combining (19) and (1), we deduce that

$$e(x)+\left(I_n\left(\int_{-1}^{x} k(x, s)u_n(s)ds\right) - \int_{-1}^{x} k(x, s)u(s)ds\right) = I_n(f) - f(x) + I_n(Y_1), \tag{20}$$

where $e(x) = u_n(x) - u(x)$ is an error function. It follows that

$$\begin{aligned} & e(x) + I_n\left(\int_{-1}^{x} k(x, s)u_n(s)ds\right) - I_n\left(\int_{-1}^{x} k(x, s)u(s)ds\right) \\ = & -I_n\left(\int_{-1}^{x} k(x, s)u(s)ds\right) + \int_{-1}^{x} k(x, s)u(s)ds + I_n(f) - f(x) + I_n(Y_1) \end{aligned} \tag{21}$$

that is

$$\begin{aligned} & e(x) + I_n\left(\int_{-1}^{x} k(x, s)e(s)ds\right) \\ = & I_n\left(f(x) - \int_{-1}^{x} k(x, s)u(s)ds\right) + \left(\int_{-1}^{x} k(x, s)u(s)ds - f(x)\right) + I_n(Y_1) \end{aligned} \tag{22}$$

from (1) we have

$$f(x) - \int_{-1}^{x} k(x, t)u(t)dt = u(x),$$

then (22) can turn into

$$e(x) + I_n\left(\int_{-1}^{x} k(x, s)e(s)ds\right) = I_n(u) - u(x) + I_n(Y_1). \tag{23}$$

Adding an integration to both sides of (23) and changing the order ,we have

$$\begin{aligned} & e(x) + \int_{-1}^{x} k(x, s)e(s)ds \\ = & \int_{-1}^{x} k(x, s)e(s)ds - I_n\left(\int_{-1}^{x} k(x, s)e(s)ds\right) + (I_n(u) - u)(x) + I_n(Y_1) \end{aligned} \tag{24}$$

Denoting

$$\begin{aligned}
Y_2(x) : &= (I_n(u) - u)(x), \\
Y_3(x) : &= \int_{-1}^{x} k(x,s)e(s)ds - I_n\left(\int_{-1}^{x} k(x_i,s)e(s)ds\right), \\
G(x) : &= I_n(Y_1) + Y_2(x) + Y_3(x),
\end{aligned}$$

then (24)can be write as

$$e(x) + \int_{-1}^{x} k(x,s)e(s)ds = I_n(Y_1) + Y_2(x) + Y_3(x) = G(x) \tag{25}$$

so we have

$$|e(x)| \le \max_{(x,s)\in[-1,1]} |k(x,s)| \int_{-1}^{x} |e(s)|ds + |G(x)|, \tag{26}$$

from Lemma 3, we can deduce that

$$\|e\|_{L^\infty(I)} \le C\|G(x)\|_{L\infty(I)} \le C\left(\|I_n(Y_1)\|_{L^\infty(I)} + \|Y_2(x)\|_{L^\infty(I)} + \|Y_3(x)\|_{L^\infty(I)}\right). \tag{27}$$

Now, we come to estimate each term of the right hand side of the above inequality one by one. First, for the evaluate of $Y_1(x)$, with the help of Lemmas 1 and Theorem 1, we get

$$\begin{aligned}
|Y_1(x)| &\le Cn^{l-m}\|k(x,s(x,\theta)u_n(s(x,\theta))\|_{H^m(I)} \\
&\le Cn^{l-m} \max_{(x,s)\in[-1,1]} |k(x,s)| \|u_n\|_{H^m(I)} \\
&\le Cn^{l-m}\|u_n\|_{H^m(I)}
\end{aligned} \tag{28}$$

then reckon $\|I_n(Y_1)\|_{L^\infty(I)}$, using Lemmas 4 , we have

$$\|I_n(Y_1)\|_{L^\infty(I)} \le \|I_n\|_{L^\infty(I)}\|Y_1\|_{L^\infty(I)} \le Cn^{l-m}(logn)\|u_n\|_{H^m(I)} \tag{29}$$

For the evaluate of $\|Y_2\|_{L^\infty(I)}$, applying Lemmas 1 and Lemmas 3 , we have

$$\begin{aligned}
\|Y_2\|_{L^\infty(I)} &\le C\|Y_2\|^{1/2}_{L^2(I)}\|Y_2\|^{1/2}_{H^1(I)} \\
&\le \left(Cn^{-m}\|u\|_{H^m(I)}Cn^{1-m}\|u\|_{H^m(I)}\right)^{1/2} \\
&\le Cn^{1/2-m}\|u\|_{H^m(I)}
\end{aligned} \tag{30}$$

similarly, we let $m = 1$, for the evaluate of $\|Y_3\|_{L^\infty(I)}$

$$\begin{aligned}
\|Y_3\|_{L^\infty(I)} &\le C\|Y_3\|^{1/2}_{L^2(I)}\|Y_3\|^{1/2}_{H^1(I)} \\
&\le Cn^{-1/2}\left\|\int_{-1}^{x} k(x,s)e(s)ds\right\|_{H^1(I)} \\
&\le Cn^{-1/2}\left\|k(x,x)e(x) - \int_{-1}^{x} k_x(x,s)e(s)ds\right\|_{L^2(I)}
\end{aligned}$$

$$\leq Cn^{-1/2}\left(\max_{\|x\|\leq 1}\|k(x,x)\| + \max_{\|x\|\leq 1}\|\partial_x k\|_{L^2(I)}\right)\|e\|_{L^2(I)}$$
$$\leq Cn^{-1/2}\|e(x)\|_{L^2(I)}$$
$$\leq Cn^{-1/2}\|e(x)\|_{L^\infty(I)} \tag{31}$$

From the above estimates, when n is sufficiently large, we can get

$$\|e\|_{L^\infty(I)}$$
$$\leq C_1 n^{l-m}(logn)\left(\|u\|_{H^m(I)} + \|e\|_{L^\infty(I)}\right) + C_2 n^{1/2-m}\|u\|_{H^m(I)}$$
$$+ C_3 n^{-1/2}\|e\|_{L^\infty(I)}$$

this implies

$$\|e\|_{L^\infty(I)} \leq Cn^{l-m}(logn)\|u\|_{H^m(I)} + Cn^{1/2-m}\|u\|_{H^m(I)}$$

□

4. Numerical examples

Example 4.1. Consider the Volterra integral equation

$$u(x) + \int_{-1}^{x} k(x,s)u(s)ds = f(x), x \in [-1,1].$$

Where $k(x,s) = xs, f(x) = e^{-x^2} + \frac{1}{2}xe^{-1} - \frac{1}{2}xe^{-x^2}$, the exact solution is $u = e^{-x^2}$. The numerical results which obtained by Clenshaw-Curtis collocation method are shown in Figure 1. These result indicate that the desired spectral accuracy is obtained when $n = 20$.

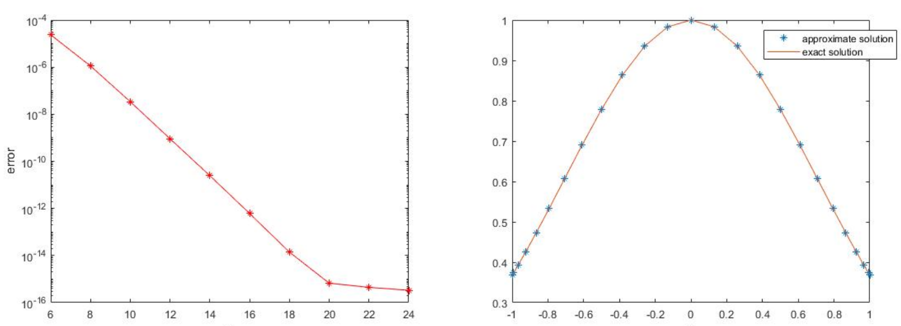

Figure 1. The errors versus the number of collocation points in L^∞ norm(left). Compassion between approximate solutin and the exact solution(right).

Example 4.2. Consider the Volterra integral equation with the proportional delay

$$u(x) = f(t) + \int_0^x k(x,s)u(qs)ds, x \in [-1,1].$$

Where $q = \frac{1}{2}, k(x,s) = e^{xs}, f(x) = e^{4x} + \frac{1}{x+2}(e^{-x-4} - e^{x^2+2x-2})$, the exact solution is $u = e^{4x}$. Numerical results are displayed in Figure 2.

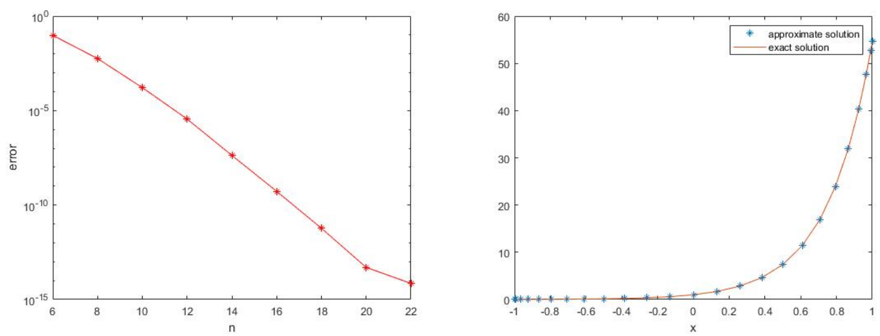

FIGURE 2. The errors versus the number of collocation points in L^∞ norm(left). Compassion between approximate solutin and the exact solution(right).

Example 4.3. Consider the Volterra integral equation

$$u(x) + \int_{-1}^x k(x,s)u(s)ds = f(x), x \in [-1,1].$$

Where $k(x,s) = e^{xs}, f(x) = e^{4x} + \frac{1}{x+4}(e^{x(x+4)} - e^{-(x+4)})$, the exact solution is$u = e^{4x}$. We use the numerical scheme (10).The result are displayed in Figure 3.

Compared with Example5.1 in Tang Tao's literature [16], the convergence accuracy obtained by Clenshaw-Curtis collocation method is better than that obtained by Gauss-Legendre collocation method. Numerical errors with several values of are displayed in the following table

n	6	8	10	12	14
Eg.3	3.36e-02	1.2e-03	3.31e-05	6.53e-07	1.01e-08
Eg.5.1 in [16]	3.66e-01	1.88e-2	6.57e-04	1.65e-05	3.11e-07
n	16	18	20	22	24
Eg.3	1.24e-10	1.26e-12	4.26e-14	4.27e-14	3.55e-14
Eg.5.1 in [16]	4.57e-09	5.37e-11	5.19e-11	5.68e-14	4.26e-14

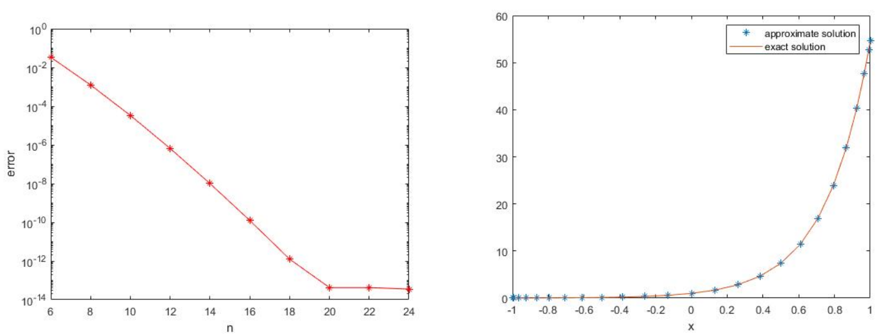

FIGURE 3. The errors versus the number of collocation points in L^∞ norm(left). Compassion between approximate solutin and the exact solution(right).

5. Conclusions

In this paper, Clenshaw-Curtis spectral collocation method is used to solve the second kind Volterra integral equations. Clenshaw-Curtis collocation points and integral weights are easier to calculate than Gauss points and integral weights. Clenshaw-Curtis quadrature formula is used to discretize the integral term in the equation to get the collocation solution. Finally the result of the numerical examples illustrate the efficiency of the method.

REFERENCES

1. S. Langdon, S.N. Chandler-Wilde, *A wavenumber independent boundary element method for an acoustic scattering problem*, SIAM J. Numer. Anal. **42** (2006), 2450-2477.
2. S.N. Chandler-Wilde, I.G. Graham, S. Langdon, E.A. Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numerica **21** (2012), 89-305.
3. G. Bao, W. Sun, *A fast algorithm for the electromagnetic scattering from a large cavity*, SIAM J. Sci. Comput. **27** (2005), 553-574.
4. P.J. Davis, D.B. Duncan, *Stability and convergence of collocation schemes for retarded potential integral equations*, SIAM J. Numer. Anal. **42** (2004), 1167-1188.
5. M. Condon, Alfredo. Deano, A. Iserles, K. Maczynski, X. Tao, *On numerical methods for highly oscillatory problems in circuit simulation*, Compel. **28** (2009), 1607-1618.
6. J. Zou, M. Li, S.H. Chang, *Calculation of the magnetic field in air produced by the underground conductor using Pollaczek integral*, IEEE Trans. EMC. **54** (2012), 198-204.
7. H. Brunner, P. Davis, D. Duncan, *Discontinuous Galerkin approximations for Volterra integral equations of the first kind*, IMA J. Numer. Anal. **29** (2009), 856-881.
8. H. Wang, S. Xiang, *Asymptotic expansion and Filon-type methods for a Volterra integral equation with a highly oscillatory kernel*, IMA J. Numer. Anal. **31** (2011), 469-490.
9. J. Ma, C. Fang, and S. Xiang, *Modified asymptotic orders of the direct Filon method for a class of Volterra integral equations*, J. Comput. Appl. Math. **281** (2015), 120-125.

10. H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, New York,2004.
11. H. Brunner. *On Volterra integral operators with highly oscillatory kernels*, Discrete Contin. Dyn. Syst. **34** (2014), 915-929.
12. Junjie Ma, Shuhuang Xiang, *A collocation boundary value method for linear volterra integral equations*, Journal of Scientific Computing **71** (2017), 1-20.
13. H. Chen, C. Zhang, *Boundary value methods for Volterra integral and integro-differential equations*, Appl. Math. Comput. **218** (2011), 2619-2630.
14. S. Xiang, Q. Wu, *Numerical solutions to Volterra integral equations of the second kind with oscillatory trigonometric kernels*, Appl. Math. Comput. **223** (2013), 34-44.
15. Q. Wu, *On graded meshes for weakly singular Volterra integral equations with oscillatory trigonometric kernels*, J. Comput. Appl. Math. **263** (2014), 370-376.
16. Tao Tang, Xiang Xu, Jin Cheng, *On spectral methods for Volterra integral equations and convergence analysis*, J. Comput. Math. **6** (2008), 825-837.
17. L.N. Trefethen, *Is Gauss Quadrature Better than Clenshaw¨CCurtis?*, SIAM Review **50** (2008), 67-87.
18. J. Waldvogel, *Fast Construction of the Fejr and ClenshawCurtis Quadrature Rules*, BIT Numer. Math. **46** (2006), 195-202.
19. M.Y. Hussaini, C. Canuto, A. Quarteroni, T.A. Zang, *Spectral Methods Fundamentals in Single Domains*, Springer, Berlin, 2006.
20. D. Henry, *Geometric theory of semilinear parabolic equations*, Springer, Berlin, 1989.
21. Y. Chen, T. Tang, *Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equation with aweakly singular kernel*, Math. Comput. **79** (2010), 147-167.
22. G. Mastroianni, D. Occorsio, *Optional systems of nodes for Lagrange interpolation on bounded intervels*, Comput. Appl. Math. **134** (2001), 325-341.

Huang Chaolan is studing for her Master's degree. Her research interests are computational mathematics and spectral method.

College of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan, 414006, PR China.
e-mail: 18528305001@163.com

Fang Chunhua received her ph.D. degree from Central University. Her research interests include numerical integration and spectral method.

College of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan, 414006, PR China.
e-mail: fangchunhuamath@163.com

Wang Jianyu is studing for his Master's degree. His reseach interests are computational mathematics and iterative method.

College of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan, 414006, PR China.
e-mail: 627346502@qq.com

Wan Zhengsu received his ph.D. degree from Shanghai University. His reseach interests is numerical solution of differential equations.

College of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan, 414006, PR China.
e-mail: 307442296@qq.com