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ON THE GEOMETRY OF COMPLEX METALLIC NORDEN MANIFOLDS

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Abstract. We study almost complex metallic Norden manifolds and their adapted connections with respect to an almost complex metallic Norden structure. We study various connections like special connection of the first type, special connection of the second type, Kobayashi-Nomizu metallic Norden type connection, Yano metallic Norden type connection etc., on almost complex metallic Norden manifolds. We establish classifications of almost complex metallic Norden manifolds by using covariant derivative of the almost complex metallic Norden structure and also by using torsion tensor on the canonical connections.

1. Introduction

Through an adapted connection (which parallelizes an almost Norden structure (J,g)) on an almost Norden manifold (M,J,g), there exist *first canonical connection*, *Kobayashi-Nomizu connection*, *Yano connection* etc. All these connections are uniquely defined by means of the Levi-Civita connection of the metric g (for details, see [5]). On the other hand, connections such as *Chern connections*, *canonical* (*well adapted*) *connections* and connections with totally skew-symmetric torsion tensor (for instance, the *Bismut connections*), are defined by imposing an additional condition on the torsion, in order to become adapted connections. But a special role, through all the adapted connections, is played by the *canonical connection*. It measures the integrability of the associated $\mathcal{G}$-structure, that is, the associated $\mathcal{G}$-structure is integrable if and only if the torsion and the curvature tensor field of the canonical connection vanish. In the Kähler case, the canonical connection coincides with the Levi-Civita connection.

Moreover, Ganchev and Borisov [7] established a classification of the almost complex manifolds with a Norden metric with respect to the covariant derivative of the almost complex structure. Later, Ganchev and Mihova [9] also

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provided a characterization of the eight classes of almost Norden manifolds by means of conditions on the torsion tensor of the canonical connection and its associated 1-form.

Motivated from above mentioned studies and from the duality between the almost complex metallic Norden and almost Norden structures, we study adapted connections with respect to an almost complex metallic Norden structure and explore *special connection of the first type*, *special connection of the second type*, *Kobayashi-Nomizu metallic Norden type connection*, *Yano metallic Norden type connection* etc., on the almost complex metallic Norden manifolds. We also provide classifications of almost complex metallic Norden manifolds by using covariant derivative of the almost complex metallic Norden structure and also by using torsion tensor on the canonical connections.

2. Almost complex metallic Norden manifolds

On a smooth manifold M, Goldberg and Yano [11] defined a polynomial structure of degree d, which is a $(1,1)$-tensor field f of constant rank such that d is the smallest integer for which $f^d, f^{d-1}, \ldots, f, I$ are not independent, where I is the identity tensor field of type $(1,1)$. Then there exist real numbers $\gamma_1, \gamma_2, \ldots, \gamma_d$ such that $f^d + \gamma_d f^{d-1} + \cdots + \gamma_2 f + \gamma_1 I = 0$. For example, if a polynomial structure f satisfies

- $f^2 + I = 0$, then f is an *almost complex structure*;
- $f^2 - I = 0$, then f is an *almost product structure*;
- $f^2 - pf - qI = 0$, where p, q are positive integers, then f is a *metallic structure* [12,18];
- $f^2 - f - I = 0$, then f is a *golden structure* [3];
- $f^2 - f + \frac{3}{2}I = 0$, then f is an *almost complex golden structure* [3,19].

Definition. Let M^{2n} be a $2n$-dimensional smooth manifold. A tensor field Ψ of $(1,1)$-type on M^{2n} satisfying $\Psi^2 - p\Psi - qI = 0$, where p and q are real numbers with $p^2 + 4q < 0$, is called an almost complex metallic structure and (M^{2n}, Ψ) is called an almost complex metallic manifold (see also [3,19]).

For an almost complex metallic structure Ψ, q is always a negative real number. The complex numbers $\sigma^c_{p,q} = \frac{p \pm \sqrt{p^2+4q}}{2}$, which are the solutions of the equation $x^2 - px - q = 0$, where $p^2 + 4q < 0$, are called the *complex metallic means*. For different values of p and q, we have the following means:

- For $p = 1$ and $q = -\frac{3}{2}$, we obtain $\sigma^c_{p,q} = \frac{1 \pm \sqrt{5}\ i}{2}$, which are almost complex golden means, where $i = \sqrt{-1}$.
- For $p = 2$ and $q = -3$, we obtain $\sigma^c_{p,q} = 1 \pm \sqrt{2}\ i$, which are almost complex silver means.
- For $p = 3$ and $q = -\frac{11}{2}$, we obtain $\sigma^c_{p,q} = \frac{3 \pm \sqrt{13}\ i}{2}$, which are almost complex bronze means.

- For $p = 4$ and $q = -9$, we obtain $\sigma^c_{p,q} = 2 \pm \sqrt{5}\, i$, which are almost complex subtle means.
- For $p = 1$ and $q = -\frac{7}{2}$, we obtain $\sigma^c_{p,q} = \frac{1 \pm \sqrt{13}\, i}{2}$, which are almost complex nickel means.

Let J be an almost complex structure on M^{2n}. Then

$$\Psi^J = \frac{1}{2}\Big[pI \pm \big((p - 2\sigma^c_{p,q})i\big)J\Big], \tag{1}$$

are almost complex metallic structures on M^{2n} and are called the *almost complex metallic structures induced by* J. Conversely, if Ψ is an almost complex metallic structure on M^{2n}, then

$$J^\Psi = \pm \frac{1}{(p - 2\sigma^c_{p,q})i}(2\Psi - pI) \tag{2}$$

are almost complex structures on M^{2n} and are called the *almost complex structures induced by* Ψ. It is known that if J is an almost complex structure, then $\widetilde{J} = -J$ is also an almost complex structure and known as the *conjugate almost complex structure of* J. If Ψ is an almost complex metallic structure, then $\widetilde{\Psi} = pI - \Psi$ is also an almost complex metallic structure and known as the *conjugate almost complex metallic structure of* Ψ. Further, the conjugate almost complex structure $\widetilde{J}$ and the conjugate almost complex metallic structure $\widetilde{\Psi}$ also satisfy the relations analogous to (1) and (2), respectively. Hence, an almost complex structure J (respectively, $\widetilde{J}$) defines a J (respectively, $\widetilde{J}$)–associated almost complex metallic structure Ψ^J (respectively, $\Psi^{\widetilde{J}}$) and vice-versa. Furthermore $\Psi^{J^\Psi} = \Psi$ and $J^{\Psi^J} = J$, therefore, there is a one-to-one correspondence between almost complex metallic structures and almost complex structures on M^{2n}.

Let φ and θ be tensor fields of the type $(1,1)$ and $(0,s)$ on M, respectively. Then θ is said to be a *pure tensor field with respect to* φ if

$$\theta(\varphi X_1, X_2, \ldots, X_s) = \theta(X_1, \varphi X_2, \ldots, X_s) = \cdots = \theta(X_1, X_2, \ldots, \varphi X_s)$$

for any vector fields $X_1, X_2, \ldots, X_s$ on M. Consider the Tachibana operator

$$\phi_\varphi : \mathfrak{T}^0_s(M) \to \mathfrak{T}^0_{s+1}(M)$$

given by

$$\begin{aligned}(\phi_\varphi \theta)(X, Y_1, Y_2, \ldots, Y_s) = {} & (\varphi X)(\theta(Y_1, Y_2, \ldots, Y_s)) \\ & - X(\theta(\varphi Y_1, Y_2, \ldots, Y_s)) \\ & + \theta((L_{Y_1}\varphi)X, Y_2, \ldots, Y_s) \\ & \cdots \\ & + \theta(Y_1, Y_2, \ldots, (L_{Y_s}\varphi)X),\end{aligned} \tag{3}$$

where L_Y denotes the Lie differentiation with respect to Y.

An indefinite almost complex manifold (M^{2n}, J, g) is said to be an *almost Norden manifold* if the pseudo-Riemannian metric g is pure with respect to J, that is, $g(JX, Y) = g(X, JY)$ for any vector fields X, Y on M^{2n}. Moreover, if the almost complex structure J is integrable, then it is said to be a *complex structure* and (M^{2n}, J, g) is called a *Norden manifold*.

A tensor field θ on M is said to be *holomorphic* if

$$(\phi_J \theta)(X, Y_1, Y_2, \dots, Y_s) = 0$$

for any vector fields $X, Y_1, \dots, Y_s$ on M.

From [13], it is known that an almost complex Norden manifold is *holomorphic Norden* ($\nabla J = 0$) if and only if g is holomorphic ($\phi_J g = 0$), where ∇ is the Levi-Civita connection of g.

Definition. Let (M^{2n}, Ψ) be an almost complex metallic manifold and let g be a pseudo-Riemannian metric on M^{2n}. If g is pure with respect to the almost complex metallic structure Ψ, that is, $g(\Psi X, Y) = g(X, \Psi Y)$ for any vector fields X, Y on M^{2n}, then the triple (M^{2n}, Ψ, g) is called an almost complex metallic Norden manifold.

For an almost complex metallic Norden manifold, we derive

$$g(\Psi X, \Psi Y) = pg(\Psi X, Y) + qg(X, Y). \tag{4}$$

An almost complex metallic structure Ψ is integrable if and only if there exists a torsion-free linear connection such that the almost complex metallic structure Ψ is covariantly constant with respect to it. Moreover, the integrability of the structure Ψ is equivalent to the vanishing of the Nijenhuis tensor field N_Ψ of Ψ, where the N_Ψ is given by

$$N_\Psi(X, Y) = [\Psi X, \Psi Y] - \Psi[\Psi X, Y] - \Psi[X, \Psi Y] + \Psi^2[X, Y], \tag{5}$$

which is equivalent to

$$N_\Psi(X, Y) = \Psi(\nabla_Y \Psi)X - \Psi(\nabla_X \Psi)Y + (\nabla_{\Psi X} \Psi)Y - (\nabla_{\Psi Y} \Psi)X. \tag{6}$$

If the almost complex metallic structure Ψ is integrable, then Ψ is called a *complex metallic structure* and (M^{2n}, Ψ) is called a *complex metallic manifold*. Since there is a one-to-one correspondence between almost complex metallic structures and almost complex structures, the technique of ϕ-operator used for almost complex structure can be used for almost complex metallic structure. Hence, analogous to the proofs of the theorems available in [10, 13], we have the following important assertions.

Theorem 2.1. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then*

(i) Ψ *is integrable if* $\phi_\Psi g = 0$;
(ii) *the condition* $\phi_\Psi g = 0$ *is equivalent to* $\nabla \Psi = 0$, *where* ∇ *is the Levi-Civita connection of* g.

Moreover, if Ψ^J is one of the two almost complex metallic structures induced by the almost complex structure J, then from (1), we also have

$$\phi_{\Psi^J} g = \frac{(p - 2\sigma^c_{p,q})i}{2}\phi_J g. \tag{7}$$

Hence, from the assertion (i) of Theorem 2.1 and (7), it is obvious that the almost complex metallic structure Ψ^J is integrable if and only if the almost complex structure J is integrable. Since

$$(\phi_J g)(X, Y, Z) = -g((\nabla_X J)Y, Z) + g((\nabla_Y J)X, Z) + g((\nabla_Z J)X, Y),$$

it follows that $\phi_J g = 0$, which is equivalent to $\nabla J = 0$.

Theorem 2.2 ([13])**.** *An almost Norden manifold of class C^ω is a holomorphic Norden manifold if and only if the almost complex structure is parallel with respect to the Levi-Civita connection ∇.*

Hence, we have the following observation immediately.

Theorem 2.3. *An almost complex metallic Norden manifold (M^{2n}, Ψ, g) is a holomorphic metallic Norden manifold if and only if $\phi_{J^\Psi} g = 0$.*

From [13], it is known that (M^{2n}, J, g) is said to be a Kähler-Norden manifold if $\nabla J = 0$, where ∇ is the Levi-Civita connection of the Norden metric g. Hence from Theorem 2.1 and (7), we can define a Kähler-Norden metallic manifold as follows:

Definition. Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. If $\nabla\Psi = 0$, where ∇ is the Levi-Civita connection of g, then the triplet (M^{2n}, Ψ, g) is called a Kähler-Norden metallic manifold.

Curvature properties of Kähler-Norden metallic manifolds can be found in [1].

3. The twin metallic Norden metric

It is well known that an almost Hermitian structure (J, g) defines a fundamental 2-form Ω by $\Omega(X, Y) = g(JX, Y)$. But for an almost Norden manifold (M^{2n}, J, g), the pair (J, g) defines a twin Norden metric $\widetilde{g}$ by $\widetilde{g}(X, Y) = (g \circ J)(X, Y) = g(JX, Y)$, which is a symmetric $(0, 2)$-tensor field, not a 2-form. Analogously, for an almost complex metallic Norden manifold, a twin metallic Norden metric $\widetilde{g}$ is defined as

$$\widetilde{g}(X, Y) = (g \circ \Psi)(X, Y) = g(\Psi X, Y) \tag{8}$$

for any vector fields X, Y on M^{2n}. It is obvious that the twin metallic Norden metric is pure with respect to the almost complex metallic structure Ψ, that is, $\widetilde{g}(\Psi X, Y) = \widetilde{g}(X, \Psi Y)$. It should be noted that both the metrics g and $\widetilde{g}$ are necessarily of the same signature (n, n). Using (4), it follows that

$$\widetilde{g}(\Psi X, Y) = p\widetilde{g}(X, Y) + qg(X, Y). \tag{9}$$

Since the twin metallic Norden metric $\widetilde{g}$ is pure with respect to Ψ, then for the conjugate almost complex metallic structure $\widetilde{\Psi} = pI - \Psi$, we have

$$\widetilde{g}(\widetilde{\Psi}X, Y) = p\widetilde{g}(X, Y) - \widetilde{g}(\Psi X, Y) = p\widetilde{g}(X, Y) - \widetilde{g}(X, \Psi Y) = \widetilde{g}(X, \widetilde{\Psi}Y),$$

this implies that the twin metallic Norden metric $\widetilde{g}$ is also pure with respect to the conjugate almost complex metallic structure $\widetilde{\Psi}$. Moreover

$$\widetilde{g}(\widetilde{\Psi}X, Y) = -qg(X, Y) \text{ and } \widetilde{g}(\widetilde{\Psi}X, \widetilde{\Psi}Y) = -qg(\widetilde{\Psi}X, Y).$$

Next, we apply a ϕ_Ψ-operator on the twin metallic Norden metric $\widetilde{g}$ and using (3), we derive

$$\begin{aligned}(\phi_\Psi \widetilde{g})(X, Y, Z) &= (\Psi X)(\widetilde{g}(Y, Z)) - X(\widetilde{g}(\Psi Y, Z)) \\ &\quad + \widetilde{g}((L_Y\Psi)X, Z) + \widetilde{g}(Y, (L_Z\Psi)X) \\ &= (L_{\Psi X}\widetilde{g} - L_X(\widetilde{g} \circ \Psi))(Y, Z) + \widetilde{g}(Y, \Psi(L_X Z)) - \widetilde{g}(\Psi Y, L_X Z) \\ &= (\phi_\Psi g)(X, \Psi Y, Z) + g(N_\Psi(X, Y), Z), \end{aligned} \tag{10}$$

where N_Ψ is the Nijenhuis tensor field of Ψ. Hence, for a complex metallic Norden manifold, the condition $\phi_\Psi \widetilde{g} = 0$ is equivalent to $\phi_\Psi g = 0$.

Let ∇ be the Levi-Civita connection of the Norden metric g. Then

$$\nabla\widetilde{g} = (\nabla g) \circ \Psi + g \circ (\nabla\Psi) = g \circ (\nabla\Psi).$$

Hence, we immediately observe the following result.

Theorem 3.1. *If (M^{2n}, Ψ, g) is a Kähler-Norden metallic manifold, then the Levi-Civita connection of the Norden metric g coincides with the Levi-Civita of the twin metric $\widetilde{g}$.*

Thus, (Ψ, g) is a Kähler-Norden metallic structure if and only if the twin Norden metallic structure $(\Psi, \widetilde{g})$ is a Kähler-Norden metallic structure. Let the Levi-Civita connections of g and $\widetilde{g}$ be denoted by ∇ and $\widetilde{\nabla}$, respectively. Consider the tensor fields F_Ψ and $\widetilde{F}_\Psi$ of type $(0, 3)$ on M^{2n} as

$$F_\Psi(X, Y, Z) = g((\nabla_X\Psi)Y, Z), \quad \widetilde{F}_\Psi(X, Y, Z) = \widetilde{g}((\widetilde{\nabla}_X\Psi)Y, Z) \tag{11}$$

for any vector fields X, Y, Z on M^{2n}. If the tensor field F_Ψ vanishes identically, then the triplet (M^{2n}, Ψ, g) becomes a Kähler-Norden metallic manifold. These tensor fields satisfy the following properties

$$F_\Psi(X, Y, Z) = F_\Psi(X, Z, Y), \quad \widetilde{F}_\Psi(X, Y, Z) = \widetilde{F}_\Psi(X, Z, Y), \tag{12}$$

$$F_\Psi(X, \Psi Y, \Psi Z) = -qF_\Psi(X, Y, Z), \quad \widetilde{F}_\Psi(X, \Psi Y, \Psi Z) = -q\widetilde{F}_\Psi(X, Y, Z), \tag{13}$$

and moreover, using (9), we have

$$F_\Psi(X, Y, Z) = (\nabla_X\widetilde{g})(Y, Z), \quad \widetilde{F}_\Psi(X, Y, Z) = q(\widetilde{\nabla}_X g)(Y, Z).$$

4. Adapted connections

A linear connection D on an almost Norden manifold (M^{2n}, J, g) is said to be an *adapted connection* with respect to an almost Norden structure (J, g) of M^{2n} if D parallelizes both J and g, that is, $DJ = 0$ and $Dg = 0$. A necessary and sufficient condition for a linear connection D, on an almost Norden manifold (M^{2n}, J, g) with torsion, to be an adapted connection with respect to an almost Norden structure (J, g) on M^{2n} is given in [9]. Moreover if J is integrable, then for a Norden manifold, there exists a unique adapted connection D with respect to (J, g), whose torsion tensor T satisfies [9]:

$$T(X,Y) = T(JX, JY), \quad T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) = 0,$$

where $T(X,Y,Z) = g(T(X,Y),Z)$. Recently, Kumar et al. [15] extensively studied adapted connections on Kähler-Norden golden manifolds. Hence, following the above description, we define an adapted connection for an almost complex metallic Norden manifold as below.

Definition. Let D be a linear connection on an almost complex metallic Norden manifold (M^{2n}, Ψ, g). We say the connection D is an adapted connection with respect to the almost complex metallic Norden structure (Ψ, g) of (M^{2n}, Ψ, g) if D parallelizes both Ψ and g, that is, $D\Psi = 0$ and $Dg = 0$.

On an almost complex metallic Norden manifold (M^{2n}, Ψ, g), using (2), we derive

$$(D_X J^{\Psi})Y = \pm \frac{2}{(p - 2\sigma^c_{p,q})i}(D_X \Psi)Y, \quad N_{J^{\Psi}} = -\frac{4}{p^2 + 4q} N_{\Psi}, \tag{14}$$

therefore we have

$$F_{J^{\Psi}} = \pm \frac{2}{(p - 2\sigma^c_{p,q})i} F_{\Psi}.$$

This allows us to link the notion of adapted connections on almost Norden manifolds and almost complex metallic Norden manifolds, hence we have the following observation immediately.

Theorem 4.1. *Let D be a linear connection on an almost complex metallic Norden manifold (M^{2n}, Ψ, g). Then the connection D is an adapted connection with respect to the almost complex metallic Norden structure (Ψ, g) if and only if D is an adapted connection with respect to its induced almost Norden structure (J^{Ψ}, g).*

Let ∇^a be an adapted connection with respect to an almost complex metallic Norden structure (Ψ, g) of an almost complex metallic Norden manifold (M^{2n}, Ψ, g). Denote by S^a the potential tensor of ∇^a with respect to the Levi-Civita connection ∇ of g, where S^a is a $(1,2)$-tensor field. Then

$$S^a(X,Y) = \nabla^a_X Y - \nabla_X Y, \tag{15}$$

and we can parameterize all the adapted connections to (Ψ, g) as below:

Theorem 4.2. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then the set of all linear connections adapted with respect to an almost complex metallic Norden structure (Ψ, g) of M^{2n} is*

$$\Big\{\nabla + S^a : (\nabla_X \Psi)Y = \Psi S^a(X,Y) - S^a(X, \Psi Y),$$
$$g(S^a(X,Y),Z) + g(S^a(X,Z),Y) = 0\Big\} \tag{16}$$

for any vector fields X, Y, Z on M^{2n}.

Proof. The proof of the theorem follows immediately from the following expressions

$$(\nabla^a_X \Psi)Y = (\nabla_X \Psi)Y - \Psi S^a(X,Y) + S^a(X, \Psi Y),$$

and

$$(\nabla^a_X g)(Y,Z) = -g(S^a(X,Y),Z) - g(S^a(X,Z),Y). \qquad \square$$

Lemma 4.3. *Let ∇^a be an adapted connection on a Kähler-Norden metallic manifold (M^{2n}, Ψ, g) and T^a be the torsion tensor of the adapted connection on M^{2n}. Then*

(i) $g(S^a(X,Y), \Psi Z) = -g(S^a(X,Z), \Psi Y)$,
(ii) $S^a(\Psi X, Y) = \Psi S^a(Y,X) + T^a(\Psi X, Y)$.

Proof. Since ∇^a is an adapted connection on M^{2n}, from (16), we have

$$S^a(X, \Psi Y) = \Psi S^a(X,Y), \quad g(S^a(X,Y),Z) = -g(S^a(X,Z),Y). \tag{17}$$

Therefore

$$g(S^a(X,Y), \Psi Z) = -g(S^a(X, \Psi Z), Y) = -g(S^a(X,Z), \Psi Y).$$

Also

$$\begin{aligned} S^a(\Psi X, Y) &= \nabla^a_{\Psi X} Y - \nabla_{\Psi X} Y \\ &= T^a(\Psi X, Y) + \Psi(\nabla^a_Y X - \nabla_Y X) \\ &= T^a(\Psi X, Y) + \Psi S^a(Y,X). \end{aligned}$$

Hence the proof is complete. $\square$

Theorem 4.4. *Let ∇^a be an adapted connection on a Kähler-Norden metallic manifold (M^{2n}, Ψ, g) and T^a be the torsion tensor of the adapted connection on M^{2n}. Then for any vector fields X, Y, Z on M^{2n}, we have*

$$S^a(X,Y,Z) = \frac{1}{2}\big\{T^a(X,Y,Z) - T^a(Y,Z,X) + T^a(Z,X,Y)\big\}, \tag{18}$$

where $S^a(X,Y,Z) = g(S^a(X,Y),Z)$ and $T^a(X,Y,Z) = g(T^a(X,Y),Z)$.

Proof. Since T^a is the torsion tensor of the adapted connection ∇^a, from (15), we have

$$T^a(X,Y,Z) = S^a(X,Y,Z) - S^a(Y,X,Z). \tag{19}$$

Using (17), we derive

$$T^a(Y,Z,X) = -S^a(Y,X,Z) + S^a(Z,X,Y), \tag{20}$$

$$T^a(Z,X,Y) = S^a(Z,X,Y) + S^a(X,Y,Z). \tag{21}$$

Hence, the assertion follows from (19)–(21). □

Lemma 4.5. *For an almost complex metallic Norden manifold* (M^{2n},Ψ,g)*, we have*

(i) $(\nabla_X\Psi)\Psi Y = \widetilde{\Psi}(\nabla_X\Psi)Y$,
(ii) $(\nabla_X\Psi)Y = -\frac{1}{q}\Psi(\nabla_X\Psi)\Psi Y$,
(iii) $g((\nabla_X\Psi)Y,Z) = g((\nabla_X\Psi)Z,Y)$,
(iv) $g((\nabla_X\Psi)\Psi Y,Z) = g((\nabla_X\Psi)Y,\widetilde{\Psi}Z)$,
(v) $g((\nabla_X\Psi)\Psi Y,Z) = pg((\nabla_X\Psi)Y,Z) - g((\nabla_X\Psi)\Psi Z,Y)$.

For a torsion-free linear connection ∇, Kobayashi and Nomizu (see [14], Theorem 3.4, page 143) introduced a special connection $\overline{\nabla}$ on an almost complex manifold (M^{2n},J,g), which parallelizes the almost complex structure J, that is, $\overline{\nabla}J=0$, and is given by

$$\overline{\nabla}_XY = \nabla_XY - Q(X,Y)$$

for any vector fields X,Y on M^{2n}, where

$$Q(X,Y) = \frac{1}{4}\big\{(\nabla_{JY}J)X + J((\nabla_YJ)X) + 2J((\nabla_XJ)Y)\big\}.$$

Then, after simplification

$$\overline{\nabla}_XY = \nabla_XY + \frac{1}{2}(\nabla_XJ)JY + \frac{1}{4}\big\{(\nabla_YJ)JX - (\nabla_{JY}J)X\big\},$$

this further can be written as

$$\overline{\nabla}_XY = \nabla^0_XY + \frac{1}{4}\big\{(\nabla_YJ)JX - (\nabla_{JY}J)X\big\},$$

where $\nabla^0_XY = \nabla_XY + \frac{1}{2}(\nabla_XJ)JY$ and $\nabla^0J=0$.

Recently, Blaga and Nannicini [2] defined a linear connection ∇^0 on an almost complex metallic Norden manifold (M^{2n},Ψ,g) by

$$\nabla^0_XY = \nabla_XY + \frac{2}{p^2+4q}\Psi(\nabla_X\Psi)Y - \frac{p}{p^2+4q}(\nabla_X\Psi)Y \tag{22}$$

for any vector fields X,Y on M^{2n} and they showed that the connection ∇^0 is an adapted connection with respect to an almost complex metallic Norden structure (Ψ,g), where ∇ is the Levi-Civita connection of g.

Theorem 4.6. *Let* (M^{2n},Ψ,g) *be an almost complex metallic Norden manifold with connection* ∇^0 *which parallelizes* Ψ*. Then a connection* $\overline{\nabla}$ *defined by* $\overline{\nabla}_XY = \nabla^0_XY + \overline{S}(X,Y)$*, for any vector fields* X,Y *on* M^{2n}*, parallelizes* Ψ *if and only if the tensor field* $\overline{S}$ *satisfies* $\overline{S}(X,\Psi Y) = \Psi\overline{S}(X,Y)$*.*

Proof. It is immediate. □

If $\overline{\nabla}$ parallelizes Ψ, then after straightforward calculations and using Theorem 4.6 and Lemma 4.5, we derive

$$\overline{S}(X,Y) = -\frac{1}{p^2+4q}\widetilde{\Psi}(\nabla_Y \Psi)X + \frac{1}{p^2+4q}(\nabla_{\Psi Y}\Psi)X. \tag{23}$$

Hence using (22) and (23), the connection $\overline{\nabla}$ is given by

$$\begin{aligned}\overline{\nabla}_X Y = \nabla_X Y &+ \frac{2}{p^2+4q}\Psi(\nabla_X \Psi)Y - \frac{p}{p^2+4q}(\nabla_X \Psi)Y \\ &- \frac{1}{p^2+4q}\widetilde{\Psi}(\nabla_Y \Psi)X + \frac{1}{p^2+4q}(\nabla_{\Psi Y}\Psi)X.\end{aligned} \tag{24}$$

For an almost complex structure J on M^{2n}, we know

$$\Psi^J = \frac{1}{2}\Big[pI \pm \big((p-2\sigma^c_{p,q})i\big)J\Big],$$

then on substitution the last expression in (24), we obtain

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY + \frac{1}{4}\big\{(\nabla_Y J)JX - (\nabla_{JY}J)X\big\}, \tag{25}$$

which is the connection introduced by Kobayashi and Nomizu in [14]. Thus, we have the following definition:

Definition. Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold and ∇ be a Levi-Civita connection on M^{2n}. We call the connection $\overline{\nabla}$ given by (24), which parallelizes Ψ, a Kobayashi-Nomizu metallic Norden type connection on M^{2n}.

Corollary 4.7. *Let $\overline{T}$ be a torsion tensor of the Kobayashi-Nomizu metallic Norden type connection $\widetilde{\nabla}$ on an almost complex metallic Norden manifold (M^{2n}, Ψ, g). Then*

$$\overline{T}(X,Y) = -\frac{1}{p^2+4q}N_\Psi(X,Y),$$

hence an almost complex metallic structure Ψ is integrable if and only if the Kobayashi-Nomizu metallic Norden type connection $\overline{\nabla}$ on M^{2n} is torsion-free.

Corollary 4.8. *If (M^{2n}, Ψ, g) is a Kähler-Norden metallic manifold, then from (24), we have $\overline{\nabla} = \nabla$, that is, the Kobayashi-Nomizu metallic Norden type connection coincides with the Levi-Civita connection on M^{2n}.*

Remark 4.9. Etayo and Santamaría [5] defined a Yano type connection $\nabla^\star$ on an almost complex manifold (M^{2n}, J, g) as

$$\nabla^\star_X Y = \nabla_X Y + \frac{1}{2}(\nabla_Y J)JX + \frac{1}{4}\big\{(\nabla_X J)JY - (\nabla_{JX}J)Y\big\} \tag{26}$$

for any vector fields X, Y on M^{2n}. Hence using (24) and (25) in (26), we call the following connection

$$\nabla^\star_X Y = \nabla_X Y + \frac{2}{p^2+4q}\Psi(\nabla_Y \Psi)X - \frac{p}{p^2+4q}(\nabla_Y \Psi)X$$

$$-\frac{1}{p^2+4q}\widetilde{\Psi}(\nabla_X\Psi)Y+\frac{1}{p^2+4q}(\nabla_{\Psi X}\Psi)Y, \tag{27}$$

as the Yano metallic Norden type connection on an almost complex metallic Norden manifold (M^{2n},Ψ,g), where ∇ is the Levi-Civita connection on M^{2n}. If $T^\star$ is the torsion tensor of the Yano metallic Norden type connection $\nabla^\star$ on an almost complex metallic Norden manifold (M^{2n},Ψ,g), then

$$T^\star(X,Y)=\frac{1}{p^2+4q}N_\Psi(X,Y),$$

hence an almost complex metallic structure Ψ is integrable if and only if the Yano metallic Norden type connection $\nabla^\star$ on M^{2n} is torsion-free. If (M^{2n},Ψ,g) is a Kähler-Norden metallic manifold, then $\nabla^\star=\nabla$.

Let T^0 be the torsion tensor of the adapted connection ∇^0 given in (22) on (M^{2n},Ψ,g). Then it is given by

$$T^0(X,Y)=\frac{(2\Psi-pI)}{p^2+4q}\Big\{(\nabla_X\Psi)Y-(\nabla_Y\Psi)X\Big\}, \tag{28}$$

this further can be written as

$$T^0(X,Y)=\frac{1}{p^2+4q}\big\{(2\Psi-pI)(\nabla_X\Psi Y-\nabla_Y\Psi X)-(p\Psi+2qI)[X,Y]\big\}$$

and this verifies

$$T^0(\Psi X,Y)+T^0(X,\Psi Y)-pT^0(X,Y)=\frac{(2\Psi-pI)}{p^2+4q}N_\Psi(X,Y),$$

where N_Ψ is the Nijenhuis tensor field Ψ. In particular, if Ψ is integrable, then

$$T^0(\Psi X,Y)+T^0(X,\Psi Y)=pT^0(X,Y).$$

It should be observed that for $p=0$, $q=-1$ and Ψ is integrable, the adapted connection ∇^0 coincides with the adapted canonical connection defined by Ganchev et al. in [8].

Theorem 4.10. *Let ∇^0 be an adapted connection on an almost complex metallic Norden manifold (M^{2n},Ψ,g). Then (M^{2n},Ψ,g) is a Kähler-Norden metallic manifold if and only if the connection ∇^0 is torsion-free.*

Proof. Let the connection ∇^0 be torsion-free. Then from (11) and (28), we have

$$F_\Psi(X,Y,Z)=F_\Psi(Y,X,Z). \tag{29}$$

Then from (12), (13) and (29), we get

$$\begin{aligned}F_\Psi(X,Y,Z)&=-\frac{1}{q}F_\Psi(X,\Psi Y,\Psi Z)\\&=-\frac{1}{q}F_\Psi(\Psi Y,\Psi Z,X)\\&=\frac{1}{q^2}F_\Psi(\Psi Y,\Psi^2 Z,\Psi X)\end{aligned}$$

$$= \frac{p}{q^2} F_\Psi(\Psi Y, \Psi Z, \Psi X) + \frac{1}{q} F_\Psi(\Psi Y, Z, \Psi X)$$
$$= -\frac{p}{q} F_\Psi(X, Y, \Psi Z) - F_\Psi(X, Y, Z),$$

this implies

$$F_\Psi(X, Y, Z) = -\frac{p}{2q} F_\Psi(X, Y, \Psi Z). \tag{30}$$

Therefore

$$F_\Psi(X, Y, \widetilde{\Psi} Z) = -\frac{p}{2q} F_\Psi(X, Y, \Psi \widetilde{\Psi} Z),$$

this further gives

$$F_\Psi(X, Y, \Psi Z) = \frac{p}{2} F_\Psi(X, Y, Z). \tag{31}$$

Hence from (30) and (31), we have

$$F_\Psi(X, Y, Z) = -\frac{p^2}{4q} F_\Psi(X, Y, Z),$$

this implies that the tensor field F vanishes identically and hence (M^{2n}, Ψ, g) is a Kähler-Norden metallic manifold. The converse implication is trivial. □

Salimov [16] defined a special connection of the first type for anti-Hermitian (Norden) manifolds. Following the same technique in [16], we have the following definition.

Definition. A linear connection $\nabla^1_X Y = \nabla_X Y + S^1(X, Y)$ on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is said to be a special connection of the first type if it satisfies $\nabla^1 \widetilde{g} = 0$ and $g(S^1(X, Y), \Psi Z) = g(S^1(X, Z), \Psi Y)$, where ∇ is the Levi-Civita connection of g and S^1 is a $(1, 2)$-tensor field.

By taking the covariant derivative of the twin metallic Norden metric $\widetilde{g}$ with respect to ∇^1, we get

$$(\nabla^1_X \widetilde{g})(Y, Z) = (\nabla_X \widetilde{g})(Y, Z) - g(S^1(X, Y), \Psi Z) - g(S^1(X, Z), \Psi Y). \tag{32}$$

Assume ∇^1 is a special connection of the first type on an almost complex metallic Norden manifold (M^{2n}, Ψ, g). Then from (32), we have

$$g(S^1(X, Y), \Psi Z) = \frac{1}{2}(\nabla_X \widetilde{g})(Y, Z) = \frac{1}{2} g((\nabla_X \Psi)Y, Z),$$

this implies

$$S^1(X, Y) = -\frac{1}{2q} \widetilde{\Psi}(\nabla_X \Psi)Y.$$

Hence the special connection of the first type on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is given by

$$\nabla^1_X Y = \nabla_X Y - \frac{1}{2q} \widetilde{\Psi}(\nabla_X \Psi)Y \tag{33}$$

for any vector fields X, Y on M^{2n}. By straightforward calculations, we can derive

$$(\nabla^1_X \Psi)Y = \frac{1}{2}(\nabla_X \Psi)Y - \frac{1}{2q}\widetilde{\Psi}(\nabla_X \Psi)(\Psi Y), \tag{34}$$

$$\begin{aligned}(\nabla^1_X g)(Y, Z) &= \frac{1}{2q} g((\nabla_X \Psi)(\Psi Y), Z) + \frac{1}{2q} g(Y, \widetilde{\Psi}(\nabla_X \Psi)Z) \\ &= \frac{1}{2q} g(\Psi Y, (\nabla_X \Psi)Z) + \frac{1}{2q} g(\widetilde{\Psi} Y, (\nabla_X \Psi)Z) \\ &= \frac{p}{2q} g((\nabla_X \Psi)Y, Z) \neq 0, \end{aligned} \tag{35}$$

and the torsion tensor T^1 of ∇^1 is given by

$$T^1(X, Y) = -\frac{1}{2q}\widetilde{\Psi}\big\{(\nabla_X \Psi)Y - (\nabla_Y \Psi)X\big\}. \tag{36}$$

Since $(\nabla^1_X \widetilde{g})(Y, Z) = (\nabla^1_X g)(\Psi Y, Z) + g((\nabla^1_X \Psi)Y, Z)$, the special connection of the first type on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is an adapted connection with respect to an almost complex metallic Norden structure (Ψ, g) on M^{2n} if and only if $\nabla^1 g = 0$. Hence from (35) and (36), we have the following observation immediately.

Theorem 4.11. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then the special connection of the first type ∇^1 is not an adapted connection with respect to an almost complex metallic Norden structure (Ψ, g) on M^{2n}. If (M^{2n}, Ψ, g) is a Kähler-Norden metallic manifold, then ∇^1 is an adapted connection with respect to (Ψ, g) and torsion-free on M^{2n}.*

An almost complex metallic Norden manifold (M^{2n}, Ψ, g) is said to be a *Kähler-Norden-Codazzi metallic manifold* [17] if the twin Norden metallic metric $\widetilde{g}$ satisfies the following Codazzi type equation

$$(\nabla_X \widetilde{g})(Y, Z) - (\nabla_Y \widetilde{g})(X, Z) = 0, \tag{37}$$

where ∇ is the Levi-Civita connection of g. The Codazzi type equation is also equivalent to

$$(\nabla_X \Psi)Y - (\nabla_Y \Psi)X = 0. \tag{38}$$

Hence from (36) and (38), we have the following result.

Theorem 4.12. *If an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is a Kähler-Norden-Codazzi metallic manifold, then the special connection of the first type ∇^1 on M^{2n} is torsion-free.*

Using (ii) of Lemma 4.5, we can write

$$\begin{aligned} N_\Psi(X, Y) = \Psi\big\{ &- \frac{1}{q}\big((\nabla_{\Psi X} \Psi)\Psi Y - (\nabla_{\Psi Y} \Psi)\Psi X\big) \\ &- \big((\nabla_X \Psi)Y - (\nabla_Y \Psi)X\big)\big\}. \end{aligned} \tag{39}$$

Hence, we have the following observation immediately.

Theorem 4.13. *The almost complex metallic Norden structure of a Kähler-Norden-Codazzi metallic manifold is always integrable.*

Theorem 4.14. *Let* (M^{2n}, Ψ, g) *be an almost complex metallic Norden manifold with the special connection of the first type* ∇^1. *Then*

$$qT^1(X,Y) + T^1(\Psi X, \Psi Y) = \frac{1}{2}\frac{\widetilde{\Psi}}{\Psi}N_\Psi(X,Y) = -\frac{1}{2q}\widetilde{\Psi}^2 N_\Psi(X,Y).$$

Proof. Since $\widetilde{\Psi} = pI - \Psi$, we can write (36) as

$$T^1(X,Y) = -\frac{p}{2q}\big\{(\nabla_X\Psi)Y - (\nabla_Y\Psi)X\big\} - \frac{1}{2q}\big\{\Psi(\nabla_Y\Psi)X - \Psi(\nabla_X\Psi)Y\big\}.$$

Using (i) of Lemma 4.5, we can write

$$T^1(\Psi X,\Psi Y) = -\frac{p}{2q}\big\{(\nabla_{\Psi X}\Psi)\Psi Y - (\nabla_{\Psi Y}\Psi)\Psi X\big\} - \frac{1}{2}\big\{(\nabla_{\Psi X}\Psi)Y - (\nabla_{\Psi Y}\Psi)X\big\}.$$

Further using (6) and (39), we derive

$$\begin{aligned} qT^1(X,Y) + T^1(\Psi X,\Psi Y) &= \frac{p}{2}\frac{N_\Psi(X,Y)}{\Psi} - \frac{1}{2}N_\Psi(X,Y) \\ &= \frac{1}{2}\frac{\widetilde{\Psi}}{\Psi}N_\Psi(X,Y). \end{aligned}$$

This completes the proof. □

Corollary 4.15. *If the special connection of the first type* ∇^1 *on an almost complex metallic Norden manifold is torsion-free, then the almost complex metallic Norden structure of an almost complex metallic Norden manifold is always integrable.*

Theorem 4.16. *Let* (M^{2n}, Ψ, g) *be an almost complex metallic Norden manifold with the special connection of the first type* ∇^1. *Then the following relation holds*

$$\begin{aligned} \frac{1}{2q}g(N_\Psi(X,Z),Y) &= g(T^1(X,Y),Z) - g(T^1(Z,Y),X) \\ &\quad + \frac{1}{q}g(T^1(\Psi X,Y),\Psi Z) - \frac{1}{q}g(T^1(\Psi Z,Y),\Psi X) \end{aligned}$$

for any vector fields X, Y, Z *on* M^{2n}.

Proof. Using Lemma 4.5 with (36), after straightforward calculations, we derive

$$g(T^1(X,Y),Z) = -\frac{1}{2q}g(\Psi(\nabla_X\Psi)Z,Y) + \frac{1}{2q}g((\nabla_Y\Psi)\Psi X,Z),$$

$$g(T^1(Z,Y),X) = -\frac{1}{2q}g(\Psi(\nabla_Z\Psi)X,Y) + \frac{p}{2q}g((\nabla_Y\Psi)Z,X)$$

$$- \frac{1}{2q} g((\nabla_Y \Psi)\Psi X, Z),$$

$$g(T^1(\Psi X, Y), \Psi Z) = \frac{1}{2} g((\nabla_{\Psi X} \Psi)Z, Y) - \frac{1}{2} g((\nabla_Y \Psi)\Psi X, Z),$$

$$g(T^1(\Psi Z, Y), \Psi X) = \frac{1}{2} g((\nabla_{\Psi Z} \Psi)X, Y) - \frac{p}{2} g((\nabla_Y \Psi)Z, X) + \frac{1}{2} g((\nabla_Y \Psi)\Psi X, Z).$$

Thus taking into account the expression (6), the proof is complete. □

Salimov [16] defined a special connection of the second type for anti-Hermitian (Norden) manifolds. Following the same technique as in [16], we have the following definition.

Definition. A linear connection $\nabla^2_X Y = \nabla_X Y + S^2(X, Y)$ on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is said to be a special connection of the second type if it satisfies $\nabla^2 \widetilde{g} = 0$ and $g(S^2(X, Y), \Psi Z) = g(S^2(Z, Y), \Psi X)$ for any vector fields X, Y, Z on M^{2n}, where ∇ is the Levi-Civita connection of g and S^2 is a $(1, 2)$-tensor field.

Since $(\nabla^2_X \widetilde{g})(Y, Z) = (\nabla^2_X g)(\Psi Y, Z) + g((\nabla^2_X \Psi)Y, Z)$, the special connection of the second type on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) is an adapted connection with respect to an almost complex metallic Norden structure (Ψ, g) on M^{2n} if and only if $\nabla^2 g = 0$. Moreover, if (M^{2n}, Ψ, g) is a Kähler-Norden metallic manifold, then the special connection of the second type ∇^2 is an adapted connection with respect to (Ψ, g) on M^{2n}.

By taking the covariant derivative of the twin metallic Norden metric $\widetilde{g}$ with respect to ∇^2, we get

$$(\nabla^2_X \widetilde{g})(Y, Z) = (\nabla_X \widetilde{g})(Y, Z) - g(S^2(X, Y), \Psi Z) - g(S^2(X, Z), \Psi Y).$$

Since $(\nabla_X \widetilde{g})(Y, Z) = g((\nabla_X \Psi)Y, Z)$ and $\nabla^2 g = 0$, we get

$$g(S^2(X, Y), \Psi Z) + g(S^2(X, Z), \Psi Y) = g((\nabla_X \Psi)Y, Z). \tag{40}$$

This further gives

$$2g(S^2(X, Y), \Psi Z) = g((\nabla_X \Psi)Y, Z) - g((\nabla_Y \Psi)Z, X) + g((\nabla_Z \Psi)X, Y).$$

If (M^{2n}, Ψ, g) is a Kähler-Norden-Codazzi metallic manifold, then the last expression reduces to

$$2g(S^2(X, Y), \Psi Z) = g((\nabla_X \Psi)Y, Z),$$

this further implies

$$S^2(X, Y) = -\frac{1}{2q} \widetilde{\Psi}(\nabla_X \Psi)Y.$$

Hence, we have the following result.

Theorem 4.17. *If (M^{2n}, Ψ, g) is a Kähler-Norden-Codazzi metallic manifold, then the special connection of the second type ∇^2 is given by*

$$\nabla^2_X Y = \nabla_X Y - \frac{1}{2q}\widetilde{\Psi}(\nabla_X \Psi)Y \tag{41}$$

for any vector fields X, Y on M^{2n}.

From (33) and (41), it is obvious that on a Kähler-Norden-Codazzi metallic manifold, the special connections of the first type and of the second type coincide with each other. Next, denote the torsion tensor of the special connection of the second type ∇^2 by T^2. Then from (40), we obtain

$$g(T^2(X,Y), \Psi Z) = g((\nabla_X \Psi)Z, Y) - g((\nabla_Y \Psi)Z, X), \tag{42}$$

further using (iii) of Lemma 4.5, we get

$$g(T^2(X,Y), \Psi Z) = g((\nabla_X \Psi)Y, Z) - g((\nabla_Y \Psi)X, Z). \tag{43}$$

Thus, we have the following observation immediately.

Theorem 4.18. *If (M^{2n}, Ψ, g) is a Kähler-Norden-Codazzi metallic manifold, then the special connection of the second type ∇^2 on M^{2n} is torsion-free.*

Next, by replacing Z by ΨZ in (42) and further using (iii) of Lemma 4.5, we derive

$$g(T^2(X,Y), Z) = -\frac{1}{q}\big\{g((\nabla_X \Psi)\Psi Y, Z) - g((\nabla_Y \Psi)\Psi X, Z)\big\}. \tag{44}$$

Using (i) of Lemma 4.5, we can write (36) as

$$g(T^1(X,Y), Z) = -\frac{1}{2q}\big\{g((\nabla_X \Psi)\Psi Y, Z) - g((\nabla_Y \Psi)\Psi X, Z)\big\}, \tag{45}$$

hence from (44) and (45), we have the following theorem.

Theorem 4.19. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then the torsion tensors of the special connection of the first type and of the second type satisfy*

$$T^1(X,Y) = \frac{1}{2}T^2(X,Y)$$

for any vector fields X, Y on M^{2n}.

Consequently, from Theorem 4.16, we have the following result.

Theorem 4.20. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold with the special connection of the second type ∇^2. Then the following relation holds*

$$\begin{aligned} g(N_\Psi(X,Z), Y) &= qg(T^2(X,Y), Z) - qg(T^2(Z,Y), X) \\ &\quad + g(T^2(\Psi X, Y), \Psi Z) - g(T^2(\Psi Z, Y), \Psi X) \end{aligned}$$

for any vector fields X, Y, Z on M^{2n}.

Theorem 4.21. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold with the special connection of the second type ∇^2. Then the following relation holds*

$$g(N_\Psi(X,Y),Z)=g(T^2(\Psi X,Y),\Psi Z)+g(T^2(X,\Psi Y),\Psi Z)-pg(T^2(X,Y),\Psi Z)$$

for any vector fields X, Y, Z on M^{2n}.

Proof. By replacing X by ΨX in (43) and further using (i) of Lemma 4.5, we derive

$$g(T^2(\Psi X,Y),\Psi Z) = g((\nabla_{\Psi X}\Psi)Y,Z) - pg((\nabla_Y\Psi)X,Z) + g(\Psi(\nabla_Y\Psi)X,Z),$$

and similarly

$$g(T^2(X,\Psi Y),\Psi Z) = pg((\nabla_X\Psi)Y,Z) - g(\Psi(\nabla_X\Psi)Y,Z) - g((\nabla_{\Psi Y}\Psi)X,Z).$$

On using (6) and (43) in the addition of the last two expressions, the proof is immediate. □

Remark 4.22. It is important to note that from Theorems 4.20 and 4.21, we can deduce the conditions for the integrability of an almost complex metallic structure Ψ of an almost complex metallic Norden manifold with the special connection of the second type ∇^2.

5. Classification of almost Norden manifolds

Let (M^{2n}, J, g) be an almost complex manifold with a Norden metric and $\widetilde{g}$ be the associated twin metric on M^{2n}. If ∇ is the Levi-Civita connection of g, then Ganchev and Borisov [7] defined a tensor field F_J of type $(0,3)$ on M^{2n} by

$$F_J(X,Y,Z) = (\nabla_X\widetilde{g})(Y,Z) = g((\nabla_X J)Y,Z)$$

for any vector fields X, Y, Z on M^{2n}. Then the 1-form associated with F_J is given by

$$\theta_J(X) = \sum_{1\leq i,j\leq 2n} g^{ij} g((\nabla_{e_i}J)e_j, X)$$

for any vector field X on M^{2n}, where $\{e_i\}_{i=1}^{2n}$ is a local basis of M^{2n} and g^{ij} is the inverse matrix of g. Then Ganchev and Borisov [7] established a classification of the almost complex manifolds with a Norden metric with respect to the covariant derivative of the almost complex structure as below:

Theorem 5.1. *Let (M^{2n}, J, g) be an almost Norden manifold. Then the eight classes of M^{2n} are characterized by the following conditions*:

(i) *The class $\mathcal{W}_0$ or Kähler-Norden manifold*:

$$F_J(X,Y,Z) = 0$$

for any vector fields X, Y, Z on M^{2n}.

(ii) *The class $\mathcal{W}_1$ or conformal Kähler-Norden manifold*:

$$F_J(X,Y,Z) = \frac{1}{2n}\big\{g(X,Y)\theta_J(Z) + g(X,Z)\theta_J(Y) + g(X,JY)\theta_J(JZ) + g(X,JZ)\theta_J(JY)\big\}$$

for any vector fields X, Y, Z on M^{2n}.

(iii) *The class $\mathcal{W}_2$ or special Norden manifold*:

$$F_J(X,Y,JZ) + F_J(Y,Z,JX) + F_J(Z,X,JY) = 0, \quad \theta_J = 0$$

for any vector fields X, Y, Z on M^{2n} or equivalently $N_J = 0$, $\theta_J = 0$.

(iv) *The class $\mathcal{W}_3$ or quasi-Kähler-Norden manifold*:

$$F_J(X,Y,Z) + F_J(Y,Z,X) + F_J(Z,X,Y) = 0$$

for any vector fields X, Y, Z on M^{2n} or equivalently $\widetilde{N}_J = 0$.

(v) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or complex Norden manifold*:

$$F_J(X,Y,JZ) + F_J(Y,Z,JX) + F_J(Z,X,JY) = 0$$

for any vector fields X, Y, Z on M^{2n} or equivalently $N_J = 0$.

(vi) *The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ or semi-Kähler-Norden manifold*:

$$\theta_J = 0.$$

(vii) *The class $\mathcal{W}_1 \oplus \mathcal{W}_3$*:

$$\begin{aligned} &F_J(X,Y,Z) + F_J(Y,Z,X) + F_J(Z,X,Y) \\ = &\frac{1}{n}\big\{g(X,Y)\theta_J(Z) + g(Z,X)\theta_J(Y) + g(Y,Z)\theta_J(X) \\ &+ g(X,JY)\theta_J(JZ) + g(Y,JZ)\theta_J(JX) + g(Z,JX)\theta_J(JY)\big\} \end{aligned}$$

for any vector fields X, Y, Z on M^{2n}.

(viii) *The class $\mathcal{W}$ or the whole class of almost Norden manifolds*: *no condition.*

From [6], it is known that for an almost Norden manifold, there exists a local basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ of T_pM, $p \in M$, satisfying

$$JX_i = Y_i, \quad g(X_i, X_j) = g(Y_i, Y_j) = 0, \quad g(X_i, Y_j) = \partial_{ij}, \quad \forall i, j \in \{1, \dots, n\},$$

which is called an adapted local basis to (J, g) and for this adapted local basis, we have (see [4])

$$\theta_J(X) = \sum_{i=1}^{n}\big\{g((\nabla_{X_i}J)Y_i, X) + g((\nabla_{Y_i}J)X_i, X)\big\}.$$

If (M^{2n}, Ψ, g) is an almost complex metallic Norden manifold, then using the adapted local basis to the almost complex metallic Norden structure (Ψ, g), the 1-form θ_Ψ associated with F_Ψ is given by

$$\theta_\Psi(X) = \sum_{i=1}^{n}\big\{g((\nabla_{X_i}\Psi)Y_i, X) + g((\nabla_{Y_i}\Psi)X_i, X)\big\}.$$

Using (2), it is easy to show that

$$g(X, J^{\Psi}Y) = \frac{2}{(p-2\sigma^c_{p,q})i} g(X, \Psi Y) - \frac{p}{(p-2\sigma^c_{p,q})i} g(X, Y), \tag{46}$$

$$(\nabla_X J^{\Psi})Y = \frac{2}{(p-2\sigma^c_{p,q})i} (\nabla_X \Psi)Y, \tag{47}$$

this implies

$$\theta_{J^{\Psi}}(X) = \frac{2}{(p-2\sigma^c_{p,q})i} \theta_{\Psi}(X), \tag{48}$$

$$\theta_{J^{\Psi}}(J^{\Psi}X) = \Big(\frac{2}{(p-2\sigma^c_{p,q})i}\Big)^2 \theta_{\Psi}(\Psi X) - \frac{2p}{\big((p-2\sigma^c_{p,q})i\big)^2} \theta_{\Psi}(X), \tag{49}$$

and

$$F_{J^{\Psi}}(X, Y, Z) = \frac{2}{(p-2\sigma^c_{p,q})i} F_{\Psi}(X, Y, Z). \tag{50}$$

Hence using (2), (46)-(50), we have the analogous classification of the almost complex metallic Norden manifolds as below.

Theorem 5.2. *Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then the eight classes of M^{2n} are characterized by the following conditions*:

(i) *The class $\mathcal{W}_0$ or Kähler-Norden metallic manifold*:

$$F_{\Psi}(X, Y, Z) = 0$$

for any vector fields X, Y, Z on M^{2n}.

(ii) *The class $\mathcal{W}_1$ or conformal Kähler-Norden metallic manifold*:

$$\begin{aligned} &F_{\Psi}(X, Y, Z) \\ =\; &\frac{1}{n(p^2+4q)}\big\{g(X, Y)\theta_{\Psi}(p\Psi Z + 2qZ) + g(X, Z)\theta_{\Psi}(p\Psi Y + 2qY)\big\} \\ &+ \frac{1}{n(p^2+4q)}\big\{g(X, \Psi Y)\theta_{\Psi}(pZ - 2\Psi Z) + g(X, \Psi Z)\theta_{\Psi}(pY - 2\Psi Y)\big\} \end{aligned}$$

for any vector fields X, Y, Z on M^{2n}.

(iii) *The class $\mathcal{W}_2$ or special Norden metallic manifold*:

$$F_{\Psi}(X, Y, \Psi Z) + F_{\Psi}(Y, Z, \Psi X) + F_{\Psi}(Z, X, \Psi Y) = 0, \theta_{\Psi} = 0$$

for any vector fields X, Y, Z on M^{2n} or equivalently $N_{\Psi} = 0$, $\theta_{\Psi} = 0$.

(iv) *The class $\mathcal{W}_3$ or quasi-Kähler-Norden metallic manifold*:

$$F_{\Psi}(X, Y, Z) + F_{\Psi}(Y, Z, X) + F_{\Psi}(Z, X, Y) = 0$$

for any vector fields X, Y, Z on M^{2n} equivalently $\widetilde{N}_{\Psi} = 0$.

(v) *The class* $\mathcal{W}_1 \oplus \mathcal{W}_2$ *or Norden metallic manifold*:

$$F_\Psi(X,Y,\Psi Z) + F_\Psi(Y,Z,\Psi X) + F_\Psi(Z,X,\Psi Y) = 0$$

for any vector fields X, Y, Z *on* M^{2n} *or equivalently* $N_\Psi = 0$.

(vi) *The class* $\mathcal{W}_2 \oplus \mathcal{W}_3$ *or semi-Kähler-Norden metallic manifold*:

$$\theta_\Psi = 0.$$

(vii) *The class* $\mathcal{W}_1 \oplus \mathcal{W}_3$:

$$\begin{aligned} &F_\Psi(X,Y,Z) + F_\Psi(Y,Z,X) + F_\Psi(Z,X,Y) \\ = &\frac{2}{n(p^2+4q)}\big\{g(X,Y)\theta_\Psi(p\Psi Z + 2qZ) + g(Y,Z)\theta_\Psi(p\Psi X + 2qX) \\ &\qquad + g(Z,X)\theta_\Psi(p\Psi Y + 2qY)\big\} \\ &+ \frac{2}{n(p^2+4q)}\big\{g(X,\Psi Y)\theta_\Psi(pZ - 2\Psi Z) + g(Y,\Psi Z)\theta_\Psi(pX - 2\Psi X) \\ &\qquad + g(Z,\Psi X)\theta_\Psi(pY - 2\Psi Y)\big\} \end{aligned}$$

for any vector fields X, Y, Z *on* M^{2n}.

(viii) *The class* $\mathcal{W}$ *or the whole class of almost Norden metallic manifolds*: *no condition.*

It is known that an adapted connection D with respect to an almost Norden structure (J, g) on an almost Norden manifold (M^{2n}, J, g) is said to be a *canonical connection* if its torsion tensor T satisfies

$$T(X,Y,Z) + T(Y,Z,X) = T(JX,Y,JZ) + T(Y,JZ,JX)$$

for any vector fields X, Y, Z on M^{2n}. From [9], we know that on an almost Norden manifold there exists a unique canonical connection. In [9], Ganchev and Mihova defined 1-form t associated with the torsion tensor T of the canonical connection D on (M^{2n}, J, g) as

$$t(v) = \sum_{1\le i,j\le 2n} g^{ij} g(T(v,e_i),e_j),$$

where $\{e_i\}_{i=1}^{2n}$ is a local basis of M^{2n}, $p \in M^{2n}$, $v \in T_pM^{2n}$ and g^{ij} is the inverse matrix of g. Later, Ganchev and Mihova [9] provided a characterization of the eight classes of almost Norden manifolds by means of conditions on the torsion tensor of the canonical connection and its associated 1-form.

From Theorem 4.1, we know that the connection D is an adapted connection with respect to an almost complex metallic Norden structure (Ψ, g) if and only if D is an adapted connection with respect to its induced almost Norden structure (J^Ψ, g). Hence, we can define a canonical connection on an almost complex metallic Norden manifold (M^{2n}, Ψ, g) as below.

Definition. Let (M^{2n}, Ψ, g) be an almost complex metallic Norden manifold. Then a connection ∇^c on (M^{2n}, Ψ, g) is said to be a canonical connection if ∇^c

is an adapted connection with respect to the induced almost Norden structure (J^{Ψ}, g) and satisfies

$$T^c(X,Y,Z) + T^c(Y,Z,X) = T^c(J^{\Psi}X, Y, J^{\Psi}Z) + T^c(Y, J^{\Psi}Z, J^{\Psi}X) \tag{51}$$

for any vector fields X, Y, Z on M^{2n}, where T^c is the torsion tensor of ∇^c.

Inspired by the characterization of the eight classes of almost Norden manifolds given by Ganchev and Mihova [9], we also characterize analogous classes for an almost complex metallic Norden manifold (M^{2n}, Ψ, g), by means of conditions on the torsion tensor T^c of the canonical connection ∇^c, which is adapted with respect to the induced almost Norden structure (J^{Ψ}, g) on M^{2n} as below.

Theorem 5.3. *Let (Ψ, g) be an almost complex metallic Norden structure with the induced almost Norden structure (J^{Ψ}, g) on an almost complex metallic Norden manifold (M^{2n}, Ψ, g). If T^c is the torsion tensor of the canonical connection ∇^c on (M^{2n}, Ψ, g), then the eight classes for (M^{2n}, Ψ, g) can be characterized by the following conditions*:

(i) *The class $\mathcal{W}_0$ or Kähler-Norden metallic manifold*:

$$T^c(X,Y) = 0$$

for any vector fields X, Y on M^{2n}.

(ii) *The class $\mathcal{W}_1$ or conformal Kähler-Norden metallic manifold*:

$$T^c(X,Y) = \frac{1}{n(p^2+4q)}\big\{t^c(X)(p\Psi Y + 2qY) - t^c(Y)(p\Psi X + 2qX) + t^c(\Psi X)(pY - 2\Psi Y) - t^c(\Psi Y)(pX - 2\Psi X)\big\}$$

for any vector fields X, Y on M^{2n}, where t^c is 1-form associated with the torsion tensor T^c.

(iii) *The class $\mathcal{W}_2$ or special Norden metallic manifold*:

$$(p^2+2q)T^c(X,Y) = -2T^c(\Psi X, \Psi Y) + p\big\{T^c(\Psi X, Y) + T^c(X, \Psi Y)\big\},$$

and

$$t^c(X) = 0$$

for any vector fields X, Y on M^{2n}.

(iv) *The class $\mathcal{W}_3$ or quasi-Kähler-Norden metallic manifold*:

$$T^c(\Psi X, Y) + \Psi T^c(X,Y) = pT^c(X,Y)$$

for any vector fields X, Y on M^{2n}.

(v) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or Norden metallic manifold*:

$$(p^2+2q)T^c(X,Y) = -2T^c(\Psi X, \Psi Y) + p\big\{T^c(\Psi X, Y) + T^c(X, \Psi Y)\big\}$$

for any vector fields X, Y on M^{2n}.

(vi) *The class* $\mathcal{W}_2 \oplus \mathcal{W}_3$ *or semi-Kähler-Norden metallic manifold*:

$$t^c(X) = 0$$

for any vector field X *on* M^{2n}.

(vii) *The class* $\mathcal{W}_1 \oplus \mathcal{W}_3$:

$$pT^c(X,Y) - \left\{T^c(\Psi X, Y) + \Psi(T^c(X,Y))\right\} = \frac{1}{n}\left\{t^c(Y)\Psi X - t^c(\Psi Y)X\right\}$$

for any vector fields X, Y *on* M^{2n}.

(viii) *The class* $\mathcal{W}$ *or the whole class of almost Norden metallic manifolds*: *no condition.*

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