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MORE PROPERTIES OF WEIGHTED BEREZIN TRANSFORM IN THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. We exhibit various properties of the weighted Berezin operator T_α and its iteration T_α^k on $L^p(\tau)$, where $\alpha > -1$ and τ is the invariant measure on the complex unit ball B_n. Iterations of T_α on $L^1_R(\tau)$ the space of radial integrable functions have performed important roles in proving $\mathcal{M}$-harmonicity of bounded functions with invariant mean value property. We show differences between the case of $1 < p < \infty$ and $p = 1, \infty$ under the infinite iteration of T_α or the infinite summation of iterations, most of which are extensions or related assertions to the propositions of the previous results.

1. Introduction and Preliminaries

Let B_n be the unit ball of $\mathbb{C}^n$ and let ν be the Lebesgue measure normalized to $\nu(B_n) = 1$. For $\alpha > -1$, we define a positive measure ν_α by

$$d\nu_\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|z|^2)^\alpha d\nu(z)$$

so that $\nu_\alpha(B_n) = 1$. For such α and $f \in L^1(B_n, \nu_\alpha)$, the weighted Berezin transform $T_\alpha f$ on B_n is defined by

$$(T_\alpha f)(z) \ = \int_{B_n} f\big(\varphi_z(w)\big) \, d\nu_\alpha(w) \ \text{ for } z \in B_n,$$

where $\varphi_a \in \text{Aut}(B_n)$ is the canonical automorphism. Equivalently, we can write

$$(T_\alpha f)(z) \ = \int_{B_n} f(w) \, \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}} \, d\nu_\alpha(w). \tag{1}$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B_n)$ by

$$(\tilde{\Delta} f)(z) \ = \ \Delta(f \circ \varphi_z)(0).$$

The $\mathcal{M}$-harmonic functions in B_n are those for which $\tilde{\Delta} f = 0$.

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If a function $f \in L^1(B_n, \nu_\alpha)$ is $\mathcal{M}$-harmonic, then $f \circ \psi$ is also $\mathcal{M}$-harmonic for every $\psi \in \mathrm{Aut}(B_n)$. Hence for a given $\alpha > -1$, bounded $\mathcal{M}$-harmonic function f satisfies an invariant mean value property

$$\int_{B_n} (f \circ \psi)\, d\nu_\alpha = f(\psi(0)) \ \text{ for every } \psi \in \mathrm{Aut}(B_n),$$

which is equivalent to saying that

$$(T_\alpha f)(z) = f(z) \ \text{ for every } z \in B_n.$$

Conversely, Furstenberg ([2],[3]) gave an abstract proof that a bounded function on the unit ball which satisfies an invariant mean value property is $\mathcal{M}$-harmonic. Later, Ahern et al.([1]) gave an analytic proof that $f \in L^\infty(B_n)$ satisfying $T_0 f = f$ is $\mathcal{M}$-harmonic, and $f \in L^1(B_n, \nu_\alpha)$ satisfying $T_0 f = f$ has to be $\mathcal{M}$-harmonic if and only if $n \leq 11$. In the polydisc D^n, the author([4]) proved that $f \in L^p(D^n)$ which satisfies an invariant mean value property need not be $n-$harmonic when $1 < p < \infty$.

In 2010, [5] gave an analytic proof that for every given $\alpha > -1$, $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$-harmonic. In [5], the author used the spectral theory and interation of T_α on the commutative Banach algebra $L^1_R(\tau)$, the space of all radial function f on B_n integrable with respect to the invariant measure τ defined by

$$d\tau(z) \ = \ (1 - |z|^2)^{-n-1}\, d\nu(z).$$

The invariant measure τ on B_n satisfies

$$\int_{B_n} f\, d\tau \ = \ \int_{B_n} (f \circ \psi)\, d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in \mathrm{Aut}(B_n)$. Even though τ is not a finite measure on B_n so that a non-zero constant does not belong to $L^1(\tau)$, T_α on $L^\infty(B_n)$ is the adjoint of T_α on $L^1(\tau)$ in the sense that

$$\int_{B_n} (T_\alpha f) \cdot g\, d\tau \ = \ \int_{B_n} f \cdot (T_\alpha g)\, d\tau \tag{2}$$

for $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. Since $L^\infty(B_n)$ is a dual space of $L^1(\tau)$, the spectrum of T_α on $L^\infty(B_n)$ is the same as the spectrum of T_α on $L^1(B_n, \tau)$.

For $1 \leq p \leq \infty$, we denote $L^p_R(\tau)$ as the subspace of $L^p(B_n, \tau)$ which consists of radial functions, which means that $f \in L^p_R(\tau)$ if and only if $f \in L^p(\tau)$ and $f(z) = f(|z|)$ for all $z \in B_n$. In [5], the key step to the proof of the main theorem is Lemma 2.1 which states that

$$\lim_{k\to\infty} \|T^k_\alpha (I - T_\alpha)\| = 0 \quad \text{on} \quad L^1_R(\tau). \tag{3}$$

Motivated by Lemma 2.1 of [5] the author([6]) developed further theory of the iterations of T_α on $L^1(\tau)$ and $L^p(\tau)$ for $1 < p < \infty$. There are three propositions of [6] stating that

If $f \in L^1_R(\tau)$, then

$$\lim_{k\to\infty} \int_{B_n} \left|T^k_\alpha f\right|\, d\tau \ = \left|\int_{B_n} f\, d\tau\right|. \tag{4}$$

If $1 < p < \infty$ and $f \in L^p(\tau)$, then

$$\lim_{k\to\infty} \int_{B_n} |T^k_\alpha f|^p\, d\tau = 0. \tag{5}$$

If $f \in L^1(\tau)$ and $z \in B_n$, then

$$\sum_{k=0}^{\infty} |T_\alpha^k f(z)| < \infty. \tag{6}$$

The main results of this paper are related with each of three propositions of [6]. Indeed, we extend the results of of three propositions of [6] to show clear differences between the case of $1 < p < \infty$ and $p = 1, \infty$ under the infinite iteration of T_α or the infinite summation of iterations. In section 2, we propose three new propositions and a lemma about additional properties iterations of T_α on $L^p(\tau)$ for $1 \le p < \infty$ and $p = 1, \infty$. Throughout the paper α is an arbitrarily given real number with $\alpha > -1$.

2. The iterations of T_α

This section contains main results of the paper most of which are extensions or related assertions to the propositions of the previous paper [6]. Indeed in Proposition 3.2 of [6], the author showed that $T_\alpha^k f$ converges pointwise to zero in B_n even though $T_\alpha^k f$ generally does not converges to zero in norm when $f \in L^1(\tau)$. Next proposition shows much more is true for $f \in L^1(\tau)$.

Proposition 2.1. *For $f \in L^1(\tau)$, if $\{T_\alpha^k f\}$ has a subsequence that converges weakly, then*

$$\lim_{k\to\infty} \int_{B_n} |T_\alpha^k f| \, d\tau \; = \; 0.$$

Proof. Let $\{T_\alpha^{m_k} f\}$ be a subsequence that converges weakly to some $g \in L^1(\tau)$, then for every $\ell \in L^\infty(B_n)$ we get

$$\begin{aligned} \int_{B_n} (g - T_\alpha g)\cdot \ell \, d\tau & \\ &= \left(\int_{B_n} g \cdot \ell \, d\tau \; - \; \int_{B_n} (\, T_\alpha^{m_k} f \,) \cdot \ell \, d\tau \right) \\ &+ \left(\int_{B_n} (\, T_\alpha^{m_k} f \,) \cdot \ell \, d\tau \; - \; \int_{B_n} (\, T_\alpha^{m_k+1} f \,) \cdot \ell \, d\tau \right) \\ &+ \left(\int_{B_n} (\, T_\alpha^{m_k+1} f \,) \cdot \ell \, d\tau \; - \; \int_{B_n} (\, T_\alpha g \,) \cdot \ell \, d\tau \right) \end{aligned}$$

as $k \to \infty$ the first and the third terms of right side converge to zero since $T_\alpha^{m_k} \to g$ weakly and the second term converges to zero by Corollary 2.1 of [5]. Hence $T_\alpha g = g$ on B_n.

And for every $z \in B_n$ we have

$$\begin{aligned} |g(z)| \;&=\; \big|T_\alpha g(z)\big| \;\le\; \int_{B_n} |g(w)| \frac{(1-|w|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}} \, d\tau(w) \\ &\le\; c_\alpha \sup_{z\in B_n} \left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \right)^{n+1+\alpha} \int_{B_n} |g| \, d\tau \\ &=\; c_\alpha \int_{B_n} |g| \, d\tau. \end{aligned}$$

Thus g is a bounded function on B_n satisfying $T_\alpha g = g$, which means that g is $\mathcal{M}$-harmonic by Theorem 1.1 of [5]. From 4.2.3 of [7], the radialization of such g is a constant. But a non-zero constant can not belong to $L^q(\tau)$, which forces g to be the constant zero.

Thus by Masur's lemma stating that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit, for every given $\epsilon > 0$ there exists an operator

$$S \ = \ \sum_{j=1}^{N} c_j \ T_\alpha^{m_{k_j}} \ (\ 0 \le c_j \le 1 \ , \ \sum c_j = 1 \ )$$

on $L^1(\tau)$ such that $\|Sf\|_{L^1(\tau)} \ < \ \epsilon$. For $k \ge 0$,

$$\|T_\alpha^k f\|_{L^1(\tau)} \ \le \ \|T_\alpha^k Sf\|_{L^1(\tau)} \ + \ \|T_\alpha^k f - T_\alpha^k Sf\|_{L^1(\tau)}$$

where

$$\|T_\alpha^k Sf\|_{L^1(\tau)} \le \|Sf\|_{L^1(\tau)} < \epsilon$$

and by (1.3) we get

$$\lim_{k\to\infty} \|T_\alpha^k f - T_\alpha^k Sf\|_{L^1(\tau)} \ = \ 0.$$

Therefore,

$$\lim_{k\to\infty} \|T_\alpha^k f\|_{L^1(\tau)} \ = \ 0$$

and this completes the proof. □

Corollary 2.2. *There exists $f \in L^\infty(B_n)$ for which $\lim_{k\to\infty} T_\alpha^k f$ does not exist pointwise.*

Proof. Assume that lim $T_\alpha^k \ell$ exists for every $\ell \in L^\infty(B_n)$ then for any $g \in L_R^1(\tau)$ with $\int_{B_n} g \ d\tau \ne 0$,

$$\lim_{k\to\infty} \int_{B_n} \left(T_\alpha^k g\right) \ \ell \ d\tau \ = \ \lim_{k\to\infty} \int_{B_n} g \ \left(T_\alpha^k \ell\right) \ d\tau$$

exists.

This means $\{T_\alpha^k g\}$ converges weakly. Since $L^1(\tau)$ is weak complete, by Proposition 2.1,

$$\lim_{k\to\infty} \|T_\alpha^k g\|_{L^1(\tau)} \ = \ 0,$$

which contradicts the Proposition 3.2 of [6] asserting that

$$\lim_{k\to\infty} \|T_\alpha^k g\|_{L^1(\tau)} \ = \left| \int_{B_n} g \ d\tau \right| \ne 0. \tag{7}$$

□

Now we recall Lemma 2.1 of [5] which states that

$$\lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha) f\|_{L^1(\tau)} = 0 \quad \text{for all} \quad f \in L_R^1(\tau). \tag{8}$$

Thus by (2.1) and (2.2) the space $(I - T_\alpha)L_R^1(\tau)$ is not dense in $L_R^1(\tau)$. Since $L_R^\infty(B_n)$ is the dual space of $L_R^1(\tau)$ where T_α is self-adjoint, by Lemma 2.1 of [5] we also have

$$\lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha) f\|_\infty = 0 \quad \text{for all} \quad f \in L_R^\infty(B_n). \tag{9}$$

Meanwhile, if $f = c$ a nonzero constant, then $T_\alpha^k c = c$ which does not tend to zero as $k \to \infty$. Hence, we can see that $(I - T_\alpha)L_R^\infty(B_n)$ is not dense in $L_R^\infty(B_n)$ either.

However, next proposition shows that for $1 < p < \infty$, $(I - T_\alpha)L^p(\tau)$ is dense in $L^p(\tau)$. We will use the formula

$$d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z) = c_\alpha(1 - |z|^2)^{n+1+\alpha} d\tau(z),$$

where

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}.$$

Proposition 2.3. *If* $1 < p < \infty$, *then*

$$\overline{(I - T_\alpha)L^p(\tau)} = L^p(\tau).$$

Proof. Let Φ be a bounded linear functional on $L^p(\tau)$, then there is a $g \in L^q(\tau)$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\Phi(f) = \int_{B_n} fg \; d\tau,$$

for every $f \in L^p(\tau)$. Assume that $\Phi(h) = 0$ for all $h \in (I - T_\alpha)L^p(\tau)$. Then

$$\int_{B_n} (f - T_\alpha f) \; g \; d\tau = 0$$

for every $f \in L^p(\tau)$ which means, by sepf-adjointness of T_α

$$\int_{B_n} f \; (g - T_\alpha g) \; d\tau = 0$$

for every $f \in L^p(\tau)$, which implies $g = T_\alpha g$. However, for every $z \in B_n$

$$\begin{aligned}
|g(z)| &= |T_\alpha g(z)| \\
&\le \int_{B_n} |g(w)| \frac{(1-|z|^2)^{n+1+\alpha}}{|1 - \langle z, w\rangle|^{2n+2+2\alpha}} \; d\nu_\alpha(w) \\
&= c_\alpha (1-|z|^2)^{n+1+\alpha} \int_{B_n} |g(w)| \frac{(1-|w|^2)^{n+1+\alpha}}{|1 - \langle z, w\rangle|^{2n+2+2\alpha}} \; d\tau(w) \\
&\le c_\alpha (1-|z|^2)^{n+1+\alpha} \; \|g\|_{L^q(\tau)} \left(\int_{B_n} \frac{(1-|w|^2)^{p(n+1+\alpha)}(1-|w|^2)^{-n-1}}{|1 - \langle z, w\rangle|^{p(2n+2+2\alpha)}} \; d\nu(w) \right)^{\frac{1}{p}} \\
&\le C_{\alpha,p} (1-|z|^2)^{n+1+\alpha} \; \|g\|_{L^q(\tau)} (1-|z|^2)^{-n-1-\alpha} \\
&\le C_{\alpha,p} \; \|g\|_{L^q(\tau)},
\end{aligned}$$

by 1.4.10 of [7] where $C_{\alpha,p}$ is a constant independent of z. Thus g is a bounded function on B_n satisfying $T_\alpha g = g$, which means that g is $\mathcal{M}$-harmonic by Theorem 1.1 of [5]. From 4.2.3 of [7], the radialization of such g is a constant. But a non-zero constant can not belong to $L^q(\tau)$, which forces g to be the constant zero. Therefore, we have $\Phi(f) = 0$ for all $f \in L^p(\tau)$.
From Hahn-Banach theorem, we conclude that $(I - T_\alpha)L^p(\tau)$ is dense in $L^p(\tau)$. □

For $1 < p < \infty$, we have characterized the closure of $(I - T_\alpha)L^p(\tau)$ in terms of the iterations of T_α on $L^p(\tau)$. Next proposition, we characterize the space $(I - T_\alpha)L^p(\tau)$ in terms of the iterations of T_α on $L^p(\tau)$ when $1 < p \le \infty$ by using the duality property of $L^p(\tau)$.

Proposition 2.4. *For* $1 < p \leq \infty$,

$$(I - T_\alpha)L^p(\tau) = \left\{ f \in L^p(\tau) \;\Big|\; \limsup_{m\to\infty} \left\| \sum_{k=0}^{m} T_\alpha^k f \right\|_{L^p(\tau)} < \infty \right\}.$$

Proof. If $g \in L^p(\tau)$ and $f = g - T_\alpha g$, then

$$\left\| \sum_{k=0}^{m} T_\alpha^k f \right\|_{L^p(\tau)} = \|g - T_\alpha^{m+1} g\|_{L^p(\tau)} \leq 2\|g\|_{L^p(\tau)}.$$

Hence we have

$$(I - T_\alpha)L^p(\tau) \subset \left\{ f \in L^p(\tau) \;\Big|\; \limsup_{m\to\infty} \left\| \sum_{k=0}^{m} T_\alpha^k f \right\|_{L^p(\tau)} < \infty \right\}.$$

On the other hand, let $f \in L^p(\tau)$ satisfy

$$\limsup_{m\to\infty} \left\| \sum_{k=0}^{m} T_\alpha^k f \right\|_{L^p(\tau)} = M < \infty.$$

Let us denote that

$$f_k = \sum_{j=0}^{k} T_\alpha^j f.$$

Then $f_k - T_\alpha f_k = f - T_\alpha^{k+1} f$. Hence, if we define

$$F_m = \frac{1}{m+1} \sum_{k=0}^{m} f_k,$$

then we have $\|F_m\|_{L^p(\tau)} \leq M$ and

$$\begin{aligned} (I - T_\alpha)F_m &= \frac{1}{m+1} \sum_{k=0}^{m} (I - T_\alpha) f_k \\ &= \frac{1}{m+1} \sum_{k=0}^{m} (f - T_\alpha^{k+1} f) \\ &= f - \frac{1}{m+1} \sum_{k=0}^{m} T_\alpha^{k+1} f. \end{aligned}$$

Hence, as $m \to \infty$

$$\|(I - T_\alpha)F_m - f\|_{L^p(\tau)} \leq \frac{1}{m+1} M \to 0.$$

Since $\{F_m\}$ is norm bounded, it has a subspace $\{F_{m_j}\}$ that converges weak* to some $h \in L^p(\tau)$ and the operator $I - T_\alpha$ is self-adjoint, which makes $\{(I - T_\alpha)F_{m_j}\}$ converge to $(I - T_\alpha)h$ weak* in $L^p(\tau)$. Since $(I - T_\alpha)F_m$ converges to f in $L^p(\tau)$ norm, f is the unique weak* limit of $(I - T_\alpha)F_m$. Therefore,

$$f = (I - T_\alpha)h \in (I - T_\alpha)L^p(\tau).$$

This completes the proof. □

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