Korean J. Math. **30** (2022), No. 3, pp. 467–474
http://dx.doi.org/10.11568/kjm.2022.30.3.467

BOUNDEDNESS OF $\mathcal{C}^{b,c}$ OPERATORS ON BLOCH SPACES

Pankaj Kumar Nath and Sunanda Naik*

Abstract. In this article, we consider the integral operator $\mathcal{C}^{b,c}$, which is defined as follows:

$$\mathcal{C}^{b,c}(f)(z) = \int_0^z \frac{f(w) * F(1,1;c;w)}{w(1-w)^{b+1-c}} dw,$$

where $*$ denotes the Hadamard/ convolution product of power series, $F(a,b;c;z)$ is the classical hypergeometric function with $b, c > 0, b+1 > c$ and $f(0) = 0$. We investigate the boundedness of the $\mathcal{C}^{b,c}$ operators on Bloch spaces.

1. Introduction and preliminary results

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$ and $\mathcal{H}_0$ be the class of all functions $f \in H(\mathbb{D})$ with $f(0) = 0$.

For any complex number $a, b,\ c \neq -n,\ n = 0, 1, 2, \ldots$, the Gaussian/classical hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by power series expansion

$${}_2F_1(a,b;c;z) = F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!} \ (|z| < 1),$$

where (a, n) is the shifted factorial defined by Appel's symbol

$$(a,n) = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, n \in \mathbb{N} = \{1, 2, \ldots\}$$

and $(a, 0) = 1$ for $a \neq 0$, (see [1]). Obviously, $F(a, b; c; z)$ is an analytic function in $\mathbb{D}$. We refer the reader to [1] for a background on Gaussian hypergeometric functions. For the asymptotic behavior of $F(a, b; c; z)$ for z near 1, we refer to [11] which has been used for a number of investigations.

We consider the integral operator, called $\mathcal{C}^{b,c}$ operator for $b, c \in \mathbb{R}$, $b, c > 0$ with $b + 1 > c$, on the space $\mathcal{H}_0$ defined by

$$\mathcal{C}^{b,c}(f)(z) = \int_0^z \frac{f(w) * F(1,1;c;w)}{w(1-w)^{b+1-c}} dw, \tag{1}$$

where $*$ denotes the Hadamard/ convolution product of power series. That is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two analytic functions in $|z| < R$ then

Received January 3, 2022. Revised July 18, 2022. Accepted August 20, 2022.
2010 Mathematics Subject Classification: 30D05, 30H05, 33C05, 47B38, 44A35.
Key words and phrases: Generalized Cesáro operators, Bloch-type space, Compactness.
* Corresponding author.

$f * g$ is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ and this series converges for $|z| < R^2$. Moreover,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|w|=r} f(w) g(z/w) \, \frac{dw}{w}, \quad |z| < rR < R^2.$$

In particular, if f, g are in $H(\mathbb{D})$, we have

$$(f * g)(rz) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) g(ze^{-it}) \, dt, \quad 0 < r < 1. \tag{2}$$

In particular, if $b = \beta$, $c = 1$ then

$$\begin{aligned} \mathcal{C}^{\beta,1}(f)(z) &= \int_0^z \frac{f(w) * F(1,1;1;w)}{w(1-w)^{\beta}} dw \\ &= \mathcal{C}_{\beta}(f)(z) \end{aligned}$$

which is the generalized β-Cesáro operator as defined in [7]. The boundedness, compactness, essential norm and spectrum of the β-Cesáro operators are studied by authors in [7]. Moreover, boundedness of the Cesáro and related operators in various function spaces are studied in the literature; see [5, 9, 13, 15]. In this paper, we study these operators as linear operators on a-Bloch space, denoted by $\mathcal{B}_a$, and is defined for each $a > 0$ as follows:

$$\mathcal{B}_a = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_a} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^a |f'(z)| < \infty\}.$$

In particular, the spaces $\mathcal{B}_{\mathbf{a}}$ becomes the classical Lipschitz and Bloch spaces whenever $a \in (0, 1)$ and $a = 1$ respectively.
The space $\mathcal{B}_a$ is a complex Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}_a},$$

whereas $\|f\|_{\mathcal{B}_a} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^a |f'(z)|$ represents a semi-norm. By restricting this space with the condition $f(0) = 0$, for $f \in \mathcal{B}_a$, we get a space, which is a subspace of $\mathcal{B}_a$, denoted by $\mathcal{B}_a^0$. The semi-norm $\|.\|_{\mathcal{B}_a}$ on $\mathcal{B}_a$ becomes norm on $\mathcal{B}_a^0$.The spaces $\mathcal{B}_a$ and $\mathcal{B}_a^0$ together with its harmonic analog have been investigated recently by a number of authors. See for instance, see [4, 8, 10] and the references therein. Unless it is specified we consider $a > 0$ throughout this paper. More on the Bloch space can be found in [16, 17].

Main motive of this paper is to study the boundedness properties of generalized $\mathcal{C}^{b,c}$ operators on $\mathcal{B}_a^0$ which include the β-Cesáro operators as well as the classical Cesáro operator.

2. Boundedness of $\mathcal{C}^{b,c}$ operators on $\mathcal{B}_a^0$

In this section, we discuss the boundedness properties of the $\mathcal{C}^{b,c}$ operators, on $\mathcal{B}_a^0$. At the end of this section, we provide few examples to show that the $\mathcal{C}^{b,c}$ operators are unbounded linear operators on $\mathcal{B}_a^0$, under some conditions on b, c. To obtain our desired results, we need the following lemmas.

LEMMA 1. *Let $b, c > 0$ with $c \geq b$. For $f \in \mathcal{B}_a$ we have $f * F \in \mathcal{B}_a$, where $F(z) = F(1, b; c; z)$. Further*

$$\|f * F\|_{\mathcal{B}_a} \leq \frac{2^a b \|f\|_{\mathcal{B}_a}}{c}.$$

Proof. For $c = b$, proof is easy as we have $f(z) * F(1, b; b; z) = f(z)$. Now for $c > b$, using the Euler integral representation with a simple calculation (see page 336 of [2]), we have

$$f(z) * F(1, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} f(tz)dt, \tag{3}$$

where $B(b, c-b)$ is the usual beta function. Hence we get

$$\begin{aligned}(1-|z|^2)^a|(f(z) * F(1, b; c; z))'| &\leq \frac{1}{B(b, c-b)} \int_0^1 t^b(1-t)^{c-b-1}(1-|z|^2)^a|f'(tz)|dt \\ &\leq \frac{1}{B(b, c-b)} \int_0^1 t^b(1-t)^{c-b-1}(1-t^2|z|^2)^a|f'(tz)|dt.\end{aligned}$$

Since $c > b$, by taking supremum in both sides of the above inequality, we find

$$\|f * F\|_{\mathcal{B}_a} \leq \frac{2^a\|f\|_{\mathcal{B}_a}}{B(b, c-b)} \int_0^1 t^b(1-t)^{c-b-1}dt,$$

which complete the proof. □

REMARK 2. In [2], integral representation for $f * F(a, b; c; z)$ is given under certain conditions on the parameters a, b, c and was used to derive geometric properties of the Hadamard product. This representation may be used to generalize Lemma 1.

LEMMA 3. *Let $f \in \mathcal{B}_a$. Suppose $c > b > 0$ and $F(z) = F(1, b; c; z)$. Then we have the following properties:*
(i) If $a < 1$, then

$$|(f * F)(z)| \leq |f(0)| + \frac{b\|f\|_{\mathcal{B}_a}}{2c(1-a)} < \infty.$$

(ii) If $a = 1$, then

$$|(f * F)(z)| \leq |f(0)| + \frac{\|f\|_{\mathcal{B}_1}}{2} \log\left(\frac{1}{1-|z|}\right).$$

(iii) If $a > 1$, then

$$|(f * F)(z)| \leq |f(0)| + \frac{\|f\|_{\mathcal{B}_a}}{a-1} \left(\frac{1}{(1-|z|)^{a-1}} - 1\right).$$

Proof. Suppose $f \in \mathcal{B}_a$ and $z \in \mathbb{D}$. Then

$$|(f * F)(z) - (f * F)(0)| = \left|z \int_0^1 (f * F)'(zu)du\right|.$$

Using (3) and by the definition of a-Bloch space, we have

$$\begin{aligned} |(f*F)(z)-(f*F)(0)| &= \frac{|z|}{B}\int_0^1 u\left(\int_0^1 t^b(1-t)^{c-b-1}|f'(tzu)|dt\right)du \\ &\leq \frac{|z|\|f\|_{\mathcal{B}_a}}{B}\int_0^1 u\left(\int_0^1 \frac{t^b(1-t)^{c-b-1}}{(1-|z|^2t^2u^2|)^a}dt\right)du \\ &\leq \frac{b|z|\|f\|_{\mathcal{B}_a}}{c}\int_0^1 u\frac{1}{(1-|z|^2u^2)^a}\,du \\ &= \frac{b\|f\|_{\mathcal{B}_a}}{2c}\int_{1-|z|}^1 \frac{1}{u^a}\,du. \end{aligned} \tag{4}$$

Since $(f*F)(0)=f(0)$, the above inequality gives

$$|(f*F)(z)-f(0)| \leq \frac{b\|f\|_{\mathcal{B}_a}}{2c(1-a)}\left(1-\frac{1}{(1-|z|)^{a-1}}\right).$$

Further, by using triangle inequality, we obtain

$$|(f*F)(z)| \leq |f(0)| + \frac{b\|f\|_{\mathcal{B}_a}}{2c(1-a)}\left(1-\frac{1}{(1-|z|)^{a-1}}\right). \tag{5}$$

Let $a<1$. Since $1-(1-|z|)^{1-a}\leq 1$, from (5) we obtain

$$|(f*F)(z)| \leq |f(0)| + \frac{b\|f\|_{\mathcal{B}_a}}{2c(1-a)}.$$

When $a=1$, from (4), we get

$$|(f*F)(z)| \leq |(f*F)(0)| + \frac{b\|f\|_{\mathcal{B}_1}}{2c}\log\left(\frac{1}{1-|z|}\right). \tag{6}$$

Now for $a>1$, it is easily follows that

$$|(f*F)(z)| \leq |f(0)| + \|f\|_{\mathcal{B}_a}\frac{1}{a-1}\left(\frac{1}{(1-|z|)^{a-1}}-1\right). \tag{7}$$

This completes the proof of this lemma. □

For $f\in B_a^0$, Lemma 1 gives $f*F\in\mathcal{B}_a^0$, $\mathcal{C}^{b,c}(f)$ is an analytic function in $\mathbb{D}$ and $\mathcal{C}^{b,c}(f)(0)=0$. Now, we have our main result, which describes the boundedness of $\mathcal{C}^{b,c}$ operators from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$ for three different restrictions on b and c. Now onward we denote $F(z)=F(1,1;c;z)$.

Theorem 4. *Suppose b,c are positive real numbers with $c>1$. The $\mathcal{C}^{b,c}$ operator is bounded linear operator from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$, for*

(i) $b+1-c\leq a<1$

(ii) $b+1-c\leq 1<a$

(iii) $b+1-c<a=1$.

Furthermore, for $a>1$ or $a<1$, we obtain

$$\|\mathcal{C}^{b,c}(f)\|_{\mathcal{B}_a} \leq \frac{b\|f\|_{\mathcal{B}_a}}{2^{1-a}c|1-a|}. \tag{8}$$

Proof. Case(i): Suppose that $f \in \mathcal{B}_a^0$, for $a < 1$. Using (5), we have

$$\frac{|(f*F)(z)|}{|z(1-z)^{b+1-c}|} \leq \frac{\|f\|_{\mathcal{B}_a}}{|z(1-z)^{b+1-c}|}\frac{b}{2c(1-a)}\left(\frac{(1-|z|)^{a-1}-1}{(1-|z|)^{a-1}}\right).$$

Since $b+1-c \leq a < 1$ and for $a < 1$, we have $1-(1-|z|)^{1-a} \leq |z|$, proceeding as in the proof of Theorem 2.3 of [7], we have

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{b\|f\|_{\mathcal{B}_a}}{2^{1-a}c(1-a)}(1-|z|)^{a-b+c-1}.$$

Also $(1-|z|)^{a-b+c-1} < 1$ and z is arbitrary point here. Therefore (8) follows.

Case (ii): Suppose that $f \in \mathcal{B}_a^0$, for $a > 1$. From (7), we have

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{(1+|z|)^a}{|z|}\frac{b\|f\|_{\mathcal{B}_a}}{2c(a-1)}[(1-|z|)^{c-b}-(1-|z|)^{a-b-1+c}].$$

For $b+1-c \leq 1 < a$, this leads to

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{(1+|z|)^a}{|z|}\frac{b\|f\|_{\mathcal{B}_a}}{2c(a-1)}(1-|z|)^{c-b}[1-(1-|z|)^{a-1}].$$

Since $1-(1-|z|)^{a-1} \leq 1-(1-|z|)^{\lceil a\rceil}$, where $\lceil . \rceil$ is a Greatest Integer Function, we have

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{2^{a-1}}{|z|}\frac{b\|f\|_{\mathcal{B}_a}}{c(a-1)}(1-|z|)^{c-b}[1-(1-|z|)^{\lceil a\rceil}].$$

Since $(1-|z|)^{-\lceil a\rceil} = F(\lceil a\rceil;1;1;|z|)$, a simple calculation gives

$$1-(1-|z|)^{\lceil a\rceil} = \lceil a\rceil F(-\lceil a\rceil+1;1;2;|z|).$$

Thus, as $b < c$ we obtain

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{2^{a-1}\lceil a\rceil b\|f\|_{\mathcal{B}_a}}{c(a-1)}F(-\lceil a\rceil+1;1;2;|z|).$$

We use the formula for $c > b-m$, $F(-m;b;c;1) = \dfrac{(c-b)_m}{(c)_m}$. Since z is arbitrary point in $\mathbb{D}$, therefore we have

$$\|\mathcal{C}^{b,c}(f)\|_{\mathcal{B}_a} \leq \frac{2^{a-1}\lceil a\rceil\|f\|_{\mathcal{B}_a}}{c(a-1)}\sup F(-\lceil a\rceil+1;1;2;|z|) = \frac{b2^{a-1}\|f\|_{\mathcal{B}_a}}{c(a-1)}$$

which completes the proof.

Case (iii) Suppose that $f*F \in \mathcal{B}_a^0$, for $a = 1$. From (6), we have

$$\frac{|(f*F)(z)|}{|z(1-z)^{b+1-c}|} \leq \frac{\|f\|_{\mathcal{B}_1}}{2|z(1-z)^{b+1-c}|}\log\left(\frac{1}{1-|z|}\right).$$

From this we get

$$(1-|z|^2)^a\left|\frac{f(z)*F(z)}{z(1-z)^{b+1-c}}\right| \leq \frac{(1-|z|^2)^{c-b}}{|z|}\frac{\|f\|_{\mathcal{B}_1}}{2}\log\left(\frac{1}{1-|z|}\right).$$

For $b+1-c<a=1$, Since z is arbitrary point, we have

$$\|\mathcal{C}^{b,c}(f)\|_{\mathcal{B}_1} \leq \sup\left\{\frac{(1-|z|)^{c-b}}{|z|}\log\left(\frac{1}{1-|z|}\right) : z\in\mathbb{D}\right\}\|f\|_{\mathcal{B}_1}.$$

This completes the proof. □

Counterexamples. We just proved that either of the cases $b+1-c\leq a<1$ or $b+1-c\leq 1<a$ or $b+1-c<a=1$, the $\mathcal{C}^{b,c}$ operators are bounded from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$. We now show that for the remaining cases: $b+1-c>a$ or $b+1-c=a\geq 1$ and $1<b+1-c<a$, the $\mathcal{C}^{b,c}$ operators need not be bounded as the following counterexamples show.

Example 5. Let $f(z)=z^k, k\geq 1$. Then $f*F\in\mathcal{B}_a^0$, for $b+1-c>a$ since

$$(1-|z|^2)^a\left|\frac{Kz^k}{z(1-z)^{b+1-c}}\right| = K(1+|z|^2)^a\frac{|z|^{k-1}(1-|z|)^a}{|(1-z)|^{b+1-c}},$$

where $K=\dfrac{(b,k)}{(1,k)}>0$. For $z=t\in(0,1)$, we obtain

$$(1-|z|^2)^a\left|\frac{Kz^k}{z(1-z)^{b+1-c}}\right| = K(1+t)^a\frac{t^{k-1}(1-t)^a}{(1-t)^{b+1-c}} = K\frac{t^{k-1}(1+t)^a}{(1-t)^{b+1-c-a}}.$$

As t tends to 1, the right hand side term tends to ∞. Therefore, the $\mathcal{C}^{b,c}$ operator is an unbounded linear operator from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$, for $b+1-c>a$.

Example 6. Let $f(z)=\log(1-z)$, where the principal value of the branch of logarithm is chosen. Using (3), when $c>1$, $f*F\in\mathcal{B}_a^0$, for $a\geq 1$ and for $z=u\in(0,1)$, we have

$$\begin{aligned}(1-|z|^2)^a\left|\frac{\log(1-z)*F(1,1;c;z)}{z(1-z)^{b+1-c}}\right| &= (1-u^2)^a\left|\frac{\int_0^1 t^{b-1}(1-t)^{c-1-1}\log(1-tu)dt}{B(b,c-b)u(1-u)^{b+1-c}}\right|\\ &= \frac{(1+u)^a}{(1-u)^{b+1-c-a}}\left|\frac{\int_0^1 t^{b-1}(1-t)^{c-1-1}\log(1-tu)dt}{B(b,c-b)u}\right|\end{aligned}$$

Since the integral on the right hand side is finite so, for $b+1-c\geq a$, as u tends to 1, the right hand side term diverges to ∞. Therefore, the $\mathcal{C}^{b,c}$ operator is an unbounded linear operator from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$, for $b+1-c\geq a\geq 1$.

Example 7. Let $f(z)=\dfrac{z}{(1-z)^a}$, for $a>0$. Then $f*F\in\mathcal{B}_{a+1}^0$. Using (3), when $c>1$, and for $z=u\in(0,1)$, we have

$$\begin{aligned}(1-|z|^2)^{a+1}\left|\frac{f(z)*F(1,1;c;z)}{z(1-z)^{b+1-c+a}}\right| &= (1-u^2)^a\left|\frac{\int_0^1 t^{b-1}(1-t)^{c-1-1}\dfrac{tu}{(1-tu)^a}dt}{B(b,c-b)u(1-u)^{b+1-c}}\right|\\ &= \frac{(1-u^2)^{a+1}}{(1-u)^{b+1-c}}\int_0^1 t^b(1-t)^{c-1-1}(1-tu)^{-a}dt\\ &> \frac{(1-u^2)^{a+1}}{(1-u)^{b+1-c}}\int_0^1 t^b(1-t)^{c-1-1}dt.\end{aligned}$$

For $z = t \in (0,1)$, then it yields

$$(1-|z|^2)^{a+1}\left|\frac{z}{z(1-z)^{b+1-c+a}}\right| = (1+t)^{a+1}\frac{(1-t)^{a+1}}{(1-t)^{b+1-c+a}} = \frac{(1+t)^{a+1}}{(1-t)^{b-c}}.$$

Then for $b+1-c > 1$, and $c > a+1$ as t tends to 1, the right hand side term diverges to ∞. Therefore, the $\mathcal{C}^{b,c}$ operator is an unbounded linear operator from $\mathcal{B}_a^0$ to $\mathcal{B}_a^0$, for $b+1-c > 1$.

Conflicts of Interests: The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, (1999).

[2] M. R. Agrawal, P. G. Howlett, S. K. Lucas, S. Naik and S. Ponnusamy,*Boundedness of generalized Cesaro averaging operators on certain function spaces*, J. Comput. Appl. Math. **180** (2005), 333–344

[3] R. Balasubramanian, S. Ponnusamy and M. Vuorinen, *On hypergeometric functions and function spaces*, J. Comput. Appl. Math. **139** (2) (2002), 299–322.

[4] H. Deng, S. Ponnusamy, and J. Qiao, *Extreme points and support points of families of harmonic Bloch mappings*, Potential Analysis **55** (2021), 619–638.

[5] E. Diamantopoulos and A. G. Siskakis, *Composition operators and the Hilbert matrix*, Studia Math. **140** (2000), 191–198.

[6] H. Hidetaka, *Bloch-type spaces and extended Cesáro operators in the unit ball of a complex Banach space*, Preprint (https://arxiv.org/abs/1710.11347).

[7] S. Kumar and S. K. Sahoo, *Properties of β-Cesáro operators on α-Bloch Space*, Rocky Mountain J. Math. **50** (1) (2020), 1723–1743.

[8] G. Liu and S. Ponnusamy, *On Harmonic ν-Bloch and ν-Bloch-type mappings*, Results in Mathematics 73(3) (2018), Art 90, 21 pages.

[9] J. Miao, *The Cesáro operator is bounded on H^p for* $0 < p < 1$,Proc. Amer. Math. Soc. **116** (4) (1992), 1077–1079.

[10] M. Huang, S. Ponnusamy, and J. Qiao, *Extreme points and support points of harmonic alpha-Bloch Mappings*, Rocky Mountain J. Math. **50** (4) (2020), 1323–1354.

[11] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997), 278–301.

[12] A. Siskakis, *Semigroups of composition operators in Bergman spaces*, Bull. Austral.Math. Soc. **35** (1987), 397–406.

[13] A. G. Siskakis, *The Cesáro operator is bounded on H^1*, Proc. Amer. Math. Soc. **110** (4) (1990), 461–462.

[14] S. Stević, *Boundedness and Compactness of an integral operator on mixed norm spaces on the polydisc*, Sibirsk. Math. Zh. **48** (3) (2007), 694–706.

[15] J. Xiao, *Cesaŕo type operators on Hardy, BMOA and Bloch spaces*, Arch. Math. **68** (1997), 398–406.

[16] K. Zhu,*Operator Theory in Function Spaces*, Second Edition, Math. Surveys and Monographs, **138** (2007).

[17] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, USA, (2005).

Pankaj Kumar Nath
Department of Applied Sciences, Gauhati University, Assam 781014, India
E-mail: pankaj.kumar0246@gmail.com

Sunanda Naik
Department of Applied Sciences, Gauhati University, Assam 781014, India
E-mail: snaik@gauhati.ac.in