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INTUITIONISTIC Q-FUZZY PMS-IDEALS OF A PMS-ALGEBRA

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Abstract. In this paper, we apply the concept of intuitionistic Q-fuzzy set to PMS-algebras. We study the concept of intuitionistic Q-fuzzy PMS-ideals of PMS-algebras and investigate some related properties of intuitionistic Q-fuzzy PMS-ideals of PMS-algebras. We provide the relationship between an intuitionistic Q-fuzzy PMS-subalgebra and an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra. We establish a condition for an intuitionistic Q-fuzzy set in a PMS-algebra to be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra. Characterizations of intuitionistic Q-fuzzy PMS-ideals of PMS-algebras in terms of their level sets are given.

1. Introduction

The concept of fuzzy sets was introduced by L. Zadeh [18] in 1965. After that, many authors applied this concept to group and ring theory. It is now a rigorous area of research with applications in various fields. Several researchers explored on the generalization of the notation of fuzzy sets. The idea of intuitionistic fuzzy set was first introduced by Atanassov [1,2] as a generalization of the notion of fuzzy set which is very effective to ideal with vagueness. Since then many researchers applied the notion of intuitionistic fuzzy sets to various algebraic structures such as groups and rings, and introduced intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings etc. In 2007, Yamak and Yilmaz [17] introduced intuitionistic Q-fuzzy R-subgroups of near-rings. In 2006, Kim [10] studied the intuitionistic Q-fuzzy semiprime ideals of semigroups. J. D. Yadav, Y. S. Pawar [16] introduced the notion of intuitionistic Q-fuzzy ideals of near-rings and investigated some related properties. Roh et al in [12] discussed the intuitionistic Q-fuzzification of the concept of subalgebras in BCK/BCI-algebra and investigated various properties. In 2005, K. H. Kim [9] established the intuitionistic Q-fuzzification of the concept of ideals in semigroups. In 2014, S. R. Barbhuiya [3] introduced the notion of intuitionistic Q-fuzzy ideals of BG-algebra and investigated some of their basic properties. The notion of BCK-algebras was introduced by Iseki and Tanaka [7] in 1978. In 1980, Iseki [8] introduced the notion of a

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BCI-algebra which is a generalization of a BCK-algebra. In 2016, Sithar and Nagalakshmi [13] introduced a new notion called a PMS-algebra, which is a generalization of BCK/BCI/TM/KUS-algebras and investigated several properties. The fuzzification of PMS-ideals was also studied by Sithar and Nagalakshmi [14]. In 2021, Derseh et al. [5] studied the intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. In the same year they also studied intuitionistic fuzzy PMS-ideal of a PMS-algebra [4].

The rest parts of the paper are organized as follows: Section 2 contains basic definitions and fundamental concepts that we use for the development of the manuscript. In Section 3, we define the intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra using intuitionistic Q-fuzzy sets and establish several basic properties. In section 4, we study the level subsets of the intuitionistic Q-fuzzy PMS-ideals and characterize intuitionistic Q-fuzzy PMS-ideals in terms of their level sets. The last section of the manuscript ends with a conclusion and suggestions for future research.

2. Preliminaries

In this section, we present some fundamental definitions and results that are required in the development of this paper.

Definition 2.1. [13] A PMS-algebra is a nonempty set $(X, *, 0)$ with a constant element 0 and a binary operation $*$ of type (2, 0) satisfying the following axioms

(i). $0 * x = x$

(ii). $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$.

The binary relation $\leq$ is defined by $x \leq y$ if and only if $x * y = 0$, for all $x, y \in X$

Definition 2.2. [13] A nonempty subset I of a PMS-algebra X is called a PMS-ideal of X if it satisfies the following conditions:

(i). $0 \in \mathrm{I}$

(ii). $z * y, z * x \in \mathrm{I} \Rightarrow y * x \in \mathrm{I}$, for all $x, y, z \in X$.

Proposition 2.3. [13] *In any PMS-algebra $(X, *, 0)$ the following properties hold for all $x, y, z \in X$.*

(i). $x * x = 0$

(ii). $(y * x) * x = y$

(iii). $x * (y * x) = y * 0$

(iv). $(y * x) * z = (z * x) * y$

(v). $(x * y) * 0 = y * x = (0 * y) * (0 * x)$

Definition 2.4. [3,12,15] Let X and Q be nonempty sets. A Q-fuzzy set A in X defined as $A = \{<(x, q), \mu_A(x, q)> | x \in X, q \in Q\}$, where the mapping $\mu_A : X \times Q \to [0, 1]$ defines the degree of membership of the element (x, q) in $X \times Q$ to the set A.

Definition 2.5. [3,12] Let X and Q be nonempty sets. An intuitionistic Q-fuzzy set A in a set $X \times Q$ is defined as an object of the form $A = \{\langle (x, q), \mu_A(x, q), \nu_A(x, q)\rangle | x \in X, q \in Q\}$, where the functions $\mu_A : X \times Q \to [0, 1]$ and $\nu_A : X \times Q \to [0, 1]$ define the degree of membership and the degree of nonmembership respectively and satisfying the condition $0 \leq \mu_A(x, q) + \nu_A(x, q) \leq 1$, for all $x \in X$ and $q \in Q$.
Note: For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for an intuitionistic Q-fuzzy set $A = \{\langle (x, q), \mu_A(x, q), \nu_A(x, q)\rangle | x \in X, q \in Q\}$.

DEFINITION 2.6. [11] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic Q-fuzzy subsets of the set $X \times Q$, where $A = \{\langle (x, q), \mu_A(x, q), \nu_A(x, q)\rangle | x \in X, q \in Q\}$ and $B = \{\langle (x, q), \mu_B(x, q), \nu_B(x, q)\rangle | x \in X, q \in Q\}$. Then

(i). $A \subseteq B$ if and only if $\mu_A(x, q) \leq \mu_B(x, q)$ and $\nu_A(x, q) \geq \nu_B(x, q)$
(ii). $A = B$ if and only if $\mu_A(x, q) = \mu_B(x, q)$ and $\nu_A(x, q) = \nu_B(x, q)$
(iii). $A \cap B = \{\langle (x, q), min(\mu_A(x, q), \mu_B(x, q)) max(\nu_A(x, q), \nu_B(x, q))\rangle \, | x \in X, q \in Q\}$
(iv). $A \cup B = \{\langle (x, q), max(\mu_A(x, q), \mu_B(x, q)),\ min(\nu_A(x, q), \nu_B(x, q))\rangle \, | x \in X, q \in Q\}$
(v). $\bar{A} = \{\langle (x, q), \nu_A(x, q), \mu_A(x, q)\rangle \, | x \in X, q \in Q\}$
(vi). $\Box A = \{\langle (x, q), \mu_A(x, q), 1 - \mu_A(x, q)\rangle \, | x \in X, q \in Q\}$
(vii). $\Diamond A = \{\langle (x, q), 1 - \nu_A(x, q), \nu_A(x, q)\rangle \, | x \in X, q \in Q\}$

DEFINITION 2.7. [6] An intuitionistic Q-fuzzy subset (IQFS) $A = (\mu_A, \nu_A)$ in a PMS-algebra X is called an intuitionistic Q-fuzzy PMS-subalgebra (IQF PMS-SA) of X if
$\mu_A(x * y, q) \geq min\{\mu_A(x, q), \mu_A(y, q)\}$ and $\nu_A(x * y, q) \leq max\{\nu_A(x, q), \nu_A(y, q)\}$, for all $x, y \in X$ and $q \in Q$.

DEFINITION 2.8. [6] Let $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-subalgebra (IQF PMS-SA) of a PMS-algebra X. Then the level subsets
$U(\mu_A, t) = \{x \in X | \mu_A(x, q) \geq t$ and $q \in Q\}$ and $L(\nu_A, s) = \{\nu_A(x, q) \leq s$ and $q \in Q\}$, for $t, s \in [0, 1]$ are called upper level and lower level PMS-subalgebras of X respectively.

3. Intuitionistic Q-fuzzy PMS-ideals.

In this section, we attempt to study the intuitionistic Q-fuzzy PMS-ideal in a PMS-algebra and investigate some important results. Throughout this section, X denotes a PMS-algebra and Q denotes any nonempty set unless otherwise specified.

DEFINITION 3.1. Let X be a PMS-algebra and Q be any nonempty set. An intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ in X is said to be an intuitionistic Q-fuzzy PMS-ideal of X if it satisfies the following conditions.

(i). $\mu_A(0,\ q) \geq \mu_A(x,\ q)$ and $\nu_A(0,\ q) \leq \nu_A(x,\ q)$,
(ii). $\mu_A(y * x,\ q) \geq min\{\mu_A(z * y,\ q),\ \mu_A(z * x,\ q)\}$,
(iii). $\nu_A(y * x,\ q) \leq max\{\nu_A(z * y,\ q),\ \nu_A(z * x,\ q)\}, \forall\ x,\ y, z \in$ X, $q \in$ Q.

EXAMPLE 3.2. Consider $X = \{a, b, c\}$ such that $(X, *, 0)$ is a PMS-algebra with the following table and $Q = \{q\}$.

$*$	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

Define an intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ in X defined by
$\mu_A(x, q) = \begin{cases} 0.9 & if\ x = 0 \\ 0.3 & if\ x = a,\ b \end{cases}$ and $\nu_A(x, q) = \begin{cases} 0 & if\ x = 0 \\ 0.4 & if\ x = a,\ b \end{cases}$ for all $x \in X$ and $q \in$Q. By routine calculation we can show that $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X.

Theorem 3.3. *If $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X with $x \leq y$ for any $x,\ y \in$ X and $q \in$Q, then $\mu_A(x,\ q) \geq \mu_A(y,\ q)$ and $\nu_A(x,\ q) \leq \nu_A(y,\ q)$ i.e. μ_A is order reversing and ν_A is order preserving.*

Proof. Le $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X such that $x \leq y$, Then we have $x * y = 0$. Now, for all $x,\ y \in$ X and $q \in$Q.

$$\begin{aligned}
\mu_A(x,q) &= \mu_A((x*y)*y,q) \ \text{(By Proposition 2.3 }(ii))\\
&= \mu_A(0*y,q) \ (\because x \leq y \Longrightarrow x*y=0.)\\
&\geq min\{\mu_A(0*0,q), \mu_A(0*y,q)\} \ \text{(By Definition 3.1 }(ii))\\
&= min\{\mu_A(0,q), \mu_A(y,q)\}\\
&= \mu_A(y,q) \ (\text{since } \mu_A(0,q) \leq \mu_A(y,q),\ \forall\ y \in X, q \in Q)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x,q) &= \nu_A((x*y)*y,q) \ \text{(By Proposition 2.3 }(ii))\\
&= \nu_A(0*y,q)(\because x \leq y \Longrightarrow x*y=0.)\\
&\leq max\{\nu_A(0*0,q), \nu_A(0*y,q)\} \ \text{(By Definition 3.1 }(iii))\\
&= max\{\nu_A(0,q), \nu_A(y,q)\}\\
&= \nu_A(y,q) \, (\text{since } \nu_A\,(0,\ q) \leq\ \nu_A\,(y,\ q)\,,\ \forall\ y \in X,\ q \in Q)
\end{aligned}$$

Hence $\mu_A\,(x,q) \geq \mu_A\,(y,q)$ and $\nu_A(x,q \leq \nu_A(y,q)$ □

Theorem 3.4. *Every intuitionistic Q-fuzzy PMS-ideal $A = (\mu_A, \nu_A)$ of X is an intuitionistic Q-fuzzy PMS-sub algebra of X.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of X and $x, y \in X$ and $q \in$Q. Then by definition 3.1, we have

$$\begin{aligned}
\mu_A(x*y,q) &\geq min\{\mu_A(0*x,q), \mu_A(0*y,q)\}\\
&= min\{\mu_A(x,q), \mu_A(y,q)\} and\\
\nu_A(x\ *\ y,q) &\leq max\{\nu_A(0*x,q), \nu_A(0*y,q)\}\\
&= max\ \{\nu_A\,(x,q)\,,\ \nu_A(y,q)\}
\end{aligned}$$

Thus $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-subalgebra of X. □

Theorem 3.5. *If A is an intuitionistic Q-fuzzy PMS-ideal of X with $x * y \leq z$ for all $x,\ y,\ z \in$ X, then $\mu_A(x,q) \geq min\{\mu_A(y,q),\ \mu_A(z,q)\}$ and $\nu_A(x,q) \leq max\{\nu_A(y,q), \nu_A(z,q)\}$.*

Proof. Let $x, y, z \in X$ such that $x * y \leq z$. Then by the binary relation $\leq$ defined in X, we have $(x * y) * z = 0$. Then using Definition 2.1(i), Proposition 2.3 (iv) and Theorem 3.4, we have

$$\begin{aligned}
\mu_A(x,q) = \mu_A(0*x,q) &= \mu_A(((x*y)*z)*x,q)\\
&= \mu_A((z*y)*x)*x,q)\\
&= \mu_A((x*x)*(z*y),q)\\
&= \mu_A(0*(z*y),q)\\
&= \mu_A(z*y,q)\\
&\geq min\{\mu_A(z,q), \mu_A(y,q)\}\\
\Rightarrow \mu_A(x,q) &\geq min\{\mu_A(y,q), \mu_A(z,q)\},
\end{aligned}$$

Similarly, $\nu_A(x,q) \leq max\{\nu_A(y,q), \nu_A(z,q)$.

Hence $\mu_A(x,q) \geq min\{\mu_A(y,q), \mu_A\ (z,q)\}$ and $\nu_A(x,q) \leq max\{\nu_A(y,q), \nu_A(z,q)\}$, for all $x,y,z \in X$ and $q \in Q$. □

THEOREM 3.6. *Let A be an intuitionistic Q-fuzzy PMS-ideal of X such that $(...((x* x_1) * x_2)\ * ...) *\ x_n\ =\ 0$ for any $x,\ x_1,\ x_2,\ x_3, \ldots, x_n \in X$. Then $\mu_A(x,\ q)\ \geq min\{\mu_A(x_1,\ q),\ \mu_A(x_2,\ q), ..., \mu_A(x_n,\ q)\}$ and $\nu_A(x,\ q)\ \leq\ max\{\nu_A(x_1,\ q),\ \nu_A(x_2,\ q), ..., \nu_A(x_n,\ q)\}$.*

Proof. Using induction on n, and Theorem 3.3 and Theorem 3.5, we have that

$$\mu_A(x,\ q)\ \geq\ min\{\mu_A(x_1,\ q),\ \mu_A(x_2,\ q),\ ...\mu_A(x_n,\ q)\}\ \text{ and }$$
$$\nu_A(x,\ q) \leq min\{\nu_A(x_1,\ q),\ \nu_A(x_2,\ q),\ ...\nu_A(x_n,\ q)\}.$$ □

THEOREM 3.7. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of X such that $\mu_A(x*y,q) \geq \mu_A(0,q)$ and $\nu_A(x*y,q) \leq \nu_A(0,q)$. Then $\mu_A(x,q) = \mu_A(y,q)$ and $\nu_A(x,q) = \nu_A(y,q)$ for all $x,y \in X$ and $q \in$Q.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X such that $\mu_A(x*y,q) \geq \mu_A(0,\ q)$ and $\nu_A\,(x*y,q) \leq\ \nu_A\,(0,\ q)$ for all $x,y \in X$ and $q \in$Q. Now we need to show that $\mu_A\,(x,\ q) = \mu_A(y,q)$ and $\nu_A\,(x,\ q) = \nu_A\,(y,q)$. By definition 3.1 (i), we know that $\mu_A(0,q)\ \geq \mu_A(x*y,q)$ and $\nu_A\,(0,q) \leq\ \nu_A\,(x*y,\ q)$. Thus it gives that $\mu_A\,(0,\ q) =\ \mu_A(x*y\ ,\ q)$ and $\nu_A\,(0,\ q) =\ \nu_A\,(x*y,\ q)$. Now consider

$$\begin{aligned}\mu_A\,(x,\ q) &= \mu_A\,((x*y)*y,\ q)\\ &\geq min\{\mu_A(x*y,q), \mu_A(y,q)\}\ \text{(By Theorem 3.4)}\\ &= min\{\mu_A(0,q), \mu_A(y,q)\}\\ &= \mu_A(y,q)\ \text{(By Definition 3.1 } (i))\end{aligned}$$

$$\Rightarrow \mu_A(x,q) \geq \mu_A(y,\ q).$$

Conversely,

$$\begin{aligned}\mu_A(y,q) &= \mu_A((y*x)*x,q) \geq \{min\mu_A(y*x,q), \mu_A(x,q)\}\ \text{(By Theorem 3.4)}\\ &= min\,\{\mu_A\,((x*y)*0,q), \mu_A(x,\ q)\}\ \text{(By proposition 2.3 } (v))\\ &\geq min\,\{\min\{\mu_A\,(x*y,q), \mu_A(0,q)\}, \mu_A(x,\ q)\}\ \text{(By Theorem 3.4)}\\ &= min\{\mu_A(0,q), \mu_A(x,q)\}\\ &= \mu_A(x,q), \text{(By Definition 3.1 } (i))\end{aligned}$$

$$\Rightarrow\ \mu_A\,(y,\ q) \geq \mu_A\,(x,\ q)\,.$$

Therefore $\mu_A\,(x,\ q) = \mu_A\,(y,\ q)$, for all $x,y \in$ X and $q \in Q$.
Similarly, $\nu_A(x,q) = \nu_A(y,q)$, for all $x,y \in$ X and $q \in Q$. □

THEOREM 3.8. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X. If $\mu_A(x*y,q) = 1$ and $\nu_A(x*y,q) = 0$ for all $x,\ y \in X$ and $q \in$Q, then $\mu_A(x,q) = \mu_A(y,q)$ and $\nu_A(x,q) = \nu_A(y,q)$.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X such that $\mu_A(x*y,q) = 1$ and $\nu_A(x*y,q) = 0$, for all $x,\ y \in X$ and $q \in$Q. Then

$$\begin{aligned}\mu_A(x,q) = \mu_A((x*y)*y,q) &\geq \min\{\mu_A(x*y,q), \mu_A(y,q)\}\ \text{(By Theorem 3.4)}\\ &=\ min\{1,\ \mu_A\,(y,\ q)\} = \mu_A(y,\ q)\end{aligned}$$

$$\Rightarrow\ \mu_A\,(x,\ q) \geq \mu_A\,(y,\ q)\,.$$

Conversely,

$$\begin{aligned}\mu_A(y,q) &= \mu_A((y*x)*x,\ q)\\ &\geq min\{\mu_A(y*x,q),\mu_A(x,q)\}\}\text{ (By Theorem 3.4)}\\ &= min\{\mu_A((x*y)*0,q),\mu_A(x,q)\}\text{ (By proposition 2.3 }(v))\\ &\geq min\{\min\{\mu_A(x*y,q),\mu_A(0,q)\},\mu_A(x,q)\}\text{ (By Theorem 3.4)}\\ &= min\{\min\{1,\ \mu_A(0,q)\},\mu_A(x,\ q)\}\}\\ &= min\{\ \mu_A(0,q),\mu_A(x,\ q)\} = \mu_A(x,\ q)\end{aligned}$$

$\Rightarrow\ \mu_A(y,\ q) \geq \mu_A(x,\ q)$.
Therefore $\mu_A(y,q) = \mu_A(x,q)$, for all $x,y \in X$ and $q \in Q$.
Similarly, $\nu_A(x,q) = \nu_A(y,q)$, for all $x,y \in X$ and $q \in Q$. □

THEOREM 3.9. *Let A be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X and $A^+ = (\mu_{A^+},\nu_{A^+})$ be an intuitionistic Q-fuzzy set in X defined by $\mu_{A^+}(x,q) = \mu_A(x,q)+\overline{\mu}_A(0,q)$ and $\nu_{A^+}(x,q) = \nu_A(x,q)-\nu_A(0,q)$, for all $x \in X$ and $q \in Q$. Then A^+ is an intuitionistic Q-fuzzy PMS-ideal of X that contains A.*

Proof. Let $A = (\mu_A,\nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X and $A^+ = (\mu_{A^+},\nu_{A^+})$ be an intuitionistic Q-fuzzy set in X as defined in the theorem. We need to prove that $A^+ = (\mu_{A^+},\nu_{A^+})$ is an intuitionistic Q-fuzzy PMS-ideal of X.

(i). Let $x \in X$ and $q \in$ Q. Then,
$\mu_{A^+}(0,q) = \mu_A(0,q)+\overline{\mu}_A(0,q) = \mu_A(0,q)+1-\mu_A(0,q) = 1 \geq \mu_{A^+}(x,q)$ and
$\nu_{A^+}(0,q) = \nu_A(0,q)-\nu_A(0,q) = 0 \leq \nu_{A^+}(x,q)$
Hence $\mu_{A^+}(0,q) \geq \mu_{A^+}(x,q)$ and $\nu_{A^+}(0,q) \leq \nu_{A^+}(x,q)$, for all $x \in X$ and $q \in$ Q.

(ii). Let $x,y,z \in X$ and $q \in$ Q Then,

$$\begin{aligned}\mu_{A^+}(y*x,q) &= \mu_A(y*x,q)+\overline{\mu}_A(0,q)\\ &\geq min\{\mu_A(z*y,q),\mu_A(z*x,q)\}+\overline{\mu}_A(0,q)\\ &= minmu_A(z*y,q)+\overline{\mu}_A(0,q),\mu_A(z*x,q)+\overline{\mu}_A(0,)\}\\ &= min\{\mu_{A^+}(z*y,q),\mu_{A^+}(z*x,q)\}\end{aligned}$$

$$\Rightarrow \mu_{A^+}(y*x,q) \geq min\{\mu_{A^+}(z*y,q),\mu_{A^+}(z*x,q)\}$$

and

$$\begin{aligned}\nu_{A^+}(y*x,q) &= \nu_A(y*x,q)-\nu_A(0,q)\\ &\leq max\{\nu_A(z*y,q),\nu_A(z*x,q)\}-\nu_A(0,q)\\ &= max\{\nu_A(z*y,q)-\nu_A(0,q),\nu_A(z*x,q)-\nu_A(0,q)\}\\ &= max\{\nu_{A^+}(z*y,q),\nu_{A^+}(z*x,q)\}\end{aligned}$$

$$\Rightarrow \nu_{A^+}(y*x,q) \leq max\{\nu_{A^+}(z*y,q),\nu_{A^+}(z*x,q)\}$$

Therefore $A^+ = (\mu_{A^+},\nu_{A^+})$ is an intuitionistic Q-fuzzy PMS-ideal of X.
Also, since $\mu_A(x,q) \leq \mu_A(x,q)+\overline{\mu}_A(0,q) = \mu_{A^+}(x,q)$ and $\nu_A(x,q) \geq \nu_A(x)-\nu_A(0,q) = \nu_{A^+}(x,q)$ it follows that $A \subseteq A^+$. Thus $A^+ = (\mu_{A^+},\nu_{A^+})$ is an intuitionistic Q-fuzzy PMS-ideal of X which contains A. □

COROLLARY 3.10. *Let $A = (\mu_A,\nu_A)$ and $A^+ = (\mu_{A^+},\nu_{A^+})$ be defined as in Theorem 3.9. If there exists $x \in X$ such that $\mu_{A^+}(x,q) = 0$ and $\nu_{A^+}(x,q) = 1$, then $\mu_A(x,q) = 0$ and $\nu_A(x,q) = 1$.*

Proof. straight forward since $A \subseteq A^+$. □

THEOREM 3.11. *If the intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, then the intuitionistic Q-fuzzy subset $\Box A = (\mu_A, \overline{\mu}_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, where $\overline{\mu}(x,q) = 1 - \mu_A(x,q)$, for all $x \in X$ and $q \in Q$.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. Then, we have $\mu_A(0,q) \geq \mu_A(x,q)$ and $\mu_A(y{*}x,q) \geq \min\{\mu_A(z{*}\,y,q),\ \mu_A(z{*}x,q)\}$, for all $x,\ y,\ z\ \in X$ and $q \in$Q. Next we have to show $\overline{\mu}_A(0,q) \leq \overline{\mu}_A(x,q)$ and $\overline{\mu}_A(y{*}x,q) \leq \max\{\overline{\mu}_A(z*y,q),\ \overline{\mu}_A(z*x,q)\}$ for all $x,\ y,\ z\ \in X$ and $q \in$ Q. Thus,
$\overline{\mu}_A(0,q) = 1 - \mu_A(0,q) \leq 1 - \mu_A(x,q) = \overline{\mu}_A(x,q)$ and

$$\begin{aligned}\overline{\mu}_A(y*x,q) &= 1 - \mu_A(y*x,q)\\ &\leq 1 - min\{\mu_A(z*y,q), \mu_A(z*x,q)\}\\ &= \max\{1-\mu_A(z*\ y,q),\ 1-\mu_A(z*x,q)\}\\ &= \max\{\overline{\mu}_A(z\ *\ y,q),\ \overline{\mu}_A(z*x,q)\}.\end{aligned}$$

Hence $\Box A$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

REMARK 3.12. The converse of the above theorem is not necessarily true. This is shown by the following example

EXAMPLE 3.13. Let $X = \{0, a, b\}$ be a set such that $(X, *, 0)$ is a PMS-algebra with the table as in example 3.2 and Q=$\{q\}$. Define an intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ by

$$\mu_A(x,q) = \begin{cases} 0.5 & if\ x = 0,\\ 0.3 & if\ x = a,\\ 0.2 & if\ x = b.\end{cases} \text{ and } \nu_A(x,q) = \begin{cases} 0.4 & if\ x = 0,\\ 0.3 & if\ x = a,\\ 0.6 & if\ x = b.\end{cases} \text{ for all } x \in X$$

and $q \in$Q. Observe that $\nu_A(0,q) = 0.4 > 0.3 = \nu_A(a,q)$. This contradicts Definition 3.1 (i). Therefore $A = (\mu_A, \nu_A)$ is not an intuitionistic Q-fuzzy PMS-ideal of X.

Now $\overline{\mu}_A(x,q) = \begin{cases} 0.5 & if\ x = 0,\\ 0.7 & if\ x = a,\\ 0.8 & if\ x = b.\end{cases}$ for all $x \in X$ and $q \in$Q.

By routine calculation we can show that $\Box A$ is an intuitionistic Q-fuzzy PMS-ideal of X. This verifies that the converse of Theorem 3.11 is not necessarily true.

THEOREM 3.14. *If the intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, then the intuitionistic Q-fuzzy subset $\Diamond A = (\overline{\nu}_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, where $\overline{\nu}(x,q) = 1 - \nu_A(x,q)$ for all $x \in X$ and $q \in Q$.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy a PMS-ideal of X. Then, we have $\nu_A(0,q) \leq \nu_A(x,q)$ and $\nu_A(y*x,q) \leq \max\{\nu_A(z*y,q),\ \nu_A(z*x,q)\}$ for any $x, y, z \in X$ and $q \in Q$. Now we have to show that $\overline{\nu}_A(0,q) \geq \overline{\nu}_A(x,q)$ and $\overline{\nu}_A(y*x,q) \geq \min\{\overline{\nu}_A(z*y,q),\ \overline{\nu}_A(z*x,q)\}$ for all $x, y, z \in X$ and $q \in Q$.
So, $\overline{\nu}_A(0,q) = 1 - \nu_A(0,q) \geq 1 - \nu_A(x,q) = \overline{\nu}_A(x,q)$ and

$$\begin{aligned}\overline{\nu}_A(y*x,q) &= 1 - \nu_A(y*x,q)\\ &\geq 1 - max\{\nu_A(z*y,q), \nu_A(z*x,q)\}\\ &= min\{1-\nu_A(z*y,q), 1-\nu_A(z*x,q)\}\\ &= min\{\overline{\nu}_A(z*y,q), \overline{\nu}_A(z*x,q)\}.\end{aligned}$$

Hence $\Diamond A$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

REMARK 3.15. The converse of Theorem 3.14 is not necessarily true. This is shown by the following example

EXAMPLE 3.16. Let $X = \{0, a, b\}$ be a set such that $(X, 0, *)$ is a PMS-algebra with the table as in example 3.2 and Q=$\{q\}$. Define an intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ by

$$\mu_A(x, q) = \begin{cases} 0.1 & if\ x = 0, \\ 0.3 & if\ x = a, \\ 0.2 & if\ x = b. \end{cases} \quad \text{and } \nu_A(x, q) = \begin{cases} 0.1 & if\ x = 0, \\ 0.3 & if\ x = a, \\ 0.4 & if\ x = b. \end{cases} \quad \text{for all}$$

$x \in X$ and $q \in$Q. Observe that $\mu_A(0, q) = 0.1 < 0.3 = \mu_A(a, q)$. This contradicts Definition 3.1 (i). Therefore $A = (\mu_A, \nu_A)$ is not an intuitionistic Q-fuzzy PMS-ideal of X. Clearly $\Diamond A$ is an intuitionistic Q-fuzzy PMS-ideal of X. This verifies that the converse of Theorem 3.14 is not necessarily true.

By combining Theorem 3.11 and 3.14, we also get the following Theorem which is a sufficient condition for an intuitiionistic Q-fuzzy set to be an intuitiionistic Q-fuzzy PMS-ideal of X.

THEOREM 3.17. *An intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ in X is an intuitionistic Q-fuzzy PMS-ideal of X if and only if $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\nu}_A, \nu_A)$ are intuitionistic Q-fuzzy PMS-ideal of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. By Theorem 3.11 and 3.14, we conculude that $\Box A$ and $\Diamond A$ are intuitionistic Q-fuzzy PMS-ideal of X. The converse is trivial. $\Box$

THEOREM 3.18. *If the intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, then $B = (\mu_A, 0)$ and $C = (0, \overline{\mu}_A)$ are intuitionistic Q-fuzzy PMS-ideals of X.*

Proof. Let $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra X. Let $x, y, z \in X$ and $q \in$Q. Then

(i) Let $B = (\mu_B, \nu_B)$,then $\mu_B = \mu_A$ and $\nu_B = 0$. Therefore
$\mu_B(0, q) = \mu_A(0, q) \geq \mu_A(x, q) = \mu_B(x, q)$ and
$$\begin{aligned} \mu_B(y * x, q) &= \mu_A(y * x, q) \\ &\geq min\{\mu_A(z * y, q), \mu_A(z * x, q)\} \\ &= min\{\mu_B(z * y, q), \mu_B(z * x, q)\} \end{aligned}$$
Also, $\nu_B(0, q) = 0 \leq \nu_B(x, q)$ and
$\nu_B(y * x, q) = 0 \leq max\{0, 0\} = max\{\nu_B(z * y, q), \nu_B(z * x, q)\}$
Hence $B = (\mu_A, 0)$ is an intuitionistic Q-fuzzy PMS-ideal of X.

(ii) Let $C = (\mu_C, \nu_C)$, then $\mu_C = 0$ and $\nu_C = \overline{\mu}_A$. Therefore
$\mu_C(0, q) \geq min\{\mu_C(0, q), \mu_C(0, q)\} = min\{\mu_C(x, q), \mu_C(x, q)\} = \mu_C(x, q)$ and
$$\begin{aligned} \nu_C(y * x) = \overline{\mu}_A(y * x) &= 1 - \mu_A(y * x) \\ &\leq 1 - min\{\mu_A(z * y, q), \mu_A(z * x, q)\} \\ &= max\{1 - \mu_A(z * y, q), 1 - \mu_A(z * x, q)\} \\ &= max\{\nu_C(z * y), \nu_C(z * x)\} \end{aligned}$$
Hence $C = (\ 0, \overline{\mu}_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. $\Box$

4. Level Subsets of Intuitionistic Q-fuzzy PMS-ideals

In this section, we study the level subsets of intuitionistic Q-fuzzy PMS-ideals of a PMS-algebra. Characterizations of intuitionistic Q-fuzzy PMS-ideals in terms of their level subsets are given and several interesting results are investigated.

THEOREM 4.1. *If an intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X, then the sets $X_{\mu_A}^{\geq Q} = \{x \in X | \mu_A(x, q) \geq \mu_A(0, q)\}$ and $X_{\nu_A}^{\leq Q} = \{x \in X | \nu_A(x, q) \leq \nu_A(0, q)\}$ are PMS-ideals of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. As $\mu_A(0, q) \geq \mu_A(0, q)$ and $\nu_A(0, q) \leq \nu_A(0, q)$, it follows that both $X_{\mu_A}^{\geq Q}$ and $X_{\nu_A}^{\leq Q}$ contain the element 0.
Let $x, y, z \in X$ such that $z * y, z * x \in X_{\mu_A}^{\geq Q}$. Then $\mu_A(z * y, q) \geq \mu_A(0, q)$ and $\mu_A(z * x, q) \geq \mu_A(0, q)$. By Definition 3.1$(ii)$ it follows that $\mu_A(y * x, q) \geq min\{\mu_A(z * y, q), \mu_A(z * x, q)\} \geq \mu_A(0, q)$. So that $y * x \in X_{\mu_A}^{\geq Q}$. Hence $X_{\mu_A}^{\geq Q}$ is a PMS-ideal of X. Also, let $x, y, z \in X$ such that $z * y, z * x \in X_{\nu_A}^{\leq Q}$. Then $\nu_A(z * y, q) \leq \nu_A(0, q)$ and $\nu_A(z * x, q) \leq \nu_A(0, q)$. By Definition 3.1$(iii)$ it follows that $\nu_A(y * x, q) \leq max\{\nu_A(z * y, q), \nu_A(z * x, q)\} \leq \nu_A(0, q)$. So that $y * x \in X_{\nu_A}^{\leq Q}$. Hence $X_{\nu_A}^{\leq Q}$ is a PMS-ideal of X. □

THEOREM 4.2. *Let A and B be intuitionistic Q-fuzzy PMS-ideals of X such that $A \subseteq B$. If $\mu_A(0, q) = \mu_B(0, q)$ and $\nu_A(0, q) = \nu_B(0, q)$, then $X_{\mu_A}^Q \subseteq X_{\mu_B}^Q$ and $X_{\nu_A}^Q \subseteq X_{\nu_B}^Q$.*

Proof. Let $x \in X_{\mu_A}^Q$. Then $\mu_B(x, q) \geq \mu_A(x, q) = \mu_A(0, q) = \mu_B(0, q)$.
So $\mu_B(x, q) = \mu_B(0, q)$.

$$\Rightarrow x \in X_{\mu_B}^Q$$
$$\Rightarrow X_{\mu_A}^Q \subseteq X_{\mu_B}^Q$$

Similarly, let $x \in X_{\nu_A}^Q$. Then $\nu_B(x, q) \leq \nu_A(x, q) = \nu_A(0, q) = \nu_B(0, q)$.
So $\nu_B(x, q) = \nu_B(0, q)$.

$$\Rightarrow x \in X_{\nu_B}^Q$$
$$\Rightarrow X_{\nu_A}^Q \subseteq X_{\nu_B}^Q$$

Hence $X_{\mu_A}^Q \subseteq X_{\mu_B}^Q$ and $X_{\nu_A}^Q \subseteq X_{\nu_B}^Q$. □

THEOREM 4.3. *Let* I *be a nonempty subset of a PMS-algebra X and $A = (\mu_A, \nu_A)$ be an intuitionistic Q-fuzzy subset of X defined by*

$$\mu_A(x, q) = \begin{cases} 1 \; if \; x \in \mathrm{I} \\ t \; if \; x \notin \mathrm{I} \end{cases} \text{ and } \nu_A(x, q) = \begin{cases} 0 \; if \; x \in \mathrm{I} \\ s \; if \; x \notin \mathrm{I} \end{cases} \text{ for } t, s \in [0, 1] \text{ with } t + s \leq 1.$$

Then A is an intuitionistic Q-fuzzy PMS-ideal of X if and only if I *is a PMS-ideal of X.*

Proof. Suppose A= $(\mu_A, \ \nu_A)$ be an intuitionistic Q-fuzzy PMS-ideal of X. Let $x \in X$ such that $x \in$ I, then $\mu_A(x, q) = 1$ and $\nu_A(x, q) = 0$. But by Definition 3.1 (i), we have $\mu_A(0, q) \geq \mu_A(x, q)$ and $\nu_A(0, q) \leq \nu_A(x, q)$ for all $x \in X$ and $q \in$Q. Then $\mu_A(0, q) = 1$ and $\nu_A(0, q) = 0$. Hence $0 \in$ I.
Let $x, y, z \in$ X such that $z{*}y$, $z{*}x \in$ I. Then $\mu_A(y * x, q) \geq \min\{\mu_A(z * y, q), \mu_A(z * x, q)\} = \min\{1, 1\} = 1$ and $\nu_A(y * x, q) \leq \max\{\nu_A(z * y, q), \nu_A(z * x, q)\} = \min\{0, 0\} = 0$.
Hence $y * x \in$ I. Therefore I is a PMS-ideal of X.
Conversely, suppose I is a PMS-ideal of X and $x \in X$. Since $0 \in$I, $\mu_A(0, \ q) = 1 \geq \mu_A(x, \ q)$

and $\nu_A(0,\ q)$=0 $\leq \nu_A(x,\ q)$. Hence $\mu_A(0,\ q)\geq\mu_A(x,\ q)$ and $\nu_A(0,\ q)\leq\nu_A(x,\ q)$ for all $x \in X$ and $q \in Q$. Let $x, y, z \in X$. Now consider the following cases

Case(i). If $z*y, z*x \in$ I, then $y*x \in$ I. Thus $\mu_A(y*x, q)=1\geq\min\{\mu_A(z*y, q), \mu_A(z*x, q)\}$ and $\nu_A(y*x, q)=0\leq\max\{\nu_A(z*y, q), \nu_A(z*x, q)\}$.

Case(ii). If $z*y \notin$ I or $z*x \notin$ I. Then$\mu_A(y*x, q)\geq t = \min\{\mu_A(z*y, q), \mu_A(z*x, q)\}$ and $\nu_A(y*x, q)\leq s = \max\{\nu_A(z*y, q), \nu_A(z*x, q)\}$.

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

DEFINITION 4.4. Let I be any nonempty subset of a PMS-algebra X and Q is a nonempty set. Then the intuitionistic Q-fuzzy characteristic function of I is the intuitionistic Q-fuzzy subset in a PMS-algebra X, denoted by $\chi_I = \{\langle(x, q), \mu_{\chi_{\mathrm{I}}}(x, q), \nu_{\chi_{\mathrm{I}}}(x, q)\rangle | x \in X, q \in Q\}$ where $\mu_{\chi_{\mathrm{I}}} : X \times Q \to [0, 1]$ and $\nu_{\chi_I} : X \times Q \to [0, 1]$ are fuzzy subsets defined by

$$\mu_{\chi_S}(x, q) = \begin{cases} 1 & \text{if} \quad x \in \mathrm{I} \\ 0 & \text{if} \quad x \notin \mathrm{I} \end{cases} \quad \text{and} \quad \nu_{\chi_S}(x, q) = \begin{cases} 0 & \text{if} \quad x \in \mathrm{I} \\ 1 & \text{if} \quad x \notin \mathrm{I} \end{cases}$$

THEOREM 4.5. *Let* I *be any nonempty subset of a PMS-algebra X. Then the intuitionistic Q-fuzzy characteristic function* $\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ *of* I *is an intuitionistic Q-fuzzy PMS-ideal of X if and only if* I *is a PMS-ideal of X.*

Proof. Let $\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ be an intuitionistic Q-fuzzy PMS-ideal of X. Then $\mu_{\chi_{\mathrm{I}}}(0, q) \geq \mu_{\chi_{\mathrm{I}}}(x, q)$ and $\nu_{\chi_{\mathrm{I}}}(0, q) \leq \nu_{\chi_I}(x, q)$, $\mu_{\chi_{\mathrm{I}}}(y*x, q) \geq min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$ and $\nu_{\chi_{\mathrm{I}}}(y*x, q) \leq max\{\nu_{\chi_{\mathrm{I}}}(z*y, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\}$, for all $x, y, z \in X$ and $q \in Q$. Now we need to prove that I is a PMS-ideal of X. Let $x \in$X and $q \in$Q such that $x \in$ I. Then $\mu_{\chi_{\mathrm{I}}}(0, q)\geq\mu_{\chi_S}(x, q)$=1 and $\nu_{\chi_{\mathrm{I}}}(0, q)\leq\nu_{\chi_{\mathrm{I}}}(x, q)$=0. Hence 0 $\in$ I. Let $x, y, z \in X$ such that $z*y,\ z*x \in$ I. Then $\mu_{\chi_{\mathrm{I}}}(y*x, q)\geq\min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$=1 and $\nu_{\chi_{\mathrm{I}}}(y*x, q)\leq\max\{\nu_{\chi_{\mathrm{I}}}(z*y, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\} = 0$, for $q \in$Q.

$$\Rightarrow \mu_{\chi_I}(y*x, q) = 1 \text{ and } \nu_{\chi_I}(y*x, q) = 0$$
$$\Rightarrow y*x \in \mathrm{I}$$

Therefore I is a PMS-ideal of X.

Conversely, assume that I is a PMS-ideal of X. Then 0$\in$I. Thus,

$\mu_{\chi_{\mathrm{I}}}(0, q)$=1 $\geq \mu_{\chi_{\mathrm{I}}}(x, q)$ and $\nu_{\chi_{\mathrm{I}}}(0, q) = 0 \leq \nu_{\chi_{\mathrm{I}}}(x, q)$ for $x \in X$ and $q \in$ Q.

Let $x, y, z \in$ X and $q \in$ Q. Now consider the following cases.

Case (i). If $z*y, z*x \in$ I, then $y*x \in$ I, since I is a PMS-ideal of a PMS-algebra X

$\Rightarrow \mu_{\chi_{\mathrm{I}}}(z*y, q) = \mu_{\chi_{\mathrm{I}}}(z*x, q) = \mu_{\chi_{\mathrm{I}}}(y*x, q) = 1$ and
$\nu_{\chi_{\mathrm{I}}}(z*y, q) = \nu_{\chi_{\mathrm{I}}}(z*x, q) = \nu_{\chi_{\mathrm{I}}}(y*x, q) = 0$

$\Rightarrow \mu_{\chi_{\mathrm{I}}}(y*x, q) = 1 \geq min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$ and
$\nu_{\chi_{\mathrm{I}}}(y*x, q) = 0 \leq \max\{\nu_{\chi_{\mathrm{I}}}(z*x, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\}$

Hence $\mu_{\chi_{\mathrm{I}}}(y*x, q\ ) \geq min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$ and
$\nu_{\chi_{\mathrm{I}}}(y*x, q) \leq \max\{\nu_{\chi_{\mathrm{I}}}(z*y, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\}$.

Case (ii). If $z*y \notin$ I or $z*x \notin$ I, then $\mu_{\chi_I}(y*x, q\ ) \geq 0 = min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$ and $\nu_{\chi_{\mathrm{I}}}(y*x, q) \leq 1 = max\{\nu_{\chi_{\mathrm{I}}}(z*y, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\}$.

Hence $\mu_{\chi_{\mathrm{I}}}(y*x, q\ ) \geq min\{\mu_{\chi_{\mathrm{I}}}(z*y, q), \mu_{\chi_{\mathrm{I}}}(z*x, q)\}$ and
$\nu_{\chi_{\mathrm{I}}}(y*x, q) \leq max\{\nu_{\chi_{\mathrm{I}}}(z*y, q), \nu_{\chi_{\mathrm{I}}}(z*x, q)\}$.

Therefore $\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

THEOREM 4.6. *Let* I *be any nonempty subset of a PMS-algebra X and* χ_{I} *be the intuitionistic Q-fuzzy characteristic function of a nonempty subset* I *of X. Then* $\Box\chi_{\mathrm{I}} =$

$(\mu_{\chi_{\mathrm{I}}}, \overline{\mu}_{\chi_{\mathrm{I}}})$ is an intuitionistic Q-fuzzy PMS-ideal of X if and only if I *is a PMS-ideal of X.*

Proof. Let $\Box\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \overline{\mu}_{\chi_{\mathrm{I}}})$ be an intuitionistic Q-fuzzy PMS-ideal of X. Now we have to prove that I is a PMS-ideal of X.
Let $x \in X$ and $q \in Q$ such that $x \in \mathrm{I}$. Then $\mu_{\chi_{\mathrm{I}}}(x,q) = 1$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(x,q) = 0$. But by Definition 3.3(i), $\mu_{\chi_{\mathrm{I}}}(0,q) \geq \mu_{\chi_{\mathrm{I}}}(x,q)$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(0,q) \leq \overline{\mu}_{\chi_{\mathrm{I}}}(x,q)$ for all $x \in X$ and $q \in Q$. This implies that $\mu_{\chi_{\mathrm{I}}}(0,q) = 1$ and $\overline{\mu}_{\chi_I}(x,q) = 0$. Hence $0 \in \mathrm{I}$.
Let $x, y, z \in X$ and $q \in Q$ such that $z * y, z * x \in \mathrm{I}$. Then $\mu_{\chi_{\mathrm{I}}}(z * y, q) = 1 = \mu_{\chi_{\mathrm{I}}}(z * x, q)$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q) = 0 = \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q)$. But by Definition 3.3, $\mu_{\chi_{\mathrm{I}}}(y * x, q) \geq \min\{\mu_{\chi_{\mathrm{I}}}(z * y, q), \mu_{\chi_{\mathrm{I}}}(z * x, q)$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) \leq \max\left\{\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q), \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q)\right\}$ for all $x \in X$ and $q \in Q$.

$$\Rightarrow \mu_{\chi_{\mathrm{I}}}(y * x, q) = 1 \text{ and } \overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) = 0.$$
$$\Rightarrow y * x \in \mathrm{I}$$

Therefore I is a PMS-subalgebra of X.
Conversely, assume that I is a PMS-ideal of X. we need to show that $\Box\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \overline{\mu}_{\chi_{\mathrm{I}}})$ is an intuitionistic Q-fuzzy PMS-ideal of X. By Definition 2.2 (i) $0 \in \mathrm{I}$. Then $\mu_{\chi_{\mathrm{I}}}(0,q) = 1$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(0,q) = 0$. So, $\mu_{\chi_{\mathrm{I}}}(0,q) = 1 \geq \mu_{\chi_{\mathrm{I}}}(x,q)$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(0,q) = 0 \leq \overline{\mu}_{\chi_{\mathrm{I}}}(x,q)$ for all $x \in X$ and $q \in \mathrm{Q}$. Hence $\mu_{\chi_{\mathrm{I}}}(0,q) \geq \mu_{\chi_{\mathrm{I}}}(x,q)$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(0,q) \leq \overline{\mu}_{\chi_{\mathrm{I}}}(x,q)$ for all $x \in X$ and $q \in \mathrm{Q}$.
Let $x, y, z \in X$ and $q \in Q$. Then we consider the following cases.

Case (i). If $z * y, z * x \in \mathrm{I}$, then $y * x \in \mathrm{I}$. Since I is a PMS-ideal of a PMS-algebra X.
$\Rightarrow \mu_{\chi_{\mathrm{I}}}(z * y, q) = \mu_{\chi_{\mathrm{I}}}(z * x, q) = \mu_{\chi_{\mathrm{I}}}(y * x, q) = 1$ and
$\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q) = \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q) = \overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) = 0$.
$\Rightarrow \mu_{\chi_S}(y * x, q) \geq min\{\mu_{\chi_{\mathrm{I}}}(z * y, q), \mu_{\chi_I}(z * x, q)\}$ and
$\overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) \leq \max\{\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q), \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q)\}$.

Case (ii). If $z * y \notin \mathrm{I}$ or $z * x \notin \mathrm{I}$, then $\mu_{\chi_{\mathrm{I}}}(y * x, q) \geq 0 = min\{\mu_{\chi_{\mathrm{I}}}(z * y, q), \mu_{\chi_{\mathrm{I}}}(z * x, q)\}$ and $\overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) \leq 1 = max\{\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q), \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q)\}$
$\Rightarrow \mu_{\chi_{\mathrm{I}}}(y * x, q) \geq min\{\mu_{\chi_I}(z * y, q), \mu_{\chi_{\mathrm{I}}}(z * x, q)\}$ and
$\overline{\mu}_{\chi_{\mathrm{I}}}(y * x, q) \leq max\{\overline{\mu}_{\chi_{\mathrm{I}}}(z * y, q), \overline{\mu}_{\chi_{\mathrm{I}}}(z * x, q)\}$.

Therefore $\Box\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \overline{\mu}_{\chi_{\mathrm{I}}})$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

THEOREM 4.7. *Let* I *be any nonempty subset of a PMS-algebra X and $\chi_{\mathrm{I}} = (\mu_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ be an intuitionistic Q-fuzzy characteristic function of a nonempty subset* I *of X. Then $\Diamond\chi_{\mathrm{I}} = (\overline{\nu}_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ is an intuitionistic Q-fuzzy PMS-ideal of X if and only if* I *is a PMS-ideal of X.*

Proof. Let $\Diamond\chi_{\mathrm{I}} = (\overline{\nu}_{\chi_{\mathrm{I}}}, \nu_{\chi_{\mathrm{I}}})$ be an intuitionistic Q-fuzzy PMS-ideal of X. Now we need to prove that I is a PMS-ideal of X.
Let $x \in X$ and $q \in Q$ such that $x \in \mathrm{I}$. Then $\overline{\nu}_{\chi_{\mathrm{I}}}(x,q) = 1$ and $\nu_{\chi_{\mathrm{I}}}(x,q) = 0$. But by Definition 3.1(i), $\overline{\nu}_{\chi_{\mathrm{I}}}(0,q) \geq \overline{\nu}_{\chi_{\mathrm{I}}}(x,q)$ and $\nu_{\chi_{\mathrm{I}}}(0,q) \leq \nu_{\chi_{\mathrm{I}}}(x,q)$ for all $x \in X$ and $q \in Q$. This implies that $\overline{\nu}_{\chi_{\mathrm{I}}}(0,q) = 1$ and $\nu_{\chi_I}(x,q) = 0$. Hence $0 \in \mathrm{I}$.
Let $x, y, z \in X$ and $q \in Q$ such that $z * y, z * x \in \mathrm{I}$. Then $\overline{\nu}_{\chi_{\mathrm{I}}}(z * y, q) = 1 = \overline{\nu}_{\chi_{\mathrm{I}}}(z * x, q)$ and $\nu_{\chi_{\mathrm{I}}}(z * y, q) = 0 = \nu_{\chi_{\mathrm{I}}}(z * x, q)$. But by Definition 3.1, $\overline{\nu}_{\chi_{\mathrm{I}}}(y * x, q) \geq \min\{\overline{\nu}_{\chi_{\mathrm{I}}}(z * y, q), \overline{\nu}_{\chi_{\mathrm{I}}}(z * x, q)\}$ and $\nu_{\chi_{\mathrm{I}}}(y * x, q) \leq \max\{\nu_{\chi_{\mathrm{I}}}(z * y, q), \nu_{\chi_{\mathrm{I}}}(z * x, q)\}$ for all $x \in X$ and $q \in Q$.

$$\Rightarrow \overline{\nu}_{\chi_{\mathrm{I}}}(y * x, q) = 1 \text{ and } \nu_{\chi_{\mathrm{I}}}(y * x, q) = 0.$$
$$\Rightarrow x * y \in \mathrm{I}$$

Therefore I is a PMS-ideal of X.

Conversely, assume that I is a PMS-ideal of X. we need to show that $\Diamond\chi_I = (\overline{\nu}_{\chi_\mathrm{I}}, \nu_{\chi_I})$ is an intuitionistic Q-fuzzy PMS-ideal of X. By Definition 2.2 (i) $0 \in \mathrm{I}$. Then $\overline{\nu}_{\chi_\mathrm{I}}(0,q) = 1$ and $\nu_{\chi_\mathrm{I}}(0,q) = 0$. So, $\overline{\nu}_{\chi_\mathrm{I}}(0,q) = 1 \geq \overline{\nu}_{\chi_\mathrm{I}}(x,q)$ and $\nu_{\chi_\mathrm{I}}(0,q) = 0 \leq \nu_{\chi_\mathrm{I}}(x,q)$, for all $x \in X$ and $q \in Q$. Hence $\overline{\nu}_{\chi_\mathrm{I}}(0,q) \geq \overline{\nu}_{\chi_\mathrm{I}}(x,q)$ and $\nu_{\chi_\mathrm{I}}(0,q) \leq \nu_{\chi_\mathrm{I}}(x,q)$, for all $x \in X$ and $q \in Q$.
Let $x, y, z \in X$ and $q \in Q$. Then we consider the following cases

Case (i). If $z * y, z * x \in$ I, then $y * x \in \mathrm{I}$, since I is a PMS-ideal of a PMS-algebra X.
$\Rightarrow \overline{\nu}_{\chi_\mathrm{I}}(z*y,q) = \overline{\nu}_{\chi_\mathrm{I}}(z*x,q) = \overline{\nu}_{\chi_\mathrm{I}}(y*x,q) = 1$ and
$\nu_{\chi_\mathrm{I}}(z*y,q) = \nu_{\chi_\mathrm{I}}(z*x,q) = \nu_{\chi_\mathrm{I}}(y*x,q) = 0$.
$\Rightarrow \overline{\nu}_{\chi_\mathrm{I}}(y*x,q) = 1 \geq min\{\, \overline{\nu}_{\chi_\mathrm{I}}(z*y,q), \overline{\nu}_{\chi_I}(z*x,q)\}$ and
$\nu_{\chi_\mathrm{I}}(y*x,q) = 0 \leq \max\{\nu_{\chi_\mathrm{I}}(z*y,q), \nu_{\chi_\mathrm{I}}(z*x,q)\}$.
Hence $\overline{\nu}_{\chi_\mathrm{I}}(y*x,q) \geq min\{\, \overline{\nu}_{\chi_\mathrm{I}}(z*y,q), \overline{\nu}_{\chi_\mathrm{I}}(z*x,q)\}$ and
$\nu_{\chi_\mathrm{I}}(y*x,q) \leq max\{\nu_{\chi_\mathrm{I}}(z*y,q), \nu_{\chi_\mathrm{I}}(z*x,q)\}$.

Case (ii). If $z*y \notin$ I or $z*x \notin$ I, then $\overline{\nu}_{\chi_\mathrm{I}}(y*x,q) \geq 0 = min\{\overline{\nu}_{\chi_\mathrm{I}}(z*y,q), \overline{\nu}_{\chi_\mathrm{I}}(z*x,q)\}$ and
$\nu_{\chi_\mathrm{I}}(y*x,q) \leq 1 = max\{\nu_{\chi_\mathrm{I}}(z*y,q), \nu_{\chi_\mathrm{I}}(z*x\ ,q)\}$.
Hence $\overline{\nu}_{\chi_\mathrm{I}}(y*x,q) \geq min\{\overline{\nu}_{\chi_\mathrm{I}}(z*y,q), \overline{\nu}_{\chi_\mathrm{I}}(z*x,q)\}$ and
$\nu_{\chi_\mathrm{I}}(y*x,q) \leq max\{\nu_{\chi_\mathrm{I}}(z*y,q), \nu_{\chi_\mathrm{I}}(z*x\ ,q)\}$.

Therefore $\Diamond\chi_\mathrm{I} = (\overline{\nu}_{\chi_\mathrm{I}}, \nu_{\chi_\mathrm{I}})$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

THEOREM 4.8. *If $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy set in X such that the nonempty level sets $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-ideals of X for all $t, s \in$[0,1], then $A = (\mu_A,\ \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X.*

Proof. Assume that the level sets $U(\mu_A\ , t)$ and $L(\nu_A,\ s)$ are PMS-ideals of X for all $s,\ t \in [0,1]$. Let $x \in X$ and $q \in Q$ such that $\mu_A(x,q) = t$ and $\nu_A(x,q) = s$. Since $U(\ \mu_A, t)$ and $L(\nu_A,\ s)$ are PMS-ideals of X, we have $0 \in U(\mu_A, t)$ and $0 \in L(\nu_A,\ s)$. So, $\mu_A(0,q) \geq t = \mu_A(x)$ and $\nu_A(0,q) \leq s = \nu_A(x)$ for all $x \in X$ and $q \in Q$.
Assume that $\mu_A(y*x,q) \geq \min\{\mu_A(z*y,q),\ \mu_A(z*x,q)\}$ is not true. Then there exist $x_0,\ y_0,\ z_0 \in X$ such that $\mu_A(y_0*x_0,q) < \min\{\mu_A(z_0*y_0,q), \mu_A(z_0*x_0,q)\}$.
Take $t = \frac{1}{2}[\mu_A(y_0*x_0,q) + \min\{\mu_A(z_0*y_0,q), \mu_A(z_0*x_0,q)\}]$.
Thus $t \in [0,1]$ and $\mu_A(y_0*x_0,q) < t < \min\{\mu_A(z_0*y_0,q), \mu_A(z_0*x_0,q)\}$.
$\Rightarrow\ \mu_A(y_0\ *\ x_0,q) < t\ ,\ \ \mu_A(z_0\ *y_0,q) > t$ and $\mu_A(z_0\ *\ x_0,q) > t$.
$\Rightarrow\ z_0\ *y_0,\ z_0\ *\ x_0 \in U(\mu_A,\ t)$ but $y_0 *\ x_0\ \notin U(\mu_A,\ t)$.
This is a contradiction since $U(\mu_A,\ t)$ is a PMS-ideal of X.
Therefore, $\mu_A(y*x) \geq \min\{\mu_A(z*y,q),\ \mu_A(z*x,q)\}$, for any $x,\ y,\ z \in X$ and $q \in Q$. Similarly, assume that $\nu_A(y*x,q) \leq max\{\nu_A(z*y,q), \nu_A(z*x,q)\}$ not true. Then there exist $a, b, c \in X$ such that $\nu_A(b*a,q) > max\{\nu_A(c*b,q), \nu_A(c*a,q)\}$. Then by taking $s = \frac{1}{2}[\nu_A(b*a,q) + max\{\nu_A(c*b,q), \nu_A(c*a,q)\}]$, we get, $max\{\nu_A(c*b,q), \nu_A(c*a,q)\} < s < \nu_A(b*a,q)$. Therefore, $c*b \in L(\nu_A,\ s)$ and $c*a \in L(\nu_A,\ s)$ but $(b*a) \notin L(\nu_A,\ s)$.
This is a contradiction since $L(\nu_A, s)$ is a PMS-ideal of X.
Thus $\nu_A(y*x,q) \leq max\{\nu_A(z*y,q), \nu_A(z*x,q)\}$, for all $x, y, z \in X$ and $q \in Q$.
Therefore A=$(\mu_\mathrm{A}, \nu_\mathrm{A})$ is an intuitionistic Q-fuzzy PMS-ideal of X. □

THEOREM 4.9. *Let A be an intuitionistic Q-fuzzy PMS-ideal of X such that $t \in$ Im(μ_A) and $s \in$ Im(ν_A) with $t + s \leq 1$. Then the sets $U(\mu_A,\ t)$ and $L(\nu_A,\ s)$ are PMS-ideals of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X such that $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t + s \leq 1$. Then there exist $x \in X$ such that $\mu_A(x, q) = t$ and $\nu_A(x, q) = s$. By Definition 3.1(i) $\mu_A(0, q) \geq \mu_A(x, q) = t$ and $\nu_A(0, q) \leq \nu_A(x, q) = s$, for all $x \in X$. So, $0 \in U(\mu_A, t)$ and $0 \in L(\nu_A, s)$.
Also, let $x, y, z \in X$ and $q \in Q$ such that $z{*}y, z{*}x \in U(\mu_A, t)$ and $z{*}y, z{*}x \in L(\nu_A, s)$. By Definition 3.1($ii, iii$), we have $\mu_A(y * x, q) \geq \min\{\mu_A(z * y, q), \mu_A(z * x, q)\} = t$ and $\nu_A(y * x, q) \leq \max\{\mu_A(z * y, q), \nu_A(z * y, q)\} = s$. So $y * x \in U(\mu_A, t)$ and $y * x \in L(\nu_A, s)$. Therefore $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-ideals of X. □

THEOREM 4.10. *Any PMS-ideal of X can be realized as both an upper level PMS-ideal and a lower level PMS-ideal for some intuitionistic Q-fuzzy PMS-ideal of X.*

Proof. Let I be an ideal of X and A= (μ_A, ν_A) be an intuitionistic Q-fuzzy set in X defined by

$$\mu_A(x, q) = \begin{cases} t & if\ x \in \mathrm{I} \\ t_0 & otherwise \end{cases} \quad \text{and} \quad \nu_A(x, q) = \begin{cases} s & if\ x \in \mathrm{I} \\ s_0 & otherwise \end{cases}$$

for all $x \in X$ and $q \in Q$, where $t, t_0, s, s_0 \in [0, 1]$ such that $t_0 \leq t$, $s \leq s_0$ and $t + s \leq 1$, $t_0 + s_0 \leq 1$. Clearly $U(\mu_A, t) = \mathrm{I} = L(\nu_A, t)$. As $0 \in \mathrm{I}$, we have $\mu_A(0, q) = t \geq \mu_A(x, q)$ and $\nu_A(0, q) = s \geq \nu_A(x, q)$. Hence $\mu_A(0, q) \geq \mu_A(x, q)$ and $\nu_A(0, q) \leq \nu_A(x, q)$, for all $x \in X$ and $q \in Q$.
Let $x, y, z \in X$ and $q \in Q$. Now we consider the following cases

Case(i). If $z * y, z * x \in \mathrm{I}$, then $y * x \in \mathrm{I}$. Thus $\mu_A(y * x, q) = t = \min\{\mu_A(z * y, q), \mu_A(z * x, q)\}$ and $\nu_A(y * x, q) = s = \max\{\nu_A(z * y, q), \nu_A(z * x, q)\}$.

Case(ii). If $z * y \notin \mathrm{I}$ or $z * x \notin \mathrm{I}$, then $\mu_A(y * x, q) \geq t_0 = \min\{\mu_A(z * y, q), \mu_A(z * x, q)\}$ and $\nu_A(y * x, q) \leq t_0 = \max\{\nu_A(z * y, q), \nu_A(z * x, q)\}$.

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X. Therefore I is both an upper level PMS-ideal and a lower level PMS-ideal of intuitionistic Q-fuzzy PMS-ideal A of X. □

THEOREM 4.11. *Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic Q-fuzzy PMS-ideal of* X *such that* $t, s \in [0, 1]$. *Then*
(i) If $t = 1$, *then the upper level set* $U(\mu_A, t)$ *is either empty or an ideal of* X.
(ii) If $s = 0$, *then the lower level set* $L(\nu_A, s)$ *is either empty or an ideal of* X.

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic Q-fuzzy PMS-ideal of X and $t, s \in [0, 1]$. Then

(i). If there is no an element x in X such that $\mu_A(x, q) = 1$, then $U(\mu_A, t) = \emptyset$. Suppose that t = 1. Assume that $U(\mu_A, t) \neq \emptyset$. Then there exists x in X such that $x \in U(\mu_A, t)$. This implies that $\mu_A(x, q) \geq t = 1$. But by Definition 3.1 (i) we have $\mu_A(0, q) \geq \mu_A(x, q)$, for all $x \in X$ and $q \in Q$. So, $\mu_A(0, q) \geq \mu_A(x, q) \geq t = 1$. Hence $0 \in U(\mu_A, t)$.
Also, let $x, y, z \in X$ and $q \in Q$ such that $z{*}y, z{*}x \in U(\mu_A, t)$. Then $\mu_A(z * y, q) \geq t = 1$ and $\mu_A(z * x, q) \geq t = 1$. It follows that$\mu_A(z * y, q) = 1 = \mu_A(z * x, q)$. By Definition 3.1 ($ii$) $\mu_A(y * x, q) \geq \min\{\mu_A(z * y, q), \mu_A(z * x, q)\} = \min\{1, 1\} = 1$. So that $y * x \in U(\mu_A, t)$.
Therefore upper level set $U(\mu_A, t)$ is either empty or an ideal of X.

(ii). If there is no an element x in X such that $\nu_A(x, q) = 1$, then $L(\nu_A, t) = \emptyset$. Suppose that s = 0. Assume that $L(\nu_A, s) \neq \emptyset$. Then there exists x in X such that $x \in L(\nu_A, t)$. This implies that $\nu_A(x, q) \leq s = 0$. But by Definition

3.1 (i) we have $\nu_A(0,q) \leq \nu_A(x,q)$, for all $x \in X$ and $q \in Q$. So, $\nu_A(0,q) \leq \nu_A(x,q) \leq s = 0$. Hence $0 \in L(\nu_A,\ s)$.
Also, let $x, y, z \in X$ and $q \in Q$ such that $z * y, z * x \in L(\nu_A,\ s)$. Then $\nu_A(z * y, q) \geq s = 0$ and $\nu_A(z * x, q) \leq\ s = 0$. It follows that $\nu_A(z * y, q) = 0 = \nu_A(z{*}x, q)$. By Definition 3.1 (iii) $\nu_A(y * x, q) \leq \max\{\nu_A(z * y, q),\ \nu_A(z * y, q)\}$ $= \max\{0, 0\}\ = 0$. So that $y * x \in L(\nu_A,\ s)$.

Therefore lower level set $L(\nu_A,\ s)$ is either empty or an ideal of X. □

Theorem 4.12. *Let Q be a nonempty set and let $\{\mathrm{I}_i|\ i \in \Lambda\}$ be a collection of PMS-ideals of a PMS-algebra X such that*
(i). $\mathrm{X} = \bigcup_{i\in\Lambda} \mathrm{I}_i$,
(ii). $j > i \Leftrightarrow \mathrm{I}_j \subset \mathrm{I}_i$ for all $i, j \in \Lambda$, where Λ is a nonempty subset of $[0, 1]$. *Then an intuitionistic Q-fuzzy set $A = (\mu_A, \nu_A)$ in X defined by $\mu_A\ (x,q) = sup\{i \in \Lambda|\ x \in \mathrm{I}_i$ and $q \in Q\}$ and $\nu_A\ (x,q) = inf\{j \in \Lambda|\ x \in \mathrm{I}_j$ and $q \in Q\}$ is an intuitionistic fuzzy PMS-ideal of X.*

Proof. By Theorem 4.8 it is sufficient to show that $U(\mu_A, i)$ and $L(\nu_A, j)$ are PMS-ideals of X for every $i \in [\ 0,\ \mu_A(0)\ ]$ and $j \in [\nu_A(0),\ 1]$. So, to prove that $U(\mu_A, i)$ is a PMS-ideal of X, we consider the following two cases.

Case(i). Let $i\ =\ \sup\{r\ \in \Lambda|\ r < i\}$. Then we have
$x \in U(\mu_A, i)\ \Leftrightarrow\ x \in \mathrm{I}_r$ for all $r < i$

$$\Leftrightarrow x \in \bigcap_{r\ <\ i} \mathrm{I}_r$$

So that $U(\mu_A,\ i) = \bigcap_{r\ <\ i} \mathrm{I}_r$, which is a PMS-ideal of X.

Case (ii). Let $i \neq\ \sup\{r \in \Lambda|r < i\}$. That is, $i = sup\{r \in \Lambda|r \geq i\}$.
we claim that $U(\mu_A,\ i) = \bigcup_{r\ \geq\ i} \mathrm{I}_r$.
If $x \in \bigcup_{r\ \geq\ i} \mathrm{I}_r$, then $x \in \mathrm{I}_r$ for some $r \geq i$. It follows that $\mu_A(x, q) \geq r \geq i$. So that $x \in U(\mu_A,\ i)$. Hence $\bigcup_{r\ \geq\ i} \mathrm{I}_r \subseteq U(\mu_A,\ i)$.
Now assume that $x \notin \bigcup_{r\ \geq\ i} \mathrm{I}_r$. Then $x \notin \mathrm{I}_r$ for all $r \geq i$. Since $i \neq\ \sup\{r \in \mathrm{I}|r < i\}$, there exists $\epsilon > 0$ such that $(i - \epsilon,\ i) \cap \Lambda = \emptyset$. Hence $x \notin \mathrm{I}_r$ for all $r > i - \epsilon$, which means that if $x \in \mathrm{I}_r$, then $r \leq i - \epsilon$. Thus, $\mu_A(x, q) \leq i - \epsilon < i$ and so $x \notin \mathrm{U}(\mu_A,\ i)$. Therefore by contrapositive $U(\mu_A,\ i) \subseteq \bigcup_{r\ \geq\ i} \mathrm{I}_r$ and thus $U(\mu_A,\ i) = \bigcup_{r\ \geq\ i} \mathrm{I}_r$, which is a PMS-ideal of X.

To prove that $L(\nu_A, j)$ is a PMS-ideal of X for all $s \in [\nu_A(0), 1]$, we again consider the following two cases:

Case (iii). Let $j\ =\ \inf\{r\ \in \Lambda|\ j < r\}$, then we have
$x \in L(\nu_A, j) \Leftrightarrow x \in \mathrm{I}_r$ for all $j < r$

$$\Leftrightarrow x \in \bigcap_{j<r} \mathrm{I}_r$$

Hence $L(\nu_A, j) = \bigcap_{j<r} \mathrm{I}_r$. Therefore $L(\nu_A, j)$ is a PMS-ideal of X.

Case (iv). Let $j \neq\ \inf\{r \in \mathrm{I}|j < r\}$, That is, $j =\ \inf\{r \in \mathrm{I}|r \leq j\}$.
We will show that $L(\nu_A, j) = \bigcup_{r\leq j} \mathrm{I}_r$. If $x \in \bigcup_{r\leq j} \mathrm{I}_r$, then $x \in \mathrm{I}_r$ for some $r \leq j$.
It follows that $\nu_A(x, q) \leq r \leq j$. So that $x \in L(\nu_A, j)$.
Hence $\bigcup_{r\leq j} \mathrm{I}_r \subseteq L(\nu_A, j)$.
Conversely if $x \notin \bigcup_{r\leq j} \mathrm{I}_r$, then $x \notin \mathrm{I}_r$ for all $r \leq j$. Since $j \neq\ inf\{r \in \Lambda|j < r\}$, there exists $\epsilon > 0$ such that $(j,\ j + \epsilon) \cap \Lambda = \emptyset$. Hence $x \notin \mathrm{I}_r$ for all $r < s + \epsilon$, that

is, if $x \in \mathrm{I}_r$, then $r \geq j + \epsilon$. Thus $\nu_A(x, q) \geq j + \epsilon > j$. That is $x \notin L(\nu_A, j)$. Therefore by contrapositive, we have $L(\nu_A, j) \subseteq \bigcup_{r \leq j} \mathrm{I}_r$.

Consequently, $L(\nu_A, j) = \bigcup_{r \leq j} \mathrm{I}_r$.Therefore $L(\nu_A, j)$ is a PMS-ideal of X. $\square$

5. Conclusion

In this paper, we applied the idea of an intuitionistic Q-fuzzy set to PMS-ideals in PMS-algebra. We introduced the notion of an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra and investigated some associated properties. We provided the relationship between an intuitionistic Q-fuzzy PMS-subalgebra and an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra. We established a condition for an intuitionistic Q-fuzzy set in a PMS-algebra to be an intuitionistic Q-fuzzy PMS-ideal of a PMS-algebra. We characterized the intuitionistic Q-fuzzy PMS-ideals of PMS-algebra by their level sets. We hope that the findings of this work will add other dimensions to the structures of intuitionistic Q-fuzzy PMS-ideals based on intuitionistic Q-fuzzy sets and serve as the foundation for further studies. In our future works, we will extend the concept of intuitionistic fuzzy PMS-algebra to the structure of t-intuitionistic fuzzy PMS-algebra and t-Q intuitionistic fuzzy PMS-algebra to obtain other novel results. Moreover, we will develop the neutro-algebraic structures with respect to the PMS-ideals of PMS-algebra.

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