Korean J. Math. **30** (2022), No. 3, pp. 513–524
http://dx.doi.org/10.11568/kjm.2022.30.3.513

EPIS, DOMINIONS AND ZIGZAG THEOREM IN COMMUTATIVE GROUPS

Aftab Hussain Shah, Muneer Nabi*, and Shabir Ahmad Ahanger

Abstract. In this paper, we introduce the notion of tensor product in groups and prove its existence and uniqueness. Next, we provide the Isbell's zigzag theorem for dominions in commutative groups. We then show that in the category of commutative groups dominions are trivial. This enables us to deduce a well known result epis are surjective in the category of commutative groups.

1. Introduction

Isbell [6], introduced the notion of dominion in semigroups and proved the famous Isbell's Zigzag Theorem for semigroups. It gives the necessary and sufficient condition for an element of a semigroup to be in its dominion in any containing semigroup. Till now generalizations and different proofs of this theorem are provided by various authors (see [2], [4] and [8]). Recently Sohail Nasir [7], studied dominions in pomonoids and gave Zigzag Theorem for pomonoids. Further, Ahanger and Shah [1] provide short proof of Isbell's Zigzag Theorem for commutative pomonoids. P. M. Higgins [3], posed a question whether Zigzag Theorem is valid in the category of all bands or not. It is also an open problem whether Zigzag Theorem holds for groups or not. However, Schreier had shown that dominion is trivial in the category of all groups by proving that any group amalgam is embeddable in a group. In this paper we provide the zigzag theorem in the category of commutative groups and show that dominion is trivial in the category of commutative groups. This also enables us to deduce a well known result that epis in this category are precisely surjective morphisms.

2. Preliminaries

A morphism $\alpha : G \to T$ in a category $\mathcal{C}$ is called an *epimorphism* (*epi* for short) if for all morphisms $\beta, \gamma : T \to V$ with $\alpha\beta = \alpha\gamma$ implies that $\beta = \gamma$. Where, for any pair of morphisms δ, η in $\mathcal{C}$ the composition $\delta\eta$ means first δ then η. In any category of algebras, all surjective maps are epimorphisms. In some categories, such as category

Received August 12, 2022. Revised September 15, 2022. Accepted September 15, 2022.
2010 Mathematics Subject Classification: 06F05, 20M07, 20M10.
Key words and phrases: Tensor product, Commutative Groups, Epimorphisms, Dominions, Zigzag.

∗ Corresponding author.

of all groups, the converse also holds. But the converse does not hold in general in the categories of semigroups and rings. For example $i : (0, 1] \to (0, \infty)$ regarding both the intervals as multiplicative semigroups, is a non-surjective epimorphism in the category of semigroups.

Let $\mathcal{C}$ denote a category of algebras. Let G, H be in $\mathcal{C}$ such that H be a subalgebra of G. Then

$$Dom_G^{\mathcal{C}}(H) = \{g \in G : \forall\, T \in \mathcal{C}, \forall\, \alpha, \beta : G \to T,\ if\ \alpha|_H = \beta|_H \Rightarrow g\alpha = g\beta\}$$

called the *dominion* of H in G within $\mathcal{C}$. Clearly, $Dom_G^{\mathcal{C}}(H)$ is a subalgebra of G such that $H \subseteq Dom_G^{\mathcal{C}}(H) \subseteq G$. If $Dom_G^{\mathcal{C}}(H) = H$, we say that dominion of H in G is *trivial* in $\mathcal{C}$. It can be easily seen that $\alpha : G \to T$ is epi in $\mathcal{C}$ if and only if

$$Dom_T^{\mathcal{C}}(G\alpha) = T. \tag{1}$$

The most useful characterization of semigroup dominion is provided by the famous Isbell's Zigzag Theorem by Isbell in [6] and is as follows.

Theorem 2.1. ([6] Theorem 2.3) *Let U be a subsemigroup of a semigroup S and $d \in S$. Then $d \in Dom_S(U)$ if and only if $d \in U$ or there exists a system of equalities for d as follows.*

$$\begin{aligned} d &= a_0 y_1 & a_0 &= x_1 a_1 \\ a_{2i-1} y_i &= a_{2i} y_{i+1} & x_i a_{2i} &= x_{i+1} a_{2i+1}\ (i = 1, 2, \ldots, m-1) \\ a_{2m-1} y_m &= a_{2m} & x_m a_{2m} &= d, \end{aligned} \tag{2}$$

where $a_i \in U$ $(0 \le i \le 2m)$ and $x_i, y_i \in S$ $(1 \le i \le m)$.

The above system of equalities (2) is called as the *zigzag of length m* in S over U with value d.

3. Bi-Sets and Tensor Product in Groups

Definition 3.1. Let G be a group with identity 1 and X be a non-empty set. Then X is a *left G-set* if there exists an action $* : G \times X \to X$ given by $(a, x) \to a *_l x$ such that:

(i) $(ab) *_l x = a *_l (b *_l x)$ for all $a, b \in G, x \in X$;
(ii) $1 *_l x = x$ for all $x \in X$.

Dually, a non-empty set X is a *right G-set* if there exists an action $* : X \times G \to X$ given by $(x, a) \mapsto x *_r a$ such that:

(i) $x *_r (ab) = (x *_r a) *_r b$ for all $a, b \in G, x \in X$;
(ii) $x *_r 1 = x$ for all $x \in X$.

Definition 3.2. If G and G' are (not necessarily different) groups, then X is said to be (G, G')-*biset* if it is both a left G-set as well as a right G'-set and

$$(a *_l x) *_r b = a *_l (x *_r b), \text{ for all } a \in G,\ b \in G' \text{ and } x \in X.$$

Note that any group G can be considered as a (G, G')-biset, where the actions are just the multiplication of G.

REMARK 3.3. If G is a commutative group then there is no distinction between a left and a right G-set. For if, X is left G-set we may define a right action $*_r$ of G on X by

$$x *_r a = a *_l x \ (x \in X, a \in G) \tag{3}$$

and under these actions X becomes a (G, G)-biset. Note that G is also a (G, G)-biset with actions as indicated above.

DEFINITION 3.4. Let X and Y be left G-sets. Then a map $\phi : X \to Y$ satisfying $(a *_l x)\phi = a *_l (x\phi)$ (for all $a \in G,\ x \in X$) is called a *morphism (or a G-morphism or a G-map)* from X to Y.

Similarly we can define G-maps between right G-sets X and Y.

DEFINITION 3.5. Let X and Y be (G, G')-bisets. Then a map $\phi : X \to Y$ is a (G, G')-map if it is a left G-map and right G'-map such that:

$$((a *_l x) *_r a')\phi = a *_l (x\phi) *_r a',$$

for all $a \in G, a' \in G', x \in X$.

From now onwards for the sake of brevity we shall denote the left and right actions ax and xa instead of $a *_l x$ and $x *_r a$ respectively.

DEFINITION 3.6. A relation ρ on a left G-set X is called a *congruence* if ρ is an equivalence on X such that

$$(x, y) \in \rho \Rightarrow (ax, ay) \in \rho, \text{ for all } x, y \in X \text{ and } a \in G.$$

Dually we can define a congruence on a right G-set.

DEFINITION 3.7. A relation ρ on a (G, G')-biset X is called a (G, G')-congruence if it is a left G-set congruence and right G'-set congruence respectively.

Let X be a left G-set, ρ be a left G-set congruence on X and $X/\rho = \{x\rho : x \in X\}$. Then it can be easily verified that X/ρ is a left G-set with the action defined by $a(x\rho) = (ax)\rho$. The map $\rho^\natural : X \to X/\rho$ defined as $x\rho^\natural = x\rho$, for every $x \in X$ is a surjective G-map.

For any left G-set X and any right G'-set Y, it may be easily checked that $Z = X \times Y$ is a (G, G')-biset with respect to actions defined by

$$a(x, y) = (ax, y) \text{ and } (x, y)a' = (x, ya') \text{ for all } (x, y) \in Z, a \in G \text{ and } a' \in G'.$$

Let A be a (G, G')-biset, B be a (G', G'')-biset and C be a (G, G'')-biset.

DEFINITION 3.8. A (G, G'')- map $\beta : A \times B \to C$ will be called a *bimap* if for all $x \in A$, $a' \in G'$ and $y \in B$, we have

$$(xa', y)\beta = (x, a'y)\beta.$$

DEFINITION 3.9. A pair (P, ψ) consisting of a (G, G'')-biset P and a bimap $\psi : A \times B \to P$ will be called a *tensor product* of A and B over G if for every (G, G'')-biset C and every bimap $\beta : A \times B \to C$, there exists unique (G, G'')-map $\bar{\beta} : P \to C$, such that the diagram

(4)
$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & P \\ {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\bar{\beta}} & \\ C & & \end{array}$$

commutes, i.e., $\psi\bar{\beta} = \beta$.

Moreover, when $C = P$ and $\beta = \psi$, the unique $\bar{\beta}$ in the above diagram is 1_p (the identity map on P)

(5)
$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & P \\ {\scriptstyle\psi}\downarrow & \swarrow{\scriptstyle 1_p} & \\ P & & \end{array}$$

Lemma 3.10. *If there exists a tensor product of A and B over G then it is unique upto isomorphism.*

Proof. Suppose (P, ψ) and (P', ψ') are two tensor products of A and B over G. Then by definition for any (G, G'')-bisets C and C' and bimaps $\beta : A \times B \to C$ and $\beta' : A \times B \to C'$ respectively there exists a unique (G, G'')-map $\bar{\beta} : P \to C$ and $\bar{\beta}' : P' \to C'$ respectively such that the following diagrams

(6)
$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & P \\ {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\bar{\beta}} & \\ C & & \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{\psi'} & P' \\ {\scriptstyle\beta'}\downarrow & \swarrow{\scriptstyle\bar{\beta}'} & \\ C' & & \end{array}$$

commute, i.e., $\psi\bar{\beta} = \beta$ and $\psi'\bar{\beta}' = \beta'$. Then by putting $C = P'$ and $C' = P$ in the diagrams (6), we find a unique $\bar{\psi}' : P \to P'$ and $\bar{\psi} : P' \to P$ such that the following diagrams

$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & P \\ {\scriptstyle\psi'}\downarrow & \swarrow{\scriptstyle\bar{\psi}'} & \\ P' & & \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{\psi'} & P' \\ {\scriptstyle\psi}\downarrow & \swarrow{\scriptstyle\bar{\psi}} & \\ P & & \end{array}$$

commute i.e., $\psi\bar{\psi}' = \psi'$ and $\psi'\bar{\psi} = \psi$. Thus $\psi\bar{\psi}'\bar{\psi} = \psi$, and so the diagram

(7)
$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & P\ . \\ {\scriptstyle\psi}\downarrow & \swarrow{\scriptstyle\bar{\psi}'\bar{\psi}} & \\ P & & \end{array}$$

commutes. By uniqueness property in the diagram (5) $\bar{\psi}'\bar{\psi} = 1_P$. By a similar argument, $\bar{\psi}\bar{\psi}' = 1_{P'}$ and so $P \simeq P'$ as required. □

Denote $A \otimes_G B = A \times B/\tau$, where τ is the equivalence relation on $A \times B$ generated by the relation (see Howie [5])

$$T = \{((xa, y), (x, ay)) : x \in A, y \in B, a \in G\}.$$

We denote the τ-class $(a, b)\tau$ of any (a, b) by $a \otimes b$. Note that by definition of the relation τ we have

$$xa \otimes y = x \otimes ay \text{ for all } x \in A, y \in B \textit{ and } a \in G. \tag{8}$$

PROPOSITION 3.11. *Let* $x \otimes y$, $x' \otimes y' \in A \otimes_G B$. *Then* $x \otimes y = x' \otimes y'$ *if and only if either* $(x, y) = (x', y')$ *or there exists* $x_1, x_2, \ldots x_{n-1} \in A$, $y_1, y_2, \ldots, y_{n-1} \in B$, $a_1, a_2, \ldots, a_{n-1}, a_n, b_1, b_2, \ldots, b_{n-1} \in G$ *such that*

$$\begin{aligned} x &= x_1a_1 & a_1y &= b_1y_1 \\ x_1b_1 &= x_2a_2 & a_2y_1 &= b_2y_2 \\ x_ib_i &= x_{i+1}a_{i+1} & a_{i+1}y_i &= b_{i+1}y_{i+1} \quad (i = 2, \ldots, n-2), \\ x_{n-1}b_{n-1} &= x^{'}a_n & a_ny_{n-1} &= y^{'} \end{aligned} \tag{9}$$

Proof. Suppose first that we have the given sequence of equations. Then using equation (8) successively we have

$$\begin{aligned} x \otimes y &= x_1a_1 \otimes y = x_1 \otimes a_1y = x_1 \otimes b_1y_1 \\ &= x_1b_1 \otimes y_1 = x_2a_2 \otimes y_1 = x_2 \otimes a_2y_1 = x_2 \otimes b_2y_2 \\ &\ldots \\ &= x_{n-1}b_{n-1} \otimes y_{n-1} = x'a_n \otimes y_{n-1} = x' \otimes a_ny_{n-1} = x' \otimes y'. \end{aligned}$$

Conversely, suppose that $x \otimes y = x' \otimes y'$. Then by ([5] Proposition 1.4.10),

$$(x, y) = (p_1, q_1) \to (p_2, q_2) \to \ldots \to (p_{n-1}, q_{n-1}) \to (p_n, q_n) = (x', y'),$$

where $((p_{i-1}, q_{i-1}), (p_{i+1}, q_{i+1})) \in T \cup T^{-1}$. We can assume without loss of generality that the sequence begins and ends with right move $(xa, y) \to (x, ay)$.

$$\begin{aligned} (x, y) = (p_1, q_1) &= (x_1a_1, y) \to (x_1, a_1y), \\ &= (x_1, b_1y_1) \to (x_1b_1, y_1), \\ &= (x_2a_2, y_1) \to (x_2, a_2y_1), \\ &= (x_2, b_2y_2) \to (x_2b_2, y_2), \\ &\vdots \\ &= (x_{n-1}, b_{n-1}y_{n-1}) \to (x_{n-1}b_{n-1}, y_{n-1}), \\ &= (x'a_n, y_{n-1}) \to (x', a_ny_{n-1}) = (x', y'). \end{aligned}$$

This gives system of equations:

$$\begin{aligned} x &= x_1a_1 \;\; a_1y = b_1y_1 \\ x_1b_1 &= x_2a_2 \;\; a_2y_1 = b_2y_2 \\ &\vdots \\ x_{n-1}b_{n-1} &= x'a_n \;\; a_ny_{n-1} = y', \end{aligned}$$

as required. □

Proposition 3.12. *The equivalence relation τ is a (G, G'')-congruence on $A \times B$, where $A \times B$ is a (G, G'')-biset.*

Proof. Suppose $(x, y)\tau = (x', y')\tau$, i.e., $x \otimes y = x' \otimes y'$ and $g \in G, g'' \in G''$. We have

$$\begin{aligned} gx \otimes y &= gx' \otimes y' \quad \text{(by equations (9))} \\ \Rightarrow (gx, y)\tau &= (gx', y')\tau \\ \Rightarrow g(x, y)\tau &= g(x', y')\tau \quad \text{(since } A \times B \text{ is a } (G, G'')\text{-biset).} \end{aligned}$$

This implies $(g(x, y), g(x', y')) \in \tau$. Similarly $((x, y)g'')\tau = ((x', y')g'')\tau$ implies $((x, y)g'', (x', y')g'') \in \tau$. Thus τ is a (G, G'')-congruence. □

Remark 3.13. For a left G-set A and a right G'-set B, $Z = A \otimes_G B$ forms a (G, G')-biset with respect to action defined by

$$g(x \otimes y) = (gx) \otimes y, \; (x \otimes y)g' = x \otimes (yg')$$

for all $(x, y) \in Z$, $g \in G$ and $g' \in G'$.

Remark 3.14. $\tau^\natural : A \times B \to (A \times B)/\tau$ defined by $(x, y)\tau^\natural = (x, y)\tau$ is a (G, G')-map.

For this let $(x, y) \in A \times B$, $a \in G$ and $a' \in G'$. Now

$$\begin{aligned} (a(x, y))\tau^\natural &= (ax, y)\tau^\natural \quad \text{(since } A \times B \text{ is a } (G, G')\text{-biset)} \\ &= (ax, y)\tau \\ &= a(x, y)\tau \quad \text{(since } (A \times B)/\tau \text{ is a } (G, G')\text{-biset)} \\ &= a(x, y)\tau^\natural. \end{aligned}$$

Similarly, $((x, y)a')\tau^\natural = (x, y)\tau^\natural a'$.

Proposition 3.15. *Let A be a (G, G')-biset, B be a (G', G'')-biset. Then $(A \otimes_G B, \tau^\natural)$ is the tensor product of A and B over G.*

Proof. Let $(x, y) \in A \times B$ and $g \in G$. Then, we have

$$\begin{aligned} (xg, y)\tau^\natural &= (xg, y)\tau \\ &= xg \otimes y \\ &= x \otimes gy \quad \text{(by equation (8))} \\ &= (x, gy)\tau \\ &= (x, gy)\tau^\natural. \end{aligned}$$

Therefore, by Remark 3.14 $\tau^\natural$ is a bimap. Now let C be a (G, G')-biset and let $\beta : A \times B \to C$ be a bimap. Define $\bar{\beta} : A \otimes_G B \to C$ by

$$(x \otimes y)\bar{\beta} = (x, y)\beta \; (x \in A, y \in B). \tag{10}$$

Now to verify that $\bar{\beta}$ is well defined. We take $x \otimes y, x' \otimes y' \in A \otimes_G B$ such that $x \otimes y = x' \otimes y'$. We have

$$\begin{aligned}
(x \otimes y)\bar{\beta} &= (x, y)\beta \quad \text{(by equation (10))}\\
&= (x_1 a_1, y)\beta \quad \text{(by Proposition 3.11)}\\
&= (x_1, a_1 y)\beta \quad \text{(by equation (8))}\\
&= (x_1, b_1 y_1)\beta \quad \text{(by Proposition 3.11)}\\
&= (x_1 b_1, y_1)\beta \quad \text{(by equation (8))}\\
&= (x_2 a_2, y_1)\beta \quad \text{(by Proposition 3.11)}\\
&= (x_2, a_2 y_1)\beta \quad \text{(by equation (8))}\\
&= (x_2, b_2 y_2)\beta \quad \text{(by Proposition 3.11)}\\
&= (x_2 b_2, y_2)\beta \quad \text{(by equation (8))}.
\end{aligned}$$

Next for $i = 2, 3, \ldots, n-2$, we have

$$\begin{aligned}
(x_i, b_i y_i)\beta &= (x_i b_i, y_i)\beta \quad \text{(by equation (8))}\\
&= (x_{i+1} a_{i+1}, y_i)\beta \quad \text{(by Proposition 3.11)}\\
&= (x_{i+1}, a_{i+1} y_i)\beta \quad \text{(by equation (8))}\\
&= (x_{i+1}, b_{i+1} y_{i+1}) \quad \text{(by Proposition 3.11)}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
(x_{n-1}, b_{n-1} y_{n-1})\beta &= (x_{n-1} b_{n-1}, y_{n-1})\beta \quad \text{(by equation (8))}\\
&= (x' a_n, y_{n-1})\beta \quad \text{(by Proposition 3.11)}\\
&= (x', a_n y_{n-1})\beta \quad \text{(by equation (8))}\\
&= (x', y')\beta \quad \text{(by Proposition 3.11)}\\
&= (x' \otimes y')\bar{\beta} \quad \text{(by equation (10))},
\end{aligned}$$

as required. Now let $g \in G$ and $g' \in G'$. Then

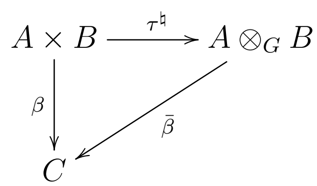
$$\begin{aligned}
(g(x \otimes y))\bar{\beta} &= (gx \otimes y)\bar{\beta} \quad \text{(by Remark 3.13)}\\
&= (gx, y)\beta \quad \text{(by equation (10))}\\
&= g(x, y)\beta \quad \text{(Since } \beta \text{ is a } (G, G')\text{-bimap)}\\
&= g(x \otimes y)\bar{\beta} \quad \text{(by equation(10))}.
\end{aligned}$$

Similarly,

$$((x \otimes y)g')\bar{\beta} = (x \otimes y)\bar{\beta} g'.$$

Therefore, $\bar{\beta}$ is a (G, G')-map. Next, we show the following diagram commutes.

(11)



For this let $(x, y) \in A \times B$. Then

$$\begin{aligned}(x, y)\tau^{\natural}\bar{\beta} &= ((x, y)\tau)\bar{\beta} \\ &= (x \otimes y)\bar{\beta} \\ &= (x, y)\beta.\end{aligned}$$

Thus, $\tau^{\natural}\bar{\beta} = \beta$.
The uniqueness of $\bar{\beta}$ follows easily. Thus $(A \otimes_G B, \tau^{\natural})$ is a tensor product of A and B over G. □

4. Zigzag Theorem for Commutative Groups

In this section we provide the Zigzag theorem for commutative groups. Throughout this section we denote by 1 the identity of the group G and $\mathcal{C}$ denotes the category of commutative groups.

To prove our main theorem we first prove the following Lemma.

LEMMA 4.1. *Let H be a subgroup of a commutative group G. Then $A = G \otimes_H G$ is a commutative group.*

Proof. Define a product on $A = G \otimes_H G$ as, for any $a \otimes b, c \otimes d$ in A

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \tag{12}$$

We claim that (12) is well defined and makes A into a commuttive group. For this take any $a \otimes b, c \otimes d$, $a' \otimes b', c' \otimes d'$ in A such that $a \otimes b = a' \otimes b'$ and $c \otimes d = c' \otimes d'$. We must get $ac \otimes bd = a'c' \otimes b'd'$. By Proposition 3.11, we have

$$\begin{aligned} a &= a_1 s_1 & s_1 b &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2 & s_2 b_1 &= t_2 b_2 \\ a_i t_i &= a_{i+1} s_{i+1} & s_{i+1} b_i &= t_{i+1} b_{i+1} \ (i = 2, ..., n-2) \\ a_{n-1} t_{n-1} &= a' s_n & s_n b_{n-1} &= b' \end{aligned} \tag{13}$$

for all $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in G$, $s_1, s_2, \ldots s_n, t_1, t_2, \ldots, t_{n-1} \in H$.

And,

$$\begin{aligned} c &= a'_1 s'_1 & s'_1 d &= t'_1 b'_1 \\ a'_1 t'_1 &= a'_2 s'_2 & s'_2 b'_1 &= t'_2 b'_2 \\ a'_i t'_i &= a'_{i+1} s'_{i+1} & s'_{i+1} b'_i &= t'_{i+1} b'_{i+1} \quad (i = 2, ..., n-2), \\ a'_{n-1} t'_{n-1} &= c' s'_n & s'_n b'_{n-1} &= d' \end{aligned} \tag{14}$$

for all $a'_1, a'_2, \ldots, a'_{n-1}, b'_1, b'_2, \ldots, b'_{n-1} \in G$, $s'_1, s'_2, \ldots s'_n, t'_1, t'_2, \ldots, t'_{n-1} \in H$.

Now from system of equations (13), (14) and commutativity of G. We have

$$\begin{aligned} ac &= (a_1a_1')(s_1s_1') & (s_1s_1')(bd) &= (t_1t_1')(b_1b_1') \\ (a_1a_1')(t_1t_1') &= (a_2a_2')(s_2s_2') & (s_2s_2')(b_1b_1') &= (t_2t_2')(b_2b_2') \\ (a_ia_i')(t_it_i') &= (a_{i+1}a_{i+1}')(s_{i+1}s_{i+1}') & (s_{i+1}s_{i+1}')(b_ib_i') &= (t_{i+1}t_{i+1}')(b_{i+1}b_{i+1}') \\ & & (i &= 2, \dots, n-2), \\ (a_{n-1}a_{n-1}')(t_{n-1}t_{n-1}') &= (a'c')(s_ns_n') & (s_ns_n')(b_{n-1}b_{n-1}') &= b'd' \end{aligned}$$

where $a_1a_1', a_2a_2', \dots, a_{n-1}a_{n-1}', b_1b_1', b_2b_2', \dots, b_{n-1}b_{n-1}' \in G$ and $s_1s_1', s_2s_2', \dots, s_ns_n', t_1t_1', t_2t_2',$
$\dots, t_{n-1}t_{n-1}' \in H$. By Proposition 3.11 we have $ac \otimes bd = a'c' \otimes b'd'$ and hence (12) is well defined. It is easily seen that (12) is also associative. Now, it remains to show the existence of identity and inverse in A. Take $a \otimes b \in A$. Then

$$(a \otimes b)(1 \otimes 1) = (a1 \otimes b1) = (a \otimes b).$$

And

$$(1 \otimes 1)(a \otimes b) = (1a \otimes 1b) = (a \otimes b).$$

Hence $(1 \otimes 1)$ is the identity of $A = G \otimes_H G$. Now let $a \otimes b \in A$, then $a^{-1} \otimes b^{-1} \in A$ and

$$\begin{aligned} (a \otimes b)(a^{-1} \otimes b^{-1}) &= (aa^{-1} \otimes bb^{-1}) = (1 \otimes 1), \\ (a^{-1} \otimes b^{-1})(a \otimes b) &= (a^{-1}a \otimes b^{-1}b) = (1 \otimes 1). \end{aligned}$$

Therefore, $a^{-1} \otimes b^{-1}$ is the inverse of $a \otimes b$ in $A = G \otimes_H G$. The commutativity of A follows simply by using commutativity of G in (12). □

Theorem 4.2. *Let H be a subgroup of a commutative group G and let $d \in G$. Then $d \in Dom_G^{\mathcal{C}}(H)$ if and only if $d \otimes 1 = 1 \otimes d$ in the tensor product $A = G \otimes_H G$.*

Proof. Suppose $d \otimes 1 = 1 \otimes d$. Let T be any commutative group and $\beta, \gamma : G \to T$ be morphisms of commutative groups such that $\beta|_H = \gamma|_H$ for all $h \in H$. We show that $d\beta = d\gamma$. First, we show T is a (H, H)-biset with respect to action defined as

(15) $\quad ht = (h\beta)t \ (= (h\gamma)t), \ th = t(h\beta) \ (= t(h\gamma))$ for all $h \in H, t \in T$.

Now for all $h_1, h_2 \in H$, $t \in T$, we have

$$\begin{aligned} (h_1h_2)t &= ((h_1h_2)\beta)t \quad \text{(by equations (15))} \\ &= (h_1\beta)(h_2\beta)t \quad \text{(since } \beta \text{ is morphism)} \\ &= (h_1\beta)(h_2t) \quad \text{(by equations (15))} \\ &= h_1(h_2t) \quad \text{(by equations (15))} \\ \text{and } 1t &= (1\beta)t = t \quad \text{for all } t \in T. \end{aligned}$$

Similarly, for all $t \in T$ and for all $h_1, h_2 \in H$ we have

$$t(h_1h_2) = (th_1)h_2 \text{ and } t1 = t.$$

Also,

$$\begin{aligned} (h_1t)h_2 &= (h_1t)(h_2\beta) \quad \text{(by equations (15))} \\ &= h_1t(h_2\beta) \\ &= h_1(th_2) \quad \text{(by equations (15))}. \end{aligned}$$

Thus, T is an (H,H)-biset. Define $\psi : G \times G \to T$ by

$$(g,g')\psi = (g\beta)(g'\gamma), \ (g,g') \in G \times G. \tag{16}$$

Then for all $g, g' \in G$, $h \in H$, we have

$$\begin{aligned}(h(g,g'))\psi &= (hg,g')\psi \\ &= ((hg)\beta)(g'\gamma) \quad \text{(by equation (16))} \\ &= (h\beta)(g\beta)(g'\gamma) \quad \text{(since } \beta \text{ is a morphism)} \\ &= h(g\beta)(g'\gamma) \quad \text{(by equations (15))} \\ &= h(g,g')\psi \quad \text{(by equation (16))}.\end{aligned}$$

Similarly, we can show

$$((g,g')h)\psi = (g,g')\psi h \text{ (for all } g,g' \in G,\, h \in H).$$

Therefore, ψ is an (H,H)-map. Next, for all $h \in H$ and $g, g' \in G$, we have

$$\begin{aligned}(gh,g')\psi &= ((gh)\beta)(g'\gamma) \\ &= (g\beta)(h\beta)(g'\gamma) \quad \text{(since } \beta \text{ is a morphism)} \\ &= (g\beta)(h\gamma)(g'\gamma) \quad \text{(since } h\beta = h\gamma) \\ &= (g\beta)(hg')\gamma \quad \text{(since } \gamma \text{ is a morphism)} \\ &= (g,hg')\psi \quad \text{(by equation (16))}.\end{aligned}$$

Therefore, ψ is a bimap. Since $(G \otimes_H G, \tau^\natural)$ is a tensor product, therefore, by Proposition 3.15, there exists a map $\bar{\psi} : G \otimes_H G \to T$ such that

$$(g \otimes g')\bar{\psi} = (g,g')\psi = (g\beta)(g'\gamma), \ g \otimes g' \in G \otimes_H G. \tag{17}$$

Now, we have

$$\begin{aligned}d\beta &= (d1)\beta \\ &= (d\beta)(1\beta) \quad \text{(since } \beta \text{ is a morphism)} \\ &= (d\beta)(1\gamma) \quad \text{(since } 1\beta = 1\gamma) \\ &= (d \otimes 1)\bar{\psi} \quad \text{(by equation (17))} \\ &= (1 \otimes d)\bar{\psi} \quad \text{(since } d \otimes 1 = 1 \otimes d) \\ &= (1\beta)(d\gamma) \quad \text{(by equation (17))} \\ &= (1\gamma)(d\gamma) \quad \text{(since } 1\beta = 1\gamma) \\ &= (1d)\gamma \quad \text{(since } \gamma \text{ is a morphism)} \\ &= d\gamma.\end{aligned}$$

Therefore, $d \in Dom_G^{\mathcal{C}}(H)$, as required.

To prove the converse part. Let $d \in Dom_G^{\mathcal{C}}(H)$, we show that $d \otimes 1 = 1 \otimes d$. Define $\beta, \gamma : G \to A$ by the rule

$$g\beta = g \otimes 1 \text{ and } g\gamma = 1 \otimes g.$$

Then β, γ are clearly commutative group morphisms. Now for any $h \in H$

$$h \otimes 1 = 1 \otimes h \Rightarrow h\beta = h\gamma.$$

Therefore, $d\beta = d\gamma$ which implies that $d \otimes 1 = 1 \otimes d$, as required. This completes the proof of the theorem. □

The next result is the zigzag theorem for commutative groups.

THEOREM 4.3. *Let H be a subgroup of a commutative group G and let $d \in G$. Then $d \in Dom_G^{\mathcal{C}}(H)$ if and only if either $d \in H$ or there exists a factorizations of d as follows.*

$$(18)\qquad \begin{aligned} d &= h_0 y_1 & h_0 &= x_1 h_1 \\ h_{2i-1} y_i &= h_{2i} y_{i+1} & x_i h_{2i} &= x_{i+1} h_{2i+1}\ (i = 1, 2, \ldots, m-1) \\ h_{2m-1} y_m &= h_{2m} & x_m h_{2m} &= d, \end{aligned}$$

where $h_i \in H$ $(0 \leq i \leq 2m)$ and $x_i, y_i \in G$ $(1 \leq i \leq m)$. The above factorizations is called as the zigzag of length *m in G over H with value d.*

Proof. The proof follows by using Theorem 4.2 and Proposition 3.11. □

COROLLARY 4.4. *Let $\mathcal{C}$ be the category of commutative groups and G, H are in $\mathcal{C}$ with H a subgroup of G. Then $Dom_G^{\mathcal{C}}(H) = H$ i.e., dominion is trivial.*

Proof. Take any $d \in Dom_G^{\mathcal{C}}(H)$. If $d \in H$, then there is nothing to prove. Otherwise, by Theorem 4.3, d has a factorization of type (18) in G over H. Now

$$\begin{aligned} d &= h_0 y_1 \quad \text{(by zigzag equations)} \\ &= h_0 h_1^{-1} h_1 y_1 \quad \text{(since } G \text{ is a group)} \\ &= h_0 h_1^{-1} h_2 y_2 \quad \text{(by zigzag equations)} \\ &= h_0 h_1^{-1} h_2 h_3^{-1} h_3 y_2 \quad \text{(since } G \text{ is a group).} \end{aligned}$$

Next for $i = 3, \cdots, m-1$, we have

$$\begin{aligned} d &= h_0 h_1^{-1} h_2 h_3^{-1} \ldots h_{2i-1}^{-1} h_{2i-1} y_i \\ &= h_0 h_1^{-1} h_2 h_3^{-1} \ldots h_{2i-1}^{-1} h_{2i} y_{i+1} \quad \text{(by zigzag equations)} \\ &= h_0 h_1^{-1} h_2 h_3^{-1} \ldots h_{2i-1}^{-1} h_{2i} h_{2i+1}^{-1} h_{2i+1} y_{i+1} \quad \text{(since } G \text{ is a group).} \end{aligned}$$

Finally, we have

$$\begin{aligned} d &= h_0 h_1^{-1} h_2 h_3^{-1} \ldots h_{2m-3}^{-1} h_{2m-2} h_{2m-1}^{-1} h_{2m-1} y_m \\ &= h_0 h_1^{-1} h_2 h_3^{-1} \ldots h_{2m-3}^{-1} h_{2m-2} h_{2m-1}^{-1} h_{2m} \quad \text{(by zigzag equations),} \end{aligned}$$

which is in H. Since $H \subseteq Dom_G^{\mathcal{C}}(H)$. Thus $Dom_G^{\mathcal{C}}(H) = H$. □

COROLLARY 4.5. *Let $\mathcal{C}$ denote the category of commutative groups then epis are surjective in $\mathcal{C}$.*

Proof. Let $G, H \in \mathcal{C}$ with H a subgroup of G. Let $\alpha : H \to G$ be an epimorphism, we must have $Im\alpha = G$. By equation (1) $Dom_G^{\mathcal{C}}(im\alpha) = G$. By Corollary 4.4 $Dom_G^{\mathcal{C}}(Im\alpha) = Im\alpha$. Thus we have $Im\alpha = G$. □

References

[1] Ahanger S. A. and Shah A. H., *On zigzag theorem for commutative pomonoids and certain closed and absolutely closed monoids and pomonoids*, Beiträge Zur Algebra und Geometrie, **61** (2020) 9–21.

[2] Alam. N. and Khan. N. M., *A note on Isbell's zigzag theorem for commutative semigroups*, Algebra Letters, **1** (1) (2012) 22–27.

[3] Higgins, P.M., *Epimorphisms and semigroup varieties*, PhD. Thesis, Department of Mathematics, Monash University Australia (1983).

[4] Howie, J.M. and Isbell, J.R., *Epimorphisms and dominions II*, J.Algebra **6**, (1967) 7–21 .

[5] Howie, J. M., *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford (1995).

[6] Isbell, J. R., *Epimorphisms and dominions*, Proceedings of the Conference on Categorical Algebra, La Jolla, (1965), Lange and Springer, Berlin, (1966), 232–246.

[7] Sohail. N., *Zigzag theorem for partially ordered monoids*, commun. Algebra, **42** (2014), 2559–2583.

[8] Stenstrom B., *Flatness and localization over monoids*, Math. Nachr. **48** (1971), 315–334.

Aftab Hussain Shah
Department of Mathematics, Central University of Kashmir,
Ganderbal 191131, India
E-mail: aftab@cukashmir.ac.in

Muneer Nabi
Department of Mathematics, Central University of Kashmir,
Ganderbal 191131, India
E-mail: muneernabi@cukashmir.ac.in

Shabir Ahmad Ahanger
Department of Mathematics, Central University of Kashmir,
Ganderbal 191131, India
E-mail: shabir@cukashmir.ac.in