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CHARACTERIZATIONS ON GEODESIC GCR-LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER STATISTICAL MANIFOLD

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Abstract. This article introduces the structure of GCR-lightlike submanifolds of an indefinite Kaehler statistical manifold and derives their geometric properties. The characterizations on totally geodesic, mixed geodesic, D-geodesic and D'-geodesic GCR-lightlike submanifolds have also been obtained.

1. INTRODUCTION

The geometry of lightlike submanifolds of semi-Riemannian manifolds introduced by [4] is a significant field of study. Various classes of lightlike submanifolds of an indefinite Kaehler manifold have since been investigated. The CR-lightlike submanifolds and SCR-lightlike submanifolds of the indefinite Kaehler manifold were developed by [4],[5]. Further, [6] introduced a class called GCR-lightlike submanifolds of an indefinite Kaehler manifold which contains SCR-lightlike and CR-lightlike submanifolds as subcases.

The statistical manifolds, which are an outcome of the inspection of geometric structures on probability distributions, were initiated by [19], and thereafter developed substantially by [1], [2],[7],[14],[8],[9],[11],[12],[15] and [21] et al. Consolidating the notion of the statistical structure with the indefinite Kaehler metric structure, we get an indefinite Kaehler statistical manifold which was introduced and studied for the CR-lightlike submanifolds and hypersurfaces by [10],[17] and [18].

Motivated by these, the geometry of GCR-lightlike submanifolds for an indefinite Kaehler statistical manifold has been studied. The geodesicity and the structure of subbundles of the tangent bundle for the GCR-submanifolds

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have been worked upon. Some characterizations on totally geodesic, mixed geodesic, D-geodesic and D'-geodesic GCR-lightlike submanifolds have been developed.

2. PRELIMINARIES

Following [4], some basic facts about the lightlike theory of submanifolds are as follows:
Consider $(\bar{M}, \bar{g})$ as an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\bar{g}$ and of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$.
Let (M, g) be a m-dimensional lightlike submanifold of $\bar{M}$. In this case, there exists a smooth distribution $Rad(TM)$ on M of rank $r > 0$, known as Radical distribution on M such that $Rad(TM_p) = TM_p \cap TM_p^{\perp}, \forall\, p \in M$ where TM_p and $TM_p^{\perp}$ are degenerate orthogonal spaces but not complementary. Then M is called an r-lightlike submanifold of $\bar{M}$.

Now, consider $S(TM)$, known as screen distribution, as a complementary distribution of radical distribution in TM i.e.,

$$TM = Rad(TM) \perp S(TM)$$

and $S(TM^{\perp})$, called screen transversal vector bundle, as a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$ i.e.,

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp})$$

As $S(TM)$ is non degenerate vector subbundle of $T\bar{M}|_M$, we have

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\bar{M}|_M$.

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}|_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}|_M = TM \oplus tr(TM),$$

$$= (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

Theorem 2.1. [4] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r- lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$.Then there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TM)|_U)$ consisting of smooth sections $\{N_1, \cdots, N_r\}$ $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad i, j = 0, 1, \cdots, r$$

where $\{\xi_1, \cdots, \xi_r\}$ *is a lightlike basis of* $\Gamma(RadTM)|_U$.

Let $\hat{\nabla}$ be the Levi-Civita connection on $\bar{M}$. We have, from the above mentioned theory, the Guass and Weingarten formulae as:

$$\hat{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall\ X, Y \in \Gamma(TM) \tag{1}$$

and

$$\hat{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall\ X \in \Gamma(TM), V \in \Gamma(tr(TM))$$

Using the projections $L : tr(TM) \to ltr(TM)$ and $S : tr(TM) \to S(TM^{\perp})$, from [4], we have the following equations from the above formulae:

$$\hat{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y)$$

$$\hat{\nabla}_X V = -A_V X + D^l_X V + D^s_X V$$

In particular,

$$\hat{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N)$$

$$\hat{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W)$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Here $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $D^l_X V = L(\nabla^{\perp}_X V)$, $D^s_X V = S(\nabla^{\perp}_X V)$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, $\nabla^s_X W, D^s(X,N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Denoting by P, the projection morphism of tangent bundle TM to the screen distribution, we consider the following decomposition:

$$\nabla_X PY = \nabla'_X PY + h'(X, PY)$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi$$

for any $X, Y \in \Gamma(TM)$,$\xi \in \Gamma(Rad(TM))$, where $\{\nabla'_X PY, A'_\xi X\}$ and $\{h'(X, PY), \nabla'^t_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively; ∇' and ∇'^t are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$ respectively. Then we have the following equations:

$$\bar{g}(h^l(X, PY), \xi) = g(A'_\xi X, PY), \quad \bar{g}(h'(X, PY), N) = g(A_N X, PY)$$

$$g(A'_\xi PX, PY) = g(PX, A'_\xi PY), \quad A'_\xi \xi = 0$$

for any $X, Y \in \Gamma((TM))$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Further, the theory of lightlike submanifolds of an indefinite statistical manifold as investigated by [7], [8], [15], [21] is as follows:

A pair $(\bar{\nabla}, \bar{g})$ is called a **statistical structure** on a semi-Riemannian manifold $\bar{M}$ such that for all $X, Y, Z \in \Gamma(T\bar{M})$

1. $\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y]$;
2. $(\bar{\nabla}_X \bar{g})(Y, Z) = (\bar{\nabla}_Y \bar{g})(X, Z)$ hold.

Then $(\bar{M}, \bar{g}, \bar{\nabla})$ is said to be an **indefinite statistical manifold**.

Moreover, there exists $\bar{\nabla}^*$ which is a dual connection of $\bar{\nabla}$ with respect to $\bar{g}$, satisfying

$$X\bar{g}(Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z).$$

If $(\bar{M}, \bar{g}, \bar{\nabla})$ is an indefinite statistical manifold, then so is $(\bar{M}, \bar{g}, \bar{\nabla}^*)$. Hence, the indefinite statistical manifold is denoted by $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Let (M, g) be a lightlike submanifold of an indefinite statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$. The geometry of the lightlike submanifolds of an indefinite statistical manifold developed heretofore gives the Gauss and Weingarten formulae on its structure as

(2)
$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + h^{*l}(X, Y) + h^{*s}(X, Y)$$

(3) $$\bar{\nabla}_X V = -A_V X + D^l_X V + D^s_X V, \quad \bar{\nabla}^*_X V = -A^*_V X + D^{*l}_X V + D^{*s}_X V,$$

(4)
$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}^*_X N = -A^*_N X + \nabla^{*l}_X N + D^{*s}(X, N)$$

(5)
$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad \bar{\nabla}^*_X W = -A^*_W X + \nabla^{*s}_X W + D^{*l}(X, W)$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Now, from the concept of indefinite statistical manifold and using the equations (2), (3), (4), (5), we have the following results:

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^{*l}(X, W)) = \bar{g}(Y, A^*_W X),$$
$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, \nabla^*_X \xi) + \bar{g}(Y, h^{*l}(X, \xi)) = 0,$$
$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A^*_W X),$$
$$\bar{g}(A_N X, PY) = \bar{g}(N, \bar{\nabla}^*_X PY),$$

and

$$\bar{g}(A_N X, N') + \bar{g}(A^*_{N'} X, N) = 0.$$

From the non-degenerate theory of submanifolds of a statistical manifold, it is known that submanifold of statistical manifold is a statistical manifold but this is not true for lightlike submanifolds since the definition of statistical manifold and the equation (2) implies

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = \bar{g}(Y, h^l(X, Z)) - \bar{g}(X, h^l(Y, Z)).$$

and

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla^*_X Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(Y, h^{*l}(X, Z))$$

Considering the projection morphism P of the tangent bundle TM to the screen distribution, we have the following decomposition w.r.t ∇ and ∇^*:

(6) $$\nabla_X PY = \nabla'_X PY + h'(X, PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X, PY)$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime t}_X \xi \tag{7}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$.

Using (2),(3),(6) and (7), we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^{*\prime}_\xi X, PY), \quad \bar{g}(h^{*l}(X, PY), \xi) = g(A'_\xi X, PY) \tag{8}$$

$$\bar{g}(h'(X, PY), N) = g(A^*_N X, PY), \quad \bar{g}(h^{*\prime}(X, PY), N) = g(A_N X, PY) \tag{9}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. As h^l and h^{*l} are symmetric, so from (8), we obtain

$$g(A'_\xi PX, PY) = g(PX, A'_\xi PY), \quad g(A^{*\prime}_\xi PX, PY) = g(PX, A^{*\prime}_\xi PY).$$

Let $\bar{\nabla}^\circ$ be the Levi-Civita connection w.r.t $\bar{g}$. Then, we have $\bar{\nabla}^\circ = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.

For a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1, 2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-Civita connection $\bar{\nabla}^\circ$ is defined as

$$K(X, Y) = K_X Y = \bar{\nabla}_X Y - \bar{\nabla}^\circ_X Y \tag{10}$$

Since $\bar{\nabla}$ and $\bar{\nabla}^\circ$ are torsion free, we have

$$K(X, Y) = K(Y, X), \quad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z)$$

for any $X, Y, Z \in \Gamma(TM)$.

Also, from (10), we have

$$\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(K(X, Y), Z) + \bar{g}(\bar{\nabla}^\circ_X Y, Z) \tag{11}$$

Now, from [21], we have the following equations for the almost Hermitian manifold:

$$(\bar{\nabla}_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y + (K_X \bar{J})Y$$

$$(\bar{\nabla}^*_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y - (K_X \bar{J})Y$$

for any $X, Y, Z \in \Gamma(TM)$.

This implies

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}^*_X \bar{J})Y = 2(\bar{\nabla}^\circ_X \bar{J})Y$$

The idea of the indefinite Kaehler statistical manifold introduced in [10] and further elaborated in [17] and [18] is as below:

Definition 2.2. *Let $(\bar{g}, \bar{J})$ be an indefinite Hermitian structure on $\bar{M}$. A triplet $(\bar{\nabla} = \bar{\nabla}^\circ + K, \bar{g}, \bar{J})$ is called an indefinite Hermitian statistical structure on $\bar{M}$ if $(\bar{\nabla}, \bar{g})$ is a statistical structure on $\bar{M}$.*

Then $(\bar{M}, \bar{\nabla}, \bar{\nabla}^*, \bar{g}, \bar{J})$ is called an indefinite Hermitian statistical manifold.

An indefinite Hermitian statistical manifold is called indefinite Kaehler statistical manifold if its almost complex structure is parallel with respect to Levi-Civita connection i.e. if,

$$(\bar{\nabla}^{\circ}_X \bar{J})Y = 0$$

Equivalently

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}^*_X \bar{J})Y = 0$$

for all $X, Y \in \Gamma(T\bar{M})$.

3. GCR - LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER STATISTICAL MANIFOLD

The class of GCR-lightlike submanifolds which contains SCR-lightlike and CR-lightlike submanifolds as subcases was introduced by [6]. In this section, we define the GCR-lightlike submanifolds for the indefinite Kaehler statistical manifold and elaborate its structure with an example. Further, some geometric properties related to the structure of these submanifolds have been introduced.

Definition 3.1. *A real lightlike submanifold* $(M, g, S(TM))$ *of an indefinite Kaehler statistical manifold* $(\bar{M}, \bar{g}, \bar{J})$ *is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:*
(i) There exists two subbundles D_1 *and* D_2 *of* $Rad(TM)$ *such that*

$$Rad(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM). \tag{12}$$

(ii) There exists two subbundles D_0 *and* D' *of* $S(TM)$ *such that*

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2. \tag{13}$$

where D_0 *is a non degenerate distribution on* M*;* L_1 *and* L_2 *are vector subbundles of* $ltr(TM)$ *and* $S(TM)^{\perp}$ *respectively.*

Then the tangent bundle TM *of* M *is decomposed as*

$$TM = D \perp D', \quad D = Rad(TM) \oplus D_0 \oplus \bar{J}D_2. \tag{14}$$

M *is called a proper* GCR*-lightlike submanifold if* $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ *and* $L_2 \neq \{0\}$.

Proposition 3.2. *A* GCR*-lightlike submanifold* M *of an indefinite Kaehler statistical manifold* $\bar{M}$ *is a* CR*-lightlike submanifold if and only if* $D_1 = 0$.

Proof. The concept of CR-lightlike submanifold implies that $\bar{J}Rad(TM) \cap Rad(TM) = \{0\}$. Hence $D_2 = Rad(TM)$ and $D_1 = 0$.
Conversely, consider a GCR lightlike submanifold such that $D_1 = \{0\}$. Then $D_2 = Rad(TM)$ and hence $\bar{J}Rad(TM) \cap Rad(TM) = \{0\}$, which shows that $\bar{J}Rad(TM) \subset S(TM)$. Thus M is a CR-lightlike submanifold. □

Inspired by [6], we consider the following example:

Example 3.3. *Consider an indefinite Kaehler manifold* $\bar{M} = (R_4^{14}, \bar{g})$ *where* $\bar{g}$ *is of signature* $(-,-,-,-,+,+,+,+,+,+,+,+,+,+)$ *with respect to the basis*
$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12}, \partial x_{13}, \partial x_{14}\}$.

As per definition (2.2), the triplet $(\bar{\nabla} = \bar{\nabla}^\circ + K, \bar{g}, \bar{J})$ *where* K *satisfies (11), defines an indefinite Kaehler statistical structure on* $\bar{M}$*. Consider a submanifold* M *of* R_4^{14} *given by the following equations*

$$x_1 = x_{14}, \quad x_2 = -x_{13}, \quad x_3 = x_{12}, \quad x_7 = \sqrt{1 - x_8^2}$$

Now TM *is spanned by* Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7,Z_8,Z_9,Z_{10}, *where*

$$Z_1 = \partial x_1 + \partial x_{14}, \quad Z_2 = \partial x_2 - \partial x_{13}, \quad Z_3 = \partial x_3 + \partial x_{12},$$

$$Z_4 = \partial x_4, \quad Z_5 = \partial x_5, \quad Z_6 = \partial x_6, \quad Z_7 = -x_8 \partial x_7 + x_7 \partial x_8,$$

$$Z_8 = \partial x_9, \quad Z_9 = \partial x_{10}, \quad Z_{10} = \partial x_{11}.$$

We see that M *is 3-lightlike with* $RadTM = Span\{Z_1, Z_2, Z_3\}$ *and* $\bar{J}Z_1 = Z_2$. *Thus,* $D_1 = Span\{Z_1, Z_2\}$. *On the other hand,*$\bar{J}Z_3 = Z_4 - Z_{10} \in \Gamma(S(TM))$ *implies that* $D_2 = Span\{Z_3\}$. *Also* $\bar{J}Z_5 = Z_6$ *and* $\bar{J}Z_8 = Z_9$. *Therefore* $D_\circ = Span\{Z_5, Z_6, Z_8, Z_9\}$.
We also have $S(TM^\perp) = Span\{W = x_7 \partial x_7 + x_8 \partial x_8\}$ *which implies that* $\bar{J}Z_7 = -W$. *Hence,* $L_2 = S(TM^\perp)$.
Further, the lightlike transversal bundle $ltr(TM)$ *is spanned by*

$$\{N_1 = \frac{1}{2}(-\partial x_1 + \partial x_{14}), \quad N_2 = \frac{1}{2}(-\partial x_2 - \partial x_{13}), \quad N_3 = \frac{1}{2}(-\partial x_3 - \partial x_{12})\}$$

It follows that $Span\{N_1, N_2\}$ *is invariant with respect to* $\bar{J}$, $\bar{J}N_3 = -\frac{1}{2}Z_4 - \frac{1}{2}Z_{10}$. *Hence,* $L_1 = Span\{N_3\}$ *and* $D' = Span\{\bar{J}N_3, \bar{J}W\}$. *This shows that* M *is a proper* GCR*-lightlike submanifold of the indefinite Kaehler statistical manifold* R_4^{14}.

Considering the structure of GCR*-lightlike submanifolds of the indefinite Kaehler statistical manifold, if* Q, P_1 *and* P_2 *are the projections on* D, $\bar{J}(L_1) = M_1 \subset D'$ *and* $\bar{J}(L_2) = M_2 \subset D'$ *respectively, then for any* $X \in \Gamma(TM)$, *we have*

$$X = QX + P_1 X + P_2 X, \tag{15}$$

Applying $\bar{J}$ *on above equation*

$$\bar{J}X = TX + wP_1 X + wP_2 X, \tag{16}$$

which can be written as

$$\bar{J}X = TX + \omega X, \tag{17}$$

where TX and ωX are the tangential and transversal components of $\bar{J}X$, respectively.
Similarly

$$\bar{J}V = BV + CV \tag{18}$$

for any $V \in \Gamma(ltr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively.

On differentiating (16) and using (2), (3), (4), (5) and (18) $\forall\, X, Y \in \Gamma(TM)$, we obtain

(19)
$$\nabla_X TY - T\nabla_X Y + \nabla^*_X TY - T\nabla^*_X Y = A_{wP_1Y}X + A_{wP_2Y}X + A^*_{wP_1Y}X + A^*_{wP_2Y}X + Bh(X,Y) + Bh^*(X,Y)$$

(20)
$$D^s(X, wP_1Y) + D^{*s}(X, wP_1Y) = -\nabla^s_X wP_2Y - \nabla^{*s}_X wP_2Y + wP_2\nabla_X Y + wP_2\nabla^*_X Y - h^s(X,TY) - h^{*s}(X,TY) + Ch^s(X,Y) + Ch^{*s}(X,Y)$$

(21)
$$D^l(X, wP_2Y) + D^{*l}(X, wP_2Y) = -\nabla^l_X wP_1Y - \nabla^{*l}_X wP_1Y + wP_1\nabla_X Y + wP_1\nabla^*_X Y - h^l(X,TY) - h^{*l}(X,TY) + Ch^l(X,Y) + Ch^{*l}(X,Y)$$

Now the structure of indefinite Kaehler statistical manifold leads to the following lemmas:

Lemma 3.4. *Let M be a GCR-lightlike submanifold of indefinite Kaehler statistical manifold $\bar{M}$. Then we have*

$$\nabla_X TY - T\nabla_X Y + \nabla^*_X TY - T\nabla^*_X Y = A_{wY}X + A^*_{wY}X + Bh(X,Y) + Bh^*(X,Y)$$

$$\nabla^{\perp}_X wY - w\nabla_X Y + \nabla^{*\perp}_X wY - w\nabla^*_X Y = Ch(X,Y) + Ch^*(X,Y) - h(X,TY) - h^*(X,TY)$$

for any $X, Y \in \Gamma(TM)$.

Lemma 3.5. *Let M be a GCR-lightlike submanifold of indefinite Kaehler statistical manifold $\bar{M}$. Then we have*

$$\nabla_X BV - B\nabla^{\perp}_X V + \nabla^*_X BV - B\nabla^{*\perp}_X V = A_{CV}X + A^*_{CV}X - TA_VX - TA^*_VX$$

$$\nabla^{\perp}_X CV - C\nabla^{\perp}_X V + \nabla^{*\perp}_X CV - C\nabla^{*\perp}_X V = -wA_VX - wA^*_VX - h(X,BV) - h^*(X,BV)$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$.

Theorem 3.6. *Let $\bar{M}$ be an indefinite Kaehler statistical manifold and M be a GCR-lightlike submanifold of $\bar{M}$. Then the induced connection $\nabla_X Y \in \Gamma(Rad(TM))$ if and only if the following holds*

$$A'_{\bar{J}Y}X - A'^*_{\bar{J}Y}X - \nabla'^t_X \bar{J}Y - \nabla'^{*t}_X \bar{J}Y \in \Gamma(\bar{J}D_2 \perp D_1), \quad \textit{when } Y \in \Gamma(D_1)$$

$$\nabla'_X \bar{J}Y + \nabla'^*_X \bar{J}Y + h'(X,\bar{J}Y) + h'^*(X,\bar{J}Y) \in \Gamma(\bar{J}D_2 \perp D_1), \quad \textit{when } Y \in \Gamma(D_2)$$

$$\nabla^*_X Y \in \Gamma(Rad(TM)), \quad Bh(X,\bar{J}Y) + Bh^*(X,\bar{J}Y) = 0, \quad \textit{when } Y \in \Gamma(Rad(TM))$$

Proof: Since $\bar{J}$ is the almost complex structure of $\bar{M}$, we have

$$\bar{\nabla}_X Y = -\bar{\nabla}_X \bar{J}^2 Y \ ; \quad Y \in \Gamma(Rad(TM)), \quad X \in \Gamma(TM)$$

The Kaehler statistical character of the manifold $\bar{M}$ and the equation (2) give

$$\bar{\nabla}_X Y = -\bar{J}\bar{\nabla}_X \bar{J}Y - \bar{J}\bar{\nabla}^*_X \bar{J}Y + \bar{\nabla}^*_X \bar{J}^2 Y \tag{22}$$

$$\nabla_X Y + h(X,Y) = -\bar{J}(\nabla_X \bar{J}Y + h(X,\bar{J}Y)) - \bar{J}(\nabla^*_X \bar{J}Y + h^*(X,\bar{J}Y)) - \nabla^*_X Y - h^*(X,Y)$$

for any $X \in \Gamma(TM), Y \in \Gamma(Rad(TM))$. Since M is a GCR-lightlike submanifold of $\bar{M}$, we have $Rad(TM) = D_1 \oplus D_2$. Further, using equations (7), (17), (18) and then equating the tangential part for any $Y \in \Gamma(D_1)$, we get

(23)

$$\nabla_X Y = TA'_{\bar{J}Y}X - TA'^*_{\bar{J}Y}X - T\nabla'^t_X \bar{J}Y - T\nabla'^{*t}_X \bar{J}Y - Bh(X,\bar{J}Y) - Bh^*(X,\bar{J}Y) - \nabla^*_X Y$$

Similarly, for any $Y \in \Gamma(D_2)$, using (6), (17), (18),(22), we get

(24)

$$\nabla_X Y = -T\nabla'_X \bar{J}Y - T\nabla'^*_X \bar{J}Y - Th'(X,\bar{J}Y) - Th'^*(X,\bar{J}Y) - Bh(X,\bar{J}Y) - Bh^*(X,\bar{J}Y) - \nabla^*_X Y$$

Thus from equation (23), $\nabla_X Y \in \Gamma(Rad(TM))$, if and only if

$$T(A'_{\bar{J}Y}X + A'^*_{\bar{J}Y}X - \nabla'^t_X \bar{J}Y - \nabla'^{*t}_X \bar{J}Y) \in \Gamma(\bar{J}D_2 \perp D_1)$$

$$\nabla^*_X Y \in \Gamma(Rad(TM)), \quad Bh(X,\bar{J}Y) + Bh^*(X,\bar{J}Y) = 0$$

for all $X \in \Gamma(TM), Y \in \Gamma(D_1)$. Also, from equation (24), $\nabla_X Y \in \Gamma(Rad(TM))$, if and only if

$$T(\nabla'_X \bar{J}Y + \nabla'^*_X \bar{J}Y + h'(X,\bar{J}Y) + h'^*(X,\bar{J}Y)) \in \Gamma(\bar{J}D_2 \perp D_1)$$

$$\nabla^*_X Y \in \Gamma(Rad(TM)), \quad Bh(X,\bar{J}Y) + Bh^*(X,\bar{J}Y) = 0$$

Thus the result follows from above equations.

Theorem 3.7. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then we have the following conditions:*
(i) the distribution D is integrable, if and only if

$$h(X,\bar{J}Y) + h^*(X,\bar{J}Y) = h(Y,\bar{J}X) + h^*(Y,\bar{J}X) \qquad \forall\, X,Y \in \Gamma(D).$$

(ii) The totally real distribution D' is integrable, if and only if

$$A_{\bar{J}Z}U + A^*_{\bar{J}Z}U = A_{\bar{J}U}Z + A^*_{\bar{J}U}Z \quad \forall\, U,Z \in \Gamma(D')$$

Proof: From equations (20),(21), we have

$$h(X,\bar{J}Y) + h^*(X,\bar{J}Y) - Ch(X,Y) + Ch^*(X,Y) = wP(\nabla_X Y) + wP(\nabla^*_X Y)$$

for $X,Y \in \Gamma(D)$ Now using the fact that h and h^* are symmetric and the connections ∇ and ∇^* are torsion free, it follows that

$$h(X,\bar{J}Y) + h^*(X,\bar{J}Y) - h(Y,\bar{J}X) - h^*(Y,\bar{J}X) = 2wP[X,Y]$$

which proves condition (i).
Now from (19), we obtain

$$A_{wP_1Z}U + A^*_{wP_1Z}U + A_{wP_2Z}U + A^*_{wP_2Z}U = -Bh(U,Z) - Bh^*(U,Z) \\ -T(\nabla_U Z) - T(\nabla^*_U Z)$$

Hence

$$A_{\bar{J}Z}U + A^*_{\bar{J}Z}U - A_{\bar{J}U}Z - A^*_{\bar{J}U}Z = -2T([Z,U])$$

The hypothesis leads to the condition (ii).

Theorem 3.8. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then,*

$$\nabla_X Z + \nabla^*_X Z = -TA_W X + B\nabla^s_X W + BD^l(X,W) - TA^*_W X + B\nabla^{*s}_X W + BD^{*l}(X,W)$$

for any $X \in \Gamma(TM)$, $Z \in \Gamma(\bar{J}L_2)$ and $W \in \Gamma(L_2)$.

Proof: Let $W \in \Gamma(L_2)$ so that $Z = \bar{J}W$. Since $\bar{M}$ is a Kaehler statistical manifold, therefore

$$\bar{\nabla}_X \bar{J}W + \bar{\nabla}^*_X \bar{J}W = \bar{J}\bar{\nabla}_X W + \bar{J}\bar{\nabla}^*_X W$$

Then,

$$\nabla_X Z + h(X,Z) + \nabla^*_X Z + h^*(X,Z) = \bar{J}(-A_W X + \nabla^s_X W + D^l(X,W)) \\ +\bar{J}(-A^*_W X + \nabla^{*s}_X W + D^{*l}(X,W))$$

$$= -TA_W X - wA_W X + B\nabla^s_X W + C\nabla^s_X W + BD^l(X,W) + CD^l(X,W) - TA^*_W X \\ -wA^*_W X + B\nabla^{*s}_X W + C\nabla^{*s}_X W + BD^{*l}(X,W) + CD^{*l}(X,W)$$

On equating tangential parts, we get

$$\nabla_X Z + \nabla^*_X Z = -TA_W X + B\nabla^s_X W + BD^l(X,W) - TA^*_W X + B\nabla^{*s}_X W + BD^{*l}(X,W)$$

4. GEODESICITY OF *GCR*-LIGHTLIKE SUBMANIFOLDS

In this section, we obtain the characterizations of totally geodesic, mixed geodesic, D-geodesic and D'-geodesic GCR-lightlike submanifolds .

Definition 4.1. *A GCR-lightlike submanifold of an indefinite Kaehler statistical manifold is called D-totally geodesic with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if $h(X,Y) = 0$ (respectively $h^*(X,Y) = 0$) for all $X, Y \in D$.*

Definition 4.2. *A GCR-lightlike submanifold of an indefinite Kaehler statistical manifold is called D'-totally geodesic with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if $h(X,Y) = 0$ (respectively $h^*(X,Y) = 0$) for all $X, Y \in D'$.*

Definition 4.3. *A GCR-lightlike submanifold of an indefinite Kaehler statistical manifold is called mixed totally geodesic with respect to $\bar{\nabla}$ (resp.$\bar{\nabla}^*$) if $h(X,Y) = 0$ (resp. $h^*(X,Y)= 0$) for $X \in D$ and $Y \in D'$.*

Theorem 4.4. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. Then $\nabla_X Y + \nabla^*_X Y \in \Gamma(D)\ \forall\ X, Y \in \Gamma(D)$ if M is D-geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$.*

Proof: Since $D' = \bar{J}(L_1 \perp L_2)$, therefore $\nabla_X Y + \nabla^*_X Y \in \Gamma(D) \quad \forall\ X, Y \in \Gamma(D)$ holds, if and only if, we have

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}\xi) = 0,$$
$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}W) = 0.$$

Let $X, Y \in \Gamma(D)$
Using the fact that $\bar{M}$ is a Kaehler statistical manifold, we derive

$$\begin{aligned}\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}\xi) &= \bar{g}(\bar{\nabla}_X Y - h(X,Y) + \bar{\nabla}^*_X Y - h^*(X,Y), \bar{J}\xi)\\ &= \bar{g}(\bar{\nabla}_X Y + \bar{\nabla}^*_X Y, \bar{J}\xi) - \bar{g}(h(X,Y), \bar{J}\xi) - \bar{g}(h^*(X,Y), \bar{J}\xi)\\ &= -\bar{g}(\bar{J}\bar{\nabla}_X Y + \bar{J}\bar{\nabla}^*_X Y, \xi) = -\bar{g}(\bar{\nabla}_X \bar{J}Y, \xi) - \bar{g}(\bar{\nabla}^*_X \bar{J}Y, \xi)\\ &= -\bar{g}(h(X, \bar{J}Y), \xi) - \bar{g}(h^*(X, \bar{J}Y), \xi)\end{aligned}$$

Similarly, we obtain

$$\bar{g}(\nabla_X Y + \nabla^*_X Y, \bar{J}W) = -\bar{g}(h(X, \bar{J}Y), W) - \bar{g}(h^*(X, \bar{J}Y), W)$$

Then the result follows from the hypothesis.

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is mixed geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, then $wA_{wY}X + wA^*_{wY}X = 0$, and $C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY = 0$ for any $X \in \Gamma(D), Y \in \Gamma(D')$.*

Proof: Since $\bar{M}$ is a Kaehler statistical manifold, therefore we derive

$$\begin{aligned}h(X,Y) + h^*(X,Y) &= -\bar{J}^2\bar{\nabla}_X Y - \nabla_X Y - \bar{J}^2\bar{\nabla}^*_X Y - \nabla^*_X Y\\ &= -\bar{J}(\bar{\nabla}_X \bar{J}Y) - \bar{J}(\bar{\nabla}^*_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y\\ &= -\bar{J}(-A_{\bar{J}Y}X + \nabla^{\perp}_X \bar{J}Y) - \bar{J}(-A^*_{\bar{J}Y}X + \nabla^{*\perp}_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y\\ &= \bar{J}(A_{\bar{J}Y}X) - \bar{J}(\nabla^{\perp}_X \bar{J}Y) + \bar{J}(A^*_{\bar{J}Y}X) - \bar{J}(\nabla^{*\perp}_X \bar{J}Y) - \nabla_X Y - \nabla^*_X Y\\ &= TA_{wY}X + wA_{wY}X - B\nabla^{\perp}_X wY - C\nabla^{\perp}_X wY + TA^*_{wY}X + wA^*_{wY}X - B\nabla^{*\perp}_X wY\\ &\quad - C\nabla^{*\perp}_X wY - \nabla_X Y - \nabla^*_X Y\end{aligned}$$

Equating transversal parts on both sides, we have

$$h(X,Y) + h^*(X,Y) = wA_{wY}X + wA^*_{wY}X - C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY$$

Thus, M is mixed geodesic w.r.t to $\bar{\nabla}$ and $\bar{\nabla}^*$, if and only if

$$wA_{wY}X + wA^*_{wY}X = 0, \quad C\nabla^{\perp}_X wY - C\nabla^{*\perp}_X wY = 0.$$

Theorem 4.6. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$,then*

$$(L_\xi \bar{g})(X,Y) = 0 \quad and \quad (L_W \bar{g})(X,Y) = 0$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(D_2))$ and $W \in \Gamma S(TM^{\perp})$.

Proof: For any $X, Y \in \Gamma(TM), \xi \in \Gamma(D_2)$, we have

$$\bar{g}(h(X,Y),\xi) = \bar{g}(\bar{\nabla}_X Y,\xi) - \bar{g}(\nabla_X Y,\xi)$$

The Gauss formula and the dual connections in a statistical manifold give

$$\bar{g}(h(X,Y),\xi) = X\bar{g}(Y,\xi) - \bar{g}(Y,\bar{\nabla}^*_X\xi)$$

$$= -\bar{g}(Y,[X,\xi]) - \bar{g}(Y,\bar{\nabla}^*_\xi X) = -\bar{g}(Y,[X,\xi]) - \xi\bar{g}(Y,X) + \bar{g}(\bar{\nabla}_\xi Y, X)$$

$$= -\bar{g}(Y,[X,\xi]) - \xi\bar{g}(Y,X) + \bar{g}([\xi,Y],X) + \bar{g}(\bar{\nabla}_Y\xi, X)$$

$$= -(L_\xi\bar{g})(X,Y) + Y\bar{g}(\xi,X) - \bar{g}(\xi,\bar{\nabla}^*_Y X)$$

$$\bar{g}(h(X,Y),\xi) = -(L_\xi\bar{g})(X,Y) - \bar{g}(\xi, h^*(Y,X)) \tag{25}$$

Similarly for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma S(TM^\perp)$

$$\bar{g}(h(X,Y),W) = -(L_W\bar{g})(X,Y) - \bar{g}(W, h^*(Y,X)) \tag{26}$$

The desired result follows using equations (25) and (26) along with the hypothesis and the concept of *GCR* lightlike submanifolds.

Theorem 4.7. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is mixed geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ then*

$$A'^*_\xi X + A'_\xi X \in \Gamma(\bar{J}D_2),$$

for any $X \in \Gamma(D')$, $\xi \in \Gamma D_2$.

Proof: For any $X \in \Gamma(D')$ and $\xi \in \Gamma D_2$, we have

$$h(X,\bar{J}\xi) + h^*(X,\bar{J}\xi) = \bar{\nabla}_X\bar{J}\xi - \nabla_X\bar{J}\xi + \bar{\nabla}^*_X\bar{J}\xi - \nabla^*_X\bar{J}\xi$$

The idea of Kaehler statistical manifold and equation (2) implies

$$h(X,\bar{J}\xi) + h^*(X,\bar{J}\xi) = \bar{J}\nabla_X\xi + \bar{J}h(X,\xi) + \bar{J}\nabla^*_X\xi + \bar{J}h^*(X,\xi) - \nabla_X\bar{J}\xi - \nabla^*_X\bar{J}\xi$$

The equations (3),(4) and the mixed geodesicity of M with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ give

$$-\bar{J}A'_\xi X + \bar{J}\nabla'^t_X\xi - \bar{J}A^{*\prime}_\xi X + \bar{J}\nabla^{*\prime t}_X\xi = \nabla'_X\bar{J}\xi + h'(X,\bar{J}\xi) + \nabla'^*_X\bar{J}\xi + h'^*(X,\bar{J}\xi)$$

from equation (17)

$$-TA'_\xi X - \omega A'_\xi X + \bar{J}\nabla'^t_X\xi - TA'^*_\xi X - \omega A'^*_\xi X + \bar{J}\nabla'^{*t}_X\xi = \nabla'_X\bar{J}\xi + h'(X,\bar{J}\xi)$$
$$+\nabla'^*_X\bar{J}\xi + h'^*(X,\bar{J}\xi)$$

By equating transversal components, we have

$$\omega A'_\xi X + \omega A^{*\prime}_\xi X = 0$$

Hence, $A'_\xi X + A^{*\prime}_\xi X \in \Gamma(\bar{J}D_2 \perp D_\circ)$. Also, for $Y \in \Gamma(D_\circ)$ and $\xi \in \Gamma(D_2)$, the non-degeneracy of $(D_\circ)$ implies $A'_\xi X + A^{*\prime}_\xi X \notin \Gamma(D_\circ)$.

Theorem 4.8. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is D-geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, then*

$$\bar{g}(Y, A^*_W X) + \bar{g}(Y, A_W X) = \bar{g}(Y, D^{*l}(X, W)) + \bar{g}(Y, D^l(X, W))$$
$$\bar{g}(Y, A^{*\prime}_\xi X) + \bar{g}(Y, A'_\xi X) = \bar{g}(Y, \nabla^{*\prime t}_X \xi) + \bar{g}(Y, \nabla'^t_X \xi)$$

for any $X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$

Proof: For any $X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\bar{g}(h^s(X, Y) + h^{*s}(X, Y), W) = \bar{g}(\bar{\nabla}_X Y + \bar{\nabla}^*_X Y, W) = -\bar{g}(Y, \bar{\nabla}^*_X W) - \bar{g}(Y, \bar{\nabla}_X W)$$

$$\bar{g}(h^s(X, Y) + h^{*s}(X, Y), W) = \bar{g}(Y, A^*_W X) - \bar{g}(Y, D^{*l}(X, W)) + \bar{g}(Y, A_W X) - \bar{g}(Y, D^l(X, W)) \tag{27}$$

$$\bar{g}(h^l(X, Y) + h^{*l}(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y + \bar{\nabla}^*_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}^*_X \xi) - \bar{g}(Y, \bar{\nabla}_X \xi) = -\bar{g}(Y, \nabla^*_X \xi) - \bar{g}(Y, \nabla_X \xi)$$

$$\bar{g}(h^l(X, Y) + h^{*l}(X, Y), \xi) = \bar{g}(Y, A^{*\prime}_\xi X) - \bar{g}(Y, \nabla^{*\prime t}_X \xi) + \bar{g}(Y, A'_\xi X) - \bar{g}(Y, \nabla'^t_X \xi) \tag{28}$$

So, the geodesicity of D in equations (27) and (28) gives the required equations.

Theorem 4.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is D geodesic with respect to ∇ and ∇^*, then*

$$\nabla'_X \bar{J}\xi + \nabla'^*_X \bar{J}\xi \notin (D_\circ \perp M_1), \quad A'_{\bar{J}Y} X + A^{*\prime}_{\bar{J}Y} X \notin (M_1)$$

and

$$h^l(X, \bar{J}Y) + h^{*l}(X, \bar{J}Y) \notin \Gamma(L_1)$$

for any $X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM))$.

Proof: For any $X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM))$ and $W \in \Gamma S(TM^\perp)$, we have

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(h^{*l}(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y, \xi) + \bar{g}(\bar{\nabla}^*_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}^*_X \xi) - \bar{g}(Y, \bar{\nabla}_X \xi)$$

From the concept of Kaehler statistical manifold , we get

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(h^{*l}(X, Y), \xi) = -\bar{g}(\bar{J}Y, \bar{\nabla}^*_X \bar{J}\xi + \bar{\nabla}_X \bar{J}\xi)$$
$$= -\bar{g}(\bar{J}Y, \nabla^*_X \bar{J}\xi) - \bar{g}(\bar{J}Y, h^{*l}(X, \bar{J}\xi)) - \bar{g}(\bar{J}Y, \nabla_X \bar{J}\xi) - \bar{g}(\bar{J}Y, h^l(X, \bar{J}\xi))$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(h^{*l}(X, Y), \xi) = -\bar{g}(\bar{J}Y, \nabla^{*\prime}_X \bar{J}\xi) - \bar{g}(\bar{J}Y, h^{*l}(X, \bar{J}\xi) - \bar{g}(\bar{J}Y, \nabla'_X \bar{J}\xi) - \bar{g}(\bar{J}Y, h^l(X, \bar{J}\xi)) \tag{29}$$

If $Y \in \Gamma(D_\circ)$ or $Y \in \Gamma(D_2)$, we have

$$\bar{g}(\bar{J}Y, h^{*l}(X, \bar{J}\xi)) + \bar{g}(\bar{J}Y, h^l(X, \bar{J}\xi)) = 0$$

and if $Y \in \Gamma(D_1)$ or $Y \in \Gamma(\bar{J}(D_2))$, then

$$\bar{g}(\bar{J}Y, h^l(X, \bar{J}\xi)) + \bar{g}(\bar{J}Y, h^{*l}(X, \bar{J}\xi)) = \bar{g}(\bar{J}Y, \bar{\nabla}_X \bar{J}\xi) + \bar{g}(\bar{J}Y, \bar{\nabla}^*_X \bar{J}\xi)$$
$$= -\bar{g}(\bar{J}\xi, \nabla^*_X \bar{J}Y) - \bar{g}(\bar{J}\xi, h^{*l}(X, \bar{J}Y)) - \bar{g}(\bar{J}\xi, \nabla_X \bar{J}Y) - \bar{g}(\bar{J}\xi, h^l(X, \bar{J}Y))$$
$$= -\bar{g}(\bar{J}\xi, A'^*_{\bar{J}Y}X) - \bar{g}(\bar{J}\xi, h^{*l}(X, \bar{J}Y)) - \bar{g}(\bar{J}\xi, A'_{\bar{J}Y}X) - \bar{g}(\bar{J}\xi, h^l(X, \bar{J}Y))$$

Now, from equation (29) we derive

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(h^{*l}(X, Y), \xi) = -\bar{g}(\bar{J}Y, \nabla^{*\prime}_X \bar{J}\xi) - \bar{g}(\bar{J}Y, \nabla'_X \bar{J}\xi)$$
$$-\bar{g}(\bar{J}\xi, A'^*_{\bar{J}Y}X) - \bar{g}(\bar{J}\xi, h^{*l}(X, \bar{J}Y)) - \bar{g}(\bar{J}\xi, A'_{\bar{J}Y}X) - \bar{g}(\bar{J}\xi, h^l(X, \bar{J}Y))$$

Hence, the hypothesis given leads to the desired result.

Theorem 4.10. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler statistical manifold $\bar{M}$. If M is D' geodesic with respect to ∇ and ∇^*, then $A^*_{\bar{J}Y}X + A_{\bar{J}Y}X$ have no component in M_1 for any $X, Y \in \Gamma(D'), \xi \in \Gamma(D_2)$.*

Proof: For any $X, Y \in \Gamma(D'), \xi \in \Gamma(D_2)$, we have

$$\bar{g}(h(X, Y) + h^*(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y + \bar{\nabla}^*_X Y, \xi)$$

Now, $\bar{M}$ being a Kaehler statistical manifold implies that

$$\bar{g}(h(X, Y) + h^*(X, Y), \xi) = \bar{g}(\bar{\nabla}_X \bar{J}Y + \bar{\nabla}^*_X \bar{J}Y, \bar{J}\xi)$$
$$= -\bar{g}(A_{\bar{J}Y}X + A^*_{\bar{J}Y}X, \bar{J}\xi)$$

Thus our assertion follows .

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