Honam Mathematical J. **44** (2022), No. 3, pp. 391–401
https://doi.org/10.5831/HMJ.2022.44.3.391

ON CHARACTERIZATIONS OF SPHERICAL CURVES USING FRENET LIKE CURVE FRAME

Kemal Eren, Kebire Hilal Ayvaci*, and Süleyman Şenyurt

Abstract. In this study, we investigate the explicit characterization of spherical curves using the Flc (Frenet like curve) frame in Euclidean 3-space. Firstly, the axis of curvature and the osculating sphere of a polynomial space curve are calculated using Flc frame invariants. It is then shown that the axis of curvature is on a straight line. The position vector of a spherical curve is expressed with curvatures connected to the Flc frame. Finally, a differential equation is obtained from the third order, which characterizes a spherical curve.

1. Introduction

The theory of curves is one of the most important research topics in differential geometry. Characterizations of curves are given with the help of the orthonormal Frenet frame in 3-dimensional Euclidean space. Also, the curvature of a curve and its torsion gives information about the local behavior of this curve in space. The curvature of a curve indicates the amount of deviation from the tangent line of the curve and as the amount of deviation gets smaller, the curve has a closed appearance. In case of the curvature of a space curve is non-zero, its torsion measures the amount of deviation from the osculating plane determined by the tangent and normal vectors of the curve. Thus, we can say that a moving curve is characterized with the help of a differential equation involving the curvature and torsion of the curve. If the second derivative of a space curve is zero, then the Frenet frame cannot be defined. Therefore, alternative frames are needed to solve this problem. One of these alternative frames is the Flc frame defined by Dede et al. [11]. The Flc frame is advantageous if the second or higher-order derivatives of the curve are zero. For this reason, the characterizations of spherical curves according to the Flc frame are investigated in this study. The differential equation characterizing a spherical curve is first given by Wong [16, 17]. Later, in 1971, Breuer and Gottlieb work on the

Received March 2, 2022. Revised July 25, 2022. Accepted July 26, 2022.
2020 Mathematics Subject Classification. 53A04, 39A06.
Key words and phrases. Flc frame, polynomial curve, spherical curve, the axis of curvature, osculating sphere.
*Corresponding author

solvability of the differential equation which characterizes a spherical curve [8]. Dannon searches that the spherical curves in E^4 are expressed with Frenet-like equations. Therefore, finding an integral characterization for a spherical curve in E^4 is the same as finding for the Frenet curve in E^3 [10]. Mehlum and Wimp investigate in 1985 that the position vector of any spherical curve can be given as the solution of a third-order linear differential equation [14]. The spherical curves have been examined from different perspectives [13, 15, 9]. Also, some studies of spherical indicatrices which generalizing the concept of the spherical indicatrix to the involute of a curve and Bertrand mate of a curve are given by [1, 2, 3, 4, 5, 6, 7].
The aim of our study is to characterize spherical curves when the higher order derivatives of the space curves are zero. First of all, the geometrical location of the center of the spheres with sufficiently close common three points and sufficiently close common four points with a given curve according to this frame is found and defined as the axis of curvature and the sphere of curvature, respectively. It is then shown that the axis of curvature of the given curve is on a straight line. Then the position vector of a spherical curve is expressed in terms of Flc frame invariants. Finally, using the Flc frame, it is concluded that the position vector of any spherical curve can be given as the solution of a third-order differential equation.

2. Preliminaries

In this section, we express some basic concepts that are used throughout the paper. Let $\alpha = \alpha(s)$ be a differentiable curve. The Frenet vectors of a space curve $\alpha = \alpha(s)$ are defined by

$$T(s) = \frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|}, B(s) = \frac{\alpha^{'}(s) \wedge \alpha^{''}(s)}{\|\alpha^{'}(s) \wedge \alpha^{''}(s)\|} \text{ and } N(s) = B(s) \wedge T(s).$$

Also, the Frenet formulas are given as follows:

$$T' = \kappa\nu N, \;\; N' = -\kappa\nu T + \tau\nu B, \;\; B' = -\tau\nu N \;, \;\; \|\alpha'\| = \nu,$$

where κ and τ are the curvature and torsion of the curve α [12]. But if the second or higher order derivatives of the curve are zero, then $\left\|\alpha^{'}(s) \wedge \alpha^{''}(s)\right\| = 0$. Therefore, the Frenet frame cannot be defined. To solve this problem, an alternative frame defined on the curve is needed. Therefore, in 2019, Dede et al. defined a new frame called "Flc frame" along a polynomial curve [11]. With the help of this Flc frame, it has become possible to make calculations on the curve.
Let $\alpha = \alpha(s)$ be a polynomial curve. The tangent vector, the binormal-like vector, and the normal-like vector of the Flc frame along the curve α are defined

by

$$T(s)=\frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|},\ D_1(s)=\frac{\alpha^{'}(s)\wedge\alpha^{(n)}(s)}{\left\|\alpha^{'}(s)\wedge\alpha^{(n)}(s)\right\|}\ \text{and}\ D_2(s)=D_1(s)\wedge T(s),$$

respectively, where $'$ is denoted derivative of the curve in terms of s parameter and also, $^{(n)}$ is denoted n. order derivative of the curve. The curvatures d_1, d_2, d_3 of the curve are

$$d_1=\frac{\langle T', D_2\rangle}{v},\ \ d_2{=}\frac{\langle T', D_1\rangle}{v},\ \ d_3=\frac{\left\langle {D_2}', D_1\right\rangle}{v},$$

where $\|\alpha'\|=\nu$. The derivative formulas of the Flc frame are called the Frenet-like derivative formulas and are as follows:

$$(1)\ \ T'=\nu\big(d_1D_2+d_2D_1\big),\ D_2'=\nu\big(-d_1T+d_3D_1\big),\ D_1'=-\nu\big(d_2T+d_3D_2\big).$$

Definition 2.1. *Let α be a curve given with coordinate neighborhood (I,α) in E^3. If $\alpha\subset S^2$ then the curve α is called a spherical curve of E^3* [16, 8, 9].

Definition 2.2. *The geometric locus of centers of the spheres having sufficiently close common three points with curve $\alpha\subset E^3$ at the point $\alpha(s_0)\in\alpha$ is called axis of curvature at the point $\alpha(s_0)\in\alpha$* [16, 8, 9].

Definition 2.3. *The sphere having sufficiently close common four points at $\alpha(s_0)\in\alpha$ with the curve $\alpha\subset E^3$ is called the osculating sphere or curvature sphere of the curve α at the point $\alpha(s_0)\in\alpha$* [16, 8, 9].

3. Characterizations of Spherical Curves using Flc Frame

In this section, let's examine the characterizations of spherical curves according to the Flc frame. First of all, with a polynomial curve given according to this frame, the geometric locations of the centers of the spheres with sufficiently close common three points and sufficiently close common four points are be found. Then the position vector of a spherical curve is expressed by the curvatures connected to the Flc frame. Finally, a third-order differential equation characterizing a spherical curve is obtained using the Flc frame.

Theorem 3.1. *Let $\alpha\subset E^3$ be a polynomial curve with coordinate neighborhood (I,α) in E^3. Then, the geometric locus of centers of the spheres having sufficiently close common three points with curve $\alpha\subset E^3$ at the point $\alpha(s)$ are determined by*

$$M=\alpha(s)+m_2D_2(s)+m_3D_1(s),$$

where

$$m_2(s) = \frac{d_1 \mp d_2\sqrt{-1 + d_1^2 r^2 + d_2^2 r^2}}{d_1^2 + d_2^2},$$

$$m_3(s) = \frac{d_2 \mp d_1\sqrt{-1 + d_1^2 r^2 + d_2^2 r^2}}{d_1^2 + d_2^2}$$

such that $m_i : I \longrightarrow \mathbb{R}$ *for all* $s \in I$.

Proof. Let $\alpha \subset E^3$ be a polynomial curve with coordinate neighborhood (I, α) in E^3. Let also M be the center and r be the radius of the sphere having sufficiently close common three points with α, then we can define the following function:

$$f : I \longrightarrow \mathbb{R} \tag{2}$$
$$s \longrightarrow f(s) = < M - \alpha(s), M - \alpha(s) > -r^2$$

such that

$$f(s) = f'(s) = f''(s) = 0.$$

From Eqs. (1) and (2), we obtain

$$\begin{aligned} f'(s) = 0 &\Longrightarrow < -\alpha'(s), M - \alpha(s) > + < M - \alpha(s), -\alpha'(s) >= 0, \\ &\Longrightarrow < \nu T, M - \alpha(s) >= 0, \\ &\Longrightarrow \nu < T, M - \alpha(s) >= 0, \quad (\nu \neq 0) \\ &\Longrightarrow < T, M - \alpha(s) >= 0. \qquad (3) \end{aligned}$$

Taking the derivative of Eq. (3), we have

$$< T', M - \alpha(s) > + < T, -\alpha'(s) >= 0.$$

Also, considering Eqs. (1) and (3), we find

$$\begin{aligned} < \nu(d_1 D_2 + d_2 D_1), M - \alpha(s) > + < T, -\nu T > &= 0, \\ \nu < (d_1 D_2 + d_2 D_1), M - \alpha(s) > &= \nu, \\ < d_1 D_2 + d_2 D_1, M - \alpha(s) > &= 1. \qquad (4) \end{aligned}$$

On the order hand, from the linear combination of the Flc frame's vectors T, D_2 and D_1, we can easily express

$$M - \alpha(s) = m_1(s)T + m_2(s)D_2 + m_3(s)D_1, \tag{5}$$

where $m_1, m_2, m_3 \in \mathbb{R}$. Considering Eqs. (3) and (5) together, it is found as

$$m_1(s) = < T, M - \alpha(s) >= 0. \tag{6}$$

Similarly, considering Eqs. (4) and (5) together, we get

$$\begin{aligned} m_2(s) &= < D_2, M - \alpha(s) >, \qquad (7) \\ m_3(s) &= < D_1, M - \alpha(s) > \end{aligned}$$

so, there is

$$d_1 m_2 + d_2 m_3 = 1. \tag{8}$$

Since $f(s) = 0$ and from Eqs. (2) and (5), we calculate

$$m_2^2 + m_3^2 = r^2. \tag{9}$$

The desired solution is obtained from the common solution of Eqs. (8) and (9). □

Corollary 3.2. *Let $\alpha \subset E^3$ be a polynomial curve with coordinate neighborhood (I, α) in E^3. Then the center M of the sphere having sufficiently close common three points with α is on a straight line.*

Proof. From Theorem (3.1), we can write

$$M = \alpha(s) + m_2 D_2(s) + m_3 D_1(s).$$

So, we can say that this equation is defined as a line parallel to D_1 and passing through point $c(s) = \alpha(s) + m_2 D_2(s)$. □

Theorem 3.3. *Let $\alpha \subset E^3$ be a polynomial curve with coordinate neighborhood (I, α) in E^3. Then, the geometric locus of centers of the spheres having sufficiently close common four points with curve $\alpha \subset E^3$ at the point $\alpha(s_0) \in \alpha$ are*

$$M = \alpha(s) + m_2 D_2(s) + m_3 D_1(s)$$

such that

$$m_2(s) = \frac{d_2' + \nu d_1 d_3}{(\frac{d_2}{d_1})' d_1^2 + \nu d_3 (d_1^2 + d_2^2)},$$

$$m_3(s) = \frac{\nu d_2 d_3 - d_1'}{(\frac{d_2}{d_1})' d_1^2 + \nu d_3 (d_1^2 + d_2^2)},$$

where $m_i : I \longrightarrow \mathbb{R}$, for all $s \in I$.

Proof. Since the sphere, which is called the osculating sphere, having sufficiently close common four points with curve α, we have

$$f(s) = f'(s) = f''(s) = f'''(s) = 0.$$

Also, from Eq. (4),

$$< d_1 D_2 + d_2 D_1, M - \alpha(s) >= 1.$$

From the differential of this last equation with respect to s and considering Eq. (1), we find as

$$\begin{aligned} &-\nu(d_1^2 + d_2^2) < T, M - \alpha(s) > +(d_1' - d_2 d_3 \nu) < D_2, M - \alpha(s) > \\ &+ (d_2' + d_1 d_3 \nu) < D_1, M - \alpha(s) >= 0. \end{aligned}$$

Substituting Eqs. (6) and (7) in this last equation, we determinate

$$(d_1' - d_2 d_3 \nu) m_2 + (d_2' + d_1 d_3 \nu) m_3 = 0. \tag{10}$$

If the equations (8) and (10) are solved together, the desired is found. □

Corollary 3.4. *Let $\alpha \subset E^3$ be a polynomial curve with coordinate neighborhood (I, α) in E^3. Then the radius r of the osculating sphere having sufficiently close common four points with curve α at point $\alpha(s) \in \alpha$ is*

$$r = \frac{\sqrt{(d_2' + \nu d_1 d_3)^2 + (\nu d_2 d_3 - d_1')^2}}{(\frac{d_2}{d_1})' d_1^2 + \nu d_3 (d_1^2 + d_2^2)}.$$

Proof. If the center M of the osculating sphere at point $\alpha(s) \in \alpha$ is

$$M = \alpha(s) + m_2 D_2(s) + m_3 D_1(s),$$

we can write

$$r = \|M - \alpha(s), M - \alpha(s)\| = \sqrt{m_2^2 + m_3^2}.$$

Substituting the values m_2 and m_3 in the last equation, the desired is found. □

Theorem 3.5. *Let $\alpha \subset E^3$ be a polynomial curve lying on a sphere with the center M and the radius $r > 0$, then the relation between the curvature d_1 and d_2 of α and the radius r of the sphere is given by*

$$d_1 + d_2 \geq \frac{1}{r}.$$

Proof. The sphere with the center M and the radius r can be represented by

$$\|\alpha - M\| = r$$

or

$$< \alpha - M, \alpha - M >= r^2.$$

Taking the derivative of this equation, we find

$$\nu < T, \alpha - M >= 0,$$

and since $\nu \neq 0$, we obtain

$$< T, \alpha - M >= 0.$$

Taking the derivative of this equation again, it is found as

$$\begin{aligned} < T', \alpha - M > + < T, \alpha' > &= 0, \\ \nu < d_1 D_2 + d_2 D_1, \alpha - M > + \nu &= 0, \ (\nu \neq 0) \\ < d_1 D_2 + d_2 D_1, \alpha - M > + 1 &= 0. \end{aligned}$$

Considering Cauchy-Schwartz inequality, we get

$$\begin{aligned} 1 \leq & \|d_1 D_2 + d_2 D_1\| . \|\alpha - M\| \\ 1 \leq & (d_1 \|D_2\| + d_2 \|D_1\|) r \\ 1 \leq & (d_1 + d_2) r. \end{aligned}$$

So, we have

$$\frac{1}{r} \leq d_1 + d_2.$$

$\square$

Theorem 3.6. *Let S_0^2 be a sphere centered at point 0 and $(I, \alpha) \subset S_0^2$ be a spherical curve. Then the following equations are satisfied*

$$\begin{aligned} < \alpha(s), T(s) > &= -m_1(s), \\ < \alpha(s), D_2(s) > &= -m_2(s), \\ < \alpha(s), D_1(s) > &= -m_3(s), \end{aligned}$$

where T, D_2 and D_1 are the Flc frame's vectors, $m_i : I \longrightarrow \mathbb{R}$ for all $s \in I$.

Proof. Let $\alpha \subset S_0^2$ be a spherical curve with he arc-length parameter s and be S_0^2 a sphere with the radius r. Then we can write

$$\begin{aligned} \|\overrightarrow{O\alpha(s)}\| = \|\overrightarrow{\alpha(s)}\| &= r, \\ \sqrt{< \alpha(s), \alpha(s) >} &= r, \\ < \alpha(s), \alpha(s) > &= r^2. \end{aligned} \tag{11}$$

From derivative of Eq. (11) in terms of s, it is obtained

$$\begin{aligned} < \alpha'(s), \alpha(s) > &= 0, \\ \nu < T(s), \alpha(s) > &= 0, \quad (\nu \neq 0) \\ < T(s), \alpha(s) > &= 0. \end{aligned}$$

Also, if $\alpha(s)$ is a point on the spherical curve α, the position vector of the spherical curve is written as

$$\begin{aligned} \overrightarrow{\alpha(s)O} &= m_1 T + m_2 D_2 + m_3 D_1, \\ -\alpha(s) &= m_1(s)T(s) + m_2(s)D_2(s) + m_3(s)D_1(s), \\ \alpha(s) &= -m_1(s)T(s) - m_2(s)D_2(s) - m_3(s)D_1(s). \end{aligned}$$

From the Euclidean inner product of the last equation and the vectors $T(s)$, $D_2(s)$, $D_1(s)$, the proof is completed. $\square$

Corollary 3.7. *Let S_0^2 be a sphere centered at point 0 and α be a curve on S_0^2. Then the osculating sphere of α is S_0^2 dir.*

Corollary 3.8. *Let S_0^2 be a sphere centered at point 0 and α be a curve on S_0^2, then the position vector of the spherical curve α*

$$\alpha(s) = -m_2(s)D_2(s) - m_3(s)D_1(s)$$

$$= -\frac{d_2' + \nu d_1 d_3}{(\frac{d_2}{d_1})' d_1^2 + \nu d_3(d_1^2 + d_2^2)} D_2(s) - \frac{\nu d_2 d_3 - d_1'}{(\frac{d_2}{d_1})' d_1^2 + \nu d_3(d_1^2 + d_2^2)} D_1(s).$$

Theorem 3.9. *Let $X = X(s)$ be a curve lying on a sphere with the center 0 and the radius r, then the differential equation characterizing the curve $X = X(s)$ is represented as*

$$v^2\big(d_1d_1' + d_2d_2'\big)X''' - \Big(v^2\big((d_2d_3v - d_1')^2 + (d_1d_3v + d_2')^2\big) + 3vv'(d_1d_1' + d_2d_2')\Big)X''$$
$$+ \Big((d_1d_1' + d_2d_2')\big(v^4(d_1^2 + d_2^2) + 3(v')^2 - vv''\big) + vv'\big((d_2d_3v - d_1')^2 + (d_1d_3v - d_2')^2\big)\Big)X'$$
$$- \Big(v^4\big(d_2^2d_3v + d_1^2(-d_3v + (\frac{d_2}{d_1})')\big)\Big)\Big(d_1^2d_3v + d_2(d_2d_3v - d_1') + d_1d_2'\Big)X = 0$$

where d_1, d_2 and d_3 are the curvatures of the curve.

Proof. From Corollary (3.8), the position vector of the spherical curve $X = X(s)$ can be written as

$$X(s) = -\frac{d_2' + \nu d_1d_3}{(\frac{d_2}{d_1})'d_1^2 + \nu d_3(d_1^2 + d_2^2)}D_2(s) - \frac{\nu d_2d_3 - d_1'}{(\frac{d_2}{d_1})'d_1^2 + \nu d_3(d_1^2 + d_2^2)}D_1(s). \tag{12}$$

On the other hand, taking the first, second, and third derivatives of the curve $X = X(s)$, we obtain

$$X'(s) = vT(s),$$

$$X''(s) = vv'\frac{X'(s)}{v} + v^2d_1D_2(s) + v^2d_2D_1(s), \tag{13}$$

$$\begin{aligned} X'''(s) = {} & \big(v'' - v^3d_1^2 - v^3d_2^2\big)\frac{X'(s)}{v} + \big(3vv'd_1 + v^2d_1' - v^3d_2d_3\big)D_2(s) \\ & + \big(3vv'd_2 + v^3d_1d_3 + v^2d_2'\big)D_1(s). \end{aligned} \tag{14}$$

Considering Eqs. (13) and (14) together, we calculate

$$D_2(s) = -\frac{\begin{array}{l} d_2^3X'v^4 - v(d_1d_3v + d_2')(X''v - X'v') \\ +d_2\big(X'''v^2 + d_1^2X'v^4 + 3X'(v')^2 + v(3X''v' + X'v'')\big)\end{array}}{v^4\big(d_1^2d_3v + d_2(d_2d_3v - d_1') + d_1d_2'\big)},$$

$$D_1(s) = -\frac{\begin{array}{l} d_1^3X'v^4 + v(d_2d_3v - d_1')(X''v - X'v') \\ +d_1\big(X'''v^2 + d_2^2X'v^4 + 3X'(v')^2 - v(3X''v' + X'v'')\big)\end{array}}{v^4\big(d_1^2d_3v + d_2(d_2d_3v - d_1') + d_1d_2'\big)}.$$

Substituting the values D_1 and D_2 in Eq. (12), the differential equation characterizing the curve $X = X(s)$ is found. So the proof is completed. $\square$

4. Examples

Let us consider a helical polynomial curve parameterized as $\alpha(s) = (6s, 3s^2, s^3)$. Then the Flc frame elements of α are given by

$$T(s) = \left(\frac{2}{s^2+2}, \frac{2s}{s^2+2}, \frac{s^2}{s^2+2}\right), \quad D_1(s) = \left(\frac{s}{\sqrt{s^2+1}}, \frac{-1}{\sqrt{s^2+1}}, 0\right),$$

$$D_2(s) = \left(-\frac{s^2}{\sqrt{s^2+1}(s^2+2)}, -\frac{s^3}{\sqrt{s^2+1}(s^2+2)}, \frac{2\sqrt{s^2+1}}{s^2+2}\right)$$

and the corresponding curvatures according to Flc frame are as following:

$$d_1(s) = \frac{s}{\sqrt{s^2+1}}, \quad d_2(s) = -\frac{1}{\sqrt{s^2+1}}, \quad d_3(s) = \frac{s^2}{2(s^2+1)}.$$

The center of the osculating sphere of the curve is the center of the sphere in which it is located. If the center and radius of the osculating sphere are found for any point, it is seen that this point satisfies the equation $\|\alpha(s) - M\|^2 = R^2$ for $\forall s \in I$. For $s = 1$, we find

$$T(1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \quad D_1(1) = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \quad D_2(1) \quad = \left(-\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{2\sqrt{2}}{3}\right).$$

If the curve is a spherical curve, we obtain

$$m_2(s) = \frac{3}{2}(s^2+2)^2, \quad m_3(s) = 3s(s^2+2)^2$$

and for $s = 1$, we can write

$$m_2(1) = \frac{27}{2}, \quad m_3(1) = 27.$$

Thus, the center of the osculating sphere is

$$M = \alpha(1) + m_2(1)D_2(1) + m_3(1)D_1(1)$$

$$= \left(6, -\frac{21}{2}, 28\right)$$

and the radius of the osculating sphere is $R = \frac{27\sqrt{5}}{2}$.

5. Conclusion

In this study, spherical curves are redefined according to the Flc frame and the characterizations of spherical curves are reconsidered in terms of Flc frame invariants. The axis of curvature and curvature of the sphere (osculating sphere) of the polynomial curve is found and it is shown that the axis of curvature is on a straight line. Finally, the differential equation characterizing a given spherical curve is found.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contributions Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] M. Bilici, M. Çaliskan, and I. Aydemir, *The natural lift curves and the geodesic sprays for the spherical indicatrices of the pair of evolute-involute curves*, International Journal of Applied Mathematics **11** (2002), no. 4, 415–420.

[2] M. Bilici, *Natural lift curves and the geodesic sprays for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space*, International Journal of Physical Sciences **6** (2011), no. 20, 4706–4711.

[3] M. Bilici and M. Çaliskan, *Some new results on the curvatures of the spherical indicatrices of the involutes of a spacelike curve with a spacelike binormal in Minkowski 3-space*, MathLAB Journal **2** (2019), no. 1, 110–119.

[4] M. Bilici, E. Ergün, and M. Çalişkan, *The natural lift curves for the spherical indicatrices of spacelike Bertrand couple in Minkowski 3-space*, J. Math. Comput. Sci. **4** (2014), no. 1, 85–104.

[5] M. Bilici and T. A. Ahmad, *On the natural lift curves for the involute spherical indicatrices in Minkowski 3-space*, Malaya Journal of Matematik **5** (2017), no. 2, 407–417.

[6] M. Bilici, E. Ergün, and M. Çalıskan, *A new approach to natural lift curves of the spherical indicatrices of timelike Bertrand mate of a spacelike curve in Minkowski 3-space*, International Journal of Mathematical Combinatorics **1** (2015), 35–48.

[7] M. Bilici and M. Çalışkan, *Some geometrical calculations for the spherical indicatrices of involutes of a timelike curve in Minkowski 3-space*, Journal of Advances in Mathematics **5** (2014), no. 2, 668–677.

[8] S. Breuer and D. Gottlieb, *An explicit characterization of spherical curves*, Proc. Am.Math.Soc. **27** (1971), 126–127.

[9] B. Bükcü and M. K. Karacan, *Spherical curves with modified orthogonal frame*, Journal of New Results in Science **10** (2016), 60–68.

[10] V. Dannon, *Integral characterizations and the theory of curves*, Proc. Amer. Math. Soc. **81** (1981), no. 4, 600–602.

[11] M. Dede, *A new representation of tubular surfaces*, Houston J. Math. **45** (2019), no. 3, 707–720.

[12] P. Do Carmo, *Differential Geometry of Curves and Surfaces*, IMPA, 1976.

[13] H. Kocayiğit, N. Yaz, Ç. Camci, and H. H. Hacisalihoğlu, *On the explicit characterization of spherical curves in n-dimensional Euclidean space*, Journal of Inverse and III-posed Problems **11** (2003), no. 3, 245–254.

[14] E. Mehlum and J. Wimp, *Spherical curves and quadratic relationships for special functions*, Austral. Mat. Soc. **27** (1985), no. 1, 111–124.
[15] P. B. Okullu, H. Kocayigit, and T. A. Aydin, *An explicit characterization of spherical curves according to Bishop frame and approximately solution*, Thermal Science **23** (2019), no. 1, 361–370.
[16] Y. C. Wong, *On an explicit characterization of spherical curves*, Proc. Am. Math. Soc. **34** (1972), 239–242.
[17] Y. C. Wong, *A global formulation of the condition for a curve to lie in a sphere*, Monatsh. Math. **67** (1963), 363–365.

Kemal Eren
Department of Mathematics, Sakarya University,
Sakarya, Turkey.
E-mail:kemal.eren1@ogr.sakarya.edu.tr

Kebire Hilal Ayvacı
Department of Mathematics, Ordu University,
Ordu, Turkey.
E-mail:kebirehilalayvaci@odu.edu.tr

Süleyman Şenyurt
Department of Mathematics, Ordu University,
Ordu, Turkey.
E-mail:senyurtsuleyman52@gmail.com