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CHEN INEQUALITIES ON LIGHTLIKE HYPERSURFACES OF A LORENTZIAN MANIFOLD WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we investigate k-Ricci curvature and k-scalar curvature on lightlike hypersurfaces of a real space form $\tilde{M}(c)$ of constant sectional curvature c, endowed with semi-symmetric non-metric connection. Using this curvatures, we establish some inequalities for screen homothetic lightlike hypersurface of a real space form $\tilde{M}(c)$ of constant sectional curvature c, endowed with semi-symmetric non-metric connection. Using these inequalities, we obtain some characterizations for such hypersurfaces. Considering the equality case, we obtain some results.

1. Introduction

It is well known that the geometry of lightlike submanifolds of a semi-Riemannian manifold is different from the geometry of submanifolds immersed in a Riemannian manifold, since the normal vector bundle of a lightlike submanifold intersects with the tangent bundle, making it more interesting to study. The geometry of lightlike submanifolds of a semi-Riemannian manifold is developed by Duggal-Bejancu [14] and Duggal-Şahin [16].

Hayden [19] introduced the notion of a semi-symmetric metric connection and Yano studied semi-symmetric metric connection in [30]. Nakao [25] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections. On the other hand, Agashe and Chafle introduced the notion of a semi-symmetric non-metric connection in [1] and [2] and they considered submanifolds of a Riemannian manifold endowed with a semi-symmetric non-metric connection. De and Kamilya [12] gave basic properties of a hypersurface of a Riemannian manifold with semi-symmetric non-metric connection.

According to Chen [8], one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants, namely the squared mean curvature and the main intrinsic invariants of a submanifold, namely the sectional curvatures. One of the most powerful tools to find

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relationships between intrinsic invariants and extrinsic invariants of a submanifold is provided by Chen's invariants. In 1993, Chen [7] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M = \tau(p) - \inf(K)(p), \tag{1}$$

where $\tau(p)$ is the scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.$$

In 1993, Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications (Lemma 2.1, [6]). This inequality is now well known as Chen's inequality, and in the equality case it is known as Chen's equality.

In [9, 10, 20, 24, 29] were studied similar problems for non-degenerate submanifolds of different spaces. Later, Özgür and Mihai studied Chen inequalities on submanifolds of real space forms endowed with semi-symmetric non-metric connection in [26]. In [23], Liang and Pan proved Chen's general inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection, which generalized a result of [26].

These problems in degenerate geometry were firstly studied by Gülbahar, Kılıç and Keleş in [17]. They introduced Chen-like inequalities and curvature invariants in lightlike geometry. Also, they established some inequalities between the extrinsic scalar curvatures and the intrinsic scalar curvatures. In [18], they established Chen-Ricci inequality and Chen inequality on a screen homothetic lightlike hypersurface of a Lorentzian manifold. In [27], Poyraz, Doğan, and Yaşar introduced k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Using this curvatures, they established some inequalities for lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Moreover several works in this direction is studied [21, 22].

In this paper, we investigate k-Ricci curvature and k-scalar curvature on lightlike hypersurfaces of a real space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with semi-symmetric non-metric connection. Using this curvatures, we establish some inequalities for screen homothetic lightlike hypersurface of a real space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with semi-symmetric non-metric connection. Using these inequalities, we obtain some characterizations for such hypersurfaces. Considering the equality case, we obtain some results.

2. Preliminaries

Let M be a hypersurface of a $(n+1)-$dimensional, $n>1$, semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\widetilde{g}$ of index $1 \leq \nu \leq n$. We consider

$$T_x M^{\perp} = \left\{ Y_x \in T_x\widetilde{M} \mid \widetilde{g}_x\left(Y_x, X_x\right) = 0, \forall X_x \in T_x M \right\}$$

for any $x \in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface* of $\widetilde{M}$ or equivalently, the immersion

$$i : M \to \widetilde{M}$$

is *lightlike (null, degenerate)* if $T_x M \cap T_x M^{\perp} \neq \{0\}$ at any $x \in M$.

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM named the *screen distribution* on M and denoted $S\left(TM\right)$. We have the following splitting into orthogonal direct sum:

$$TM = S\left(TM\right) \perp TM^{\perp}. \tag{2}$$

The subbundle $S\left(TM\right)$ is non-degenerate, so is $S\left(TM\right)^{\perp}$, and the following satisfies:

$$T\widetilde{M} = S\left(TM\right) \perp S\left(TM\right)^{\perp}, \tag{3}$$

where $S\left(TM\right)^{\perp}$ is the orthogonal complementary vector bundle to $S\left(TM\right)$ in $T\widetilde{M}\Big|_M$.

Let $tr(TM)$ denotes the complementary vector bundle of $TM^{\perp}$ in $S\left(TM\right)^{\perp}$. Then we have

$$S\left(TM\right)^{\perp} = TM^{\perp} \oplus tr(TM). \tag{4}$$

Let $\mathcal{U}$ be a coordinate neighborhood in M and ξ be a basis of $\Gamma(TM^{\perp}|_{\mathcal{U}})$. Then there exists a basis N of $tr\left(TM\right)|_U$ satisfying the following conditions:

$$\tilde{g}(N, \xi) = 1$$

and

$$\tilde{g}\left(N, N\right) = \tilde{g}(W, N) = 0, \quad \forall W \in \Gamma(\,S\left(TM\right)|_{\mathcal{U}}).$$

The subbundle $tr\left(TM\right)$ is named a *lightlike transversal vector bundle* of M. We note that $tr\left(TM\right)$ is never orthogonal to TM. From (2), (3) and (4) we have

$$T\widetilde{M}\Big|_M = S\left(TM\right) \perp \left(TM^{\perp} \oplus tr\left(TM\right)\right) = TM \oplus tr\left(TM\right). \tag{5}$$

For more details, we refer to [14, 16].

3. Semi-Symmetric Non-Metric Connection

Let $\widetilde{M}$ be an $(n+2)-$dimensional differentiable manifold of class C^{∞} and $\widetilde{\nabla}$ be a linear connection in $\widetilde{M}$. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ is defined by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}], \ \ \forall \widetilde{X},\widetilde{Y} \in \Gamma(T\widetilde{M})$$

satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

for a $1-$form $\widetilde{\pi}$, then the connection $\widetilde{\nabla}$ is said to be semi-symmetric (see [1, 30]).

Let $\widetilde{g}$ be a semi-Riemannian metric of index ν with $1 \leq \nu \leq n+1$ in $\widetilde{M}$ and $\widetilde{\nabla}$ be satisfy

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{g})(\widetilde{Y},\widetilde{Z}) = -\widetilde{\pi}(\widetilde{Y})(\widetilde{X},\widetilde{Z}) - \widetilde{\pi}(\widetilde{X})(\widetilde{Y},\widetilde{Z}),$$

then such a linear connection of this type is called a non-metric connection (see [1]).

We assume that the semi-Riemannian manifold $\widetilde{M}$ admits a semi-symmetric non-metric connection which is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\pi}(\widetilde{Y})\widetilde{X} \tag{6}$$

for arbitrary vector fields $\widetilde{X}$ and $\widetilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\widetilde{g}$, $\widetilde{\pi}$ is a $1-$form and $\tilde{Q}$ is the vector field defined by

$$\widetilde{g}(\tilde{Q},\widetilde{X}) = \widetilde{\pi}(\widetilde{X})$$

for an arbitrary vector field $\widetilde{X}$ of $\widetilde{M}$ (see [1] and [15]).

By using the second form of the decomposition (5), we can write

$$\tilde{Q} = \varphi\tilde{Q} + \mu N, \tag{7}$$

where $\tilde{Q}$ is a vector field and μ is a function in M.

The *Gauss formula* with respect to the induced connection ∇ on the lightlike hypersurface from the semi-symmetric non-metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + m\,(X,Y)\,N \tag{8}$$

for arbitrary vector fields X and Y of M, where m is a tensor of type $(0,2)$ of the lightlike hypersurface of M [31].

On the other hand, denoting the projection of TM on $S\,(TM)$ with respect to the decomposition (2) by P, one has the *Gauss formula* with respect to the semi-symmetric non-metric connection which is given by

$$\nabla_X PY = \overset{*}{\nabla}_X PY + D(X,PY)\xi, \tag{9}$$

where $\overset{*}{\nabla}_X PY$ belongs to $\Gamma(S\,(TM))$ and D is $1-$form on M.

Thus (7) can be written as

$$\tilde{Q} = P\varphi\tilde{Q} + \lambda\xi + \mu N, \tag{10}$$

where $\lambda = \widetilde{\pi}(N)$.

The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$ on real space form $\widetilde{M}(c)$ is defined by

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\}. \tag{11}$$

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}(c)$ can be written as [1]

$$\widetilde{R}(X,Y,Z,W) = \overset{\circ}{\widetilde{R}}(X,Y,Z,W) + s(X,Z)g(Y,W) - s(Y,Z)g(X,W), \tag{12}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$ and $(0,2)$ tensor field s which defined by

$$s(X,Y) = (\overset{\circ}{\widetilde{\nabla}}_X \pi)Y - \pi(X)\pi(Y). \tag{13}$$

Moreover, Gauss-Codazzi equations with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ can be written as [31]

$$\begin{aligned} R(X,Y,Z,PW) = {} & \widetilde{R}(X,Y,Z,PW) + \lambda m(X,Z)g(PY,PW) \\ & - \lambda m(Y,Z)g(PX,PW) + m(Y,Z)D(X,PW) \\ & - m(X,Z)D(Y,PW) \\ & + \{m(X,Z)\eta(Y) - m(Y,Z)\eta(X)\}\pi(PW), \end{aligned} \tag{14}$$

$$\begin{aligned} \widetilde{g}(\tilde{R}(X,Y)Z,\xi) \;=\; & \pi(Y)\,m(X,Z) - \pi(X)\,m(Y,Z) + (\nabla_X m)(Y,Z) \\ & -(\nabla_Y m)(X,Z) + m(Y,Z)\,\tau(X) \\ & -m(X,Z)\,\tau(Y) \end{aligned} \tag{15}$$

and

$$\widetilde{g}(\tilde{R}(X,Y)Z,N) = g(R(X,Y)Z,N) + \lambda m(Y,Z)\eta(X) - \lambda m(X,Z)\eta(Y), \tag{16}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$.

From (11), (12) and (14), we have

$$\begin{aligned} R(X,Y,Z,PW) \;=\; & c\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\} \\ & +s(X,Z)g(Y,PW) - s(Y,Z)g(X,PW)) \\ & +\lambda m(X,Z)g(PY,PW) - \lambda m(Y,Z)g(PX,PW) \\ & +m(Y,Z)D(X,PW) - m(X,Z)D(Y,PW) \\ & +\{m(X,Z)\eta(Y) - m(Y,Z)\eta(X)\}\pi(PW). \end{aligned} \tag{17}$$

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then M is named totally umbilical lightlike hypersurface if there exists a smooth function such that

$$m(X,Y)_p = Hg_p(X,Y), \qquad X,Y \in \Gamma(T_pM) \tag{18}$$

for any coordinate neighborhood U, where $H \in R$. If every points of M is umbilical, the lightlike hypsersurface M is named totally umbilical in $\widetilde{M}$ [14]. If $m = 0$, then the lightlike hypsersurface M is named totally geodesic in $\widetilde{M}$.

The mean curvature μ of M with respect to an orthonormal basis $\{e_1, ..., e_n\}$ of $\Gamma(S(TM))$ is defined by [5]

$$\mu = \frac{1}{n} tr(m) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i m(e_i, e_i), \qquad g(e_i, e_i) = \varepsilon_i. \tag{19}$$

A lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is called *screen locally conformal* if the shape operators A_N and $\overset{*}{A}_\xi$ of M and $S(TM)$, respectively, are related by

$$A_N = \varphi \overset{*}{A}_\xi, \tag{20}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ on M. In particular, if φ is a non-zero constant, M is called screen homothetic [3].

Let $\Pi = sp\{e_i, e_j\}$ be 2-dimensional non-degenerate plane of the tangent space T_pM at $p \in M$. Then the number

$$K_{ij} = \frac{g(R(e_j, e_i)e_i, e_j)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2} \tag{21}$$

is called the sectional curvature of the section Π at $p \in M$. Since the screen second fundamental form C is symmetric on a screen homothetic lightlike hypersurface, the sectional curvature K_{ij} is symmetric, that is, $K_{ij} = K_{ji}$. But, in general, the sectional curvature need not be symmetric for a lightlike hypersurface of a semi-Riemannian manifold [16].

Let $p \in M$ and ξ be null vector of T_pM. A plane Π of T_pM is said to be null plane if it contains ξ and e_i such that $g(\xi, e_i) = 0$ and $g(e_i, e_i) = \varepsilon_i = \pm 1$. One defines the null sectional curvature of Π by

$$K_i^{null} = \frac{g(R_p(e_i, \xi)\xi, e_i)}{g_p(e_i, e_i)}. \tag{22}$$

For more details related to the null sectional curvature, we refer to [4].

Denote the Ricci tensor of $\widetilde{M}$ with $\widetilde{Ric}$ and the induced Ricci type tensor of M with $R^{(0,2)}$. Then, $\widetilde{Ric}$ and $R^{(0,2)}$ are given by

$$\widetilde{Ric}(X,Y) = trace\{Z \to \widetilde{R}(Z,X)Y\}, \quad \forall X, Y \in \Gamma(T\widetilde{M}), \tag{23}$$

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall X, Y \in \Gamma(TM), \tag{24}$$

where

$$R^{(0,2)}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i,X)Y,e_i) + \tilde{g}(R(\xi,X)Y,N) \tag{25}$$

for the quasi-orthonormal frame $\{e_1,...,e_n,\xi\}$ of T_pM.

The scalar curvature τ is defined

$$\tau(p) = \sum_{i,j=1}^{n} K_{ij} + \sum_{i=1}^{n} K_i^{null} + K_{iN}, \tag{26}$$

where $K_{iN} = \tilde{g}(R(\xi,e_i)e_i,N)$ for $i \in \{1,...,n\}$ [13].

4. Chen-Like Inequalities

Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold $\widetilde{M}$ with a semi-symmetric non-metric connection. Suppose that $\{e_1,...,e_n,\xi\}$ and $\{e_1,...,e_n\}$ are basis of $\Gamma(TM)$ and an orthonormal basis of $\Gamma(S(TM))$, respectively. Similarly, for $k \le n$, $\pi_{k,\xi} = sp\{e_1,...,e_k,\xi\}$ and $\pi_k = sp\{e_1,...,e_k\}$ are $(k+1)-$dimensional degenerate plane section and $\pi_k = sp\{e_1,...,e_k\}$ is $k-$dimensional non-degenerate plane section, respectively. For a unit vector $X \in \Gamma(TM)$, the $k-$degenerate Ricci curvature and the $k-$Ricci curvature are defined by

$$Ric_{\pi_{k,\xi}}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j) + \widetilde{g}(R(\xi,X)X,N), \tag{27}$$

$$Ric_{\pi_k}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j), \tag{28}$$

respectively. Also for $p \in M$, $k-$degenerate scalar curvature and $k-$scalar curvature are determined by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{null} + K_{iN}, \tag{29}$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij}, \tag{30}$$

respectively. For $k = n$, $\pi_n = sp\{e_1,...,e_n\} = \Gamma(S(TM))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$Ric_{S(TM)}(e_1) = Ric_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + ... + K_{1n} \tag{31}$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^{n} K_{ij}, \tag{32}$$

respectively [17].

Using (17) and (32) we obtain

$$\tau_{S(TM)}(p) = n(n-1)c - (n-1)\alpha + \sum_{i,j=1}^{n} m_{ii}D_{jj} - m_{ij}D_{ji} - n(n-1)\lambda\mu, \tag{33}$$

where α is the trace of s and $m_{ij} = m(e_i, e_j)$, $D_{ij} = D(e_i, e_j)$ for $i, j \in \{1, ..., n\}$.

Let $\widetilde{M}(c)$ be a Lorentzian space form and M be a screen homothetic lightlike hypersurface of an $(n+2)$-dimensional $\widetilde{M}(c)$. Using (11), (12), (14), (16) and (33) we get the following equations:

$$\tau_{S(TM)}(p) = n(n-1)c - (n-1)\alpha + \varphi n^2\mu^2 - \varphi \sum_{i,j=1}^{n} (m_{ij})^2 - n(n-1)\lambda\mu, \tag{34}$$

$$\begin{aligned} \sum_{i=1}^{n} K_i^{null} &= \sum_{i=1}^{n} R(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^{n} \widetilde{R}(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^{n} -s(\xi, \xi) = -ns(\xi, \xi), \end{aligned} \tag{35}$$

$$\begin{aligned} \sum_{i=1}^{n} K_i^{N} &= \sum_{i=1}^{n} R(\xi, e_i, e_i, N) \\ &= \sum_{i=1}^{n} \widetilde{R}(\xi, e_i, e_i, N) - \lambda m(e_i, e_i) \\ &= \sum_{i=1}^{n} (c - s(e_i, e_i) - \lambda m(e_i, e_i)) \\ &= nc - \alpha - \lambda n\mu. \end{aligned} \tag{36}$$

From (26), (34), (35) and (36), we get the induced scalar curvature $\tau(p)$ of M as following:

$$\tau(p) = n^2c - n\alpha + \varphi n^2\mu^2 - \varphi \sum_{i,j=1}^{n} (m_{ij})^2 - ns(\xi, \xi) - n^2\lambda\mu. \tag{37}$$

Using (37) we obtain the following :

Theorem 4.1. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

$$\frac{1}{\varphi}\left(\tau(p) - n^2c + n\alpha + ns(\xi,\xi) + n^2\lambda\mu\right) \leq n^2\mu^2. \tag{38}$$

The equality of (38) holds for $p \in M$ if and only if p is a totally geodesic point.

Lemma 4.2. [28] *Let $a_1, a_2,...,a_n$, be n-real number $(n > 1)$, then*

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality if and only if $a_1 = a_2 =...= a_n$.

Theorem 4.3. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

$$\frac{1}{\varphi}\left(\tau(p) - n^2c + n\alpha + ns(\xi,\xi) + n^2\lambda\mu\right) \leq n(n-1)\mu^2. \tag{39}$$

The equality of (39) satisfies at $p \in M$ if and only if p is a totally umbilical point.

Proof. Using Lemma 4.2 one derives

$$n\mu^2 \leq \sum_{i=1}^{n} (m_{ii})^2. \tag{40}$$

After substituting (40) in (37) we find (39). The equality of (39) satisfies for all $p \in M$ if and only if

$$m_{11} = ... = m_{nn}.$$

Thus p is a totally umbilical point. □

Lemma 4.4. [11] *Let $a_1, ..., a_n$ be $n-$real numbers and define $A = \sum_{i<j}(a_i - a_j)^2$. Then*

(1) $A \geq \frac{n}{2}(a_1 - a_2)^2$ and equality holds if and only if

$$\frac{1}{2}(a_1 + a_2) = a_3 = ... = a_n.$$

(2) Let k, ℓ be integers such that $1 \leq k < \ell \leq n$ and $(k, \ell) \neq (1, 2)$. If $A = \frac{n}{2}(a_1 - a_2)^2 = \frac{n}{2}(a_k - a_1)^2$ then $a_1 = a_2 = ... = a_n$.

If the sectional curvature is screen homothetic, then the sectional curvature of lightlike hypersurface is symmetric. One defines the screen scalar curvature $r_{S(TM)}$

$$r_{S(TM)}(p) = \sum_{1\leq i<j\leq n} K_{ij} = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij} = \frac{1}{2}\tau_{S(TM)}(p). \tag{41}$$

By using (41), the equality (34) can be rewritten as follows:

$$2r_{S(TM)}(p) = n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\varphi n^2\mu^2-\varphi\sum_{i,j=1}^{n}(m_{ij})^2. \tag{42}$$

Theorem 4.5. *Let M be a screen homothetic lightlike hypersurface with $\varphi>0$ of $\bar{M}$. Then we have*

$$\begin{aligned} 2r_{S(TM)}(p) &\leq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{n^3}{n+1}\varphi\mu^2 \\ &\quad -\frac{\varphi n}{2(n+1)}\sum_{i,j=1}^{n}(m_{11}-m_{22})^2. \end{aligned} \tag{43}$$

The equality of (43) holds at $p\in M$ if and only if the mean curvature of M is equal to $\frac{n}{2}(m_{11}+m_{22})$, that is, $\mu=\frac{n}{2}(m_{11}+m_{22})$.

Proof. From the Binomial Theorem, we can write

$$\begin{aligned} &(m_{11}-m_{22})^2+...+(m_{11}-m_{nn})^2+(m_{22}-m_{33})^2+...+(m_{22}-m_{nn})^2 \\ &+...+(m_{n-1n-1}-m_{nn})^2 = n\sum_{i=1}^{n}(m_{ii})^2-2\sum_{1\leq i\neq j\leq n} m_{ii}m_{jj}. \end{aligned} \tag{44}$$

By Lemma 4.4 and (44) we have

$$\sum_{i=1}^{n}(m_{ii})^2 \geq \frac{1}{n}\sum_{i\neq j} m_{ii}m_{jj}+\frac{1}{2}(m_{11}-m_{22})^2. \tag{45}$$

On the other hand, we can write

$$\frac{1}{n}\sum_{i\neq j} m_{ii}m_{jj} = n\mu^2-\frac{1}{n}\sum_{i=1}^{n}(m_{ii})^2. \tag{46}$$

Using (45) and (46) we get

$$\sum_{i=1}^{n}(m_{ii})^2 \geq \frac{n^2}{n+1}\mu^2+\frac{n}{2(n+1)}(m_{11}-m_{22})^2. \tag{47}$$

Finally, by (42) and (47), we obtain (43).

The equality case of (43) holds then taking consideration of the case (1) of Lemma 4.4 we get $\mu=\frac{n}{2}(m_{11}+m_{22})$. The converse part of the theorem is straightforward. □

Lemma 4.6. *If $n > k \geq 2$ and $a_1, ..., a_n \in \mathbb{R}$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + a\right),$$

then

$$2 \sum_{1\leq i<j\leq k} a_i a_j \geq a$$

with equality holding if and only if

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n.$$

Theorem 4.7. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then, for each point $p \in M$ and each non-degenerate k-plane section $\Pi_k \subset TpM$ $(n > k \geq 2)$, we have*

$$\begin{aligned} \tau_{S(TM)}(p) - \tau(\pi_k) \quad \leq \quad & (n-k)\left(\frac{\varphi n^2}{(n-k+1)}\mu^2 + (n+k-1)c - \alpha\right) \\ & +\varphi \sum_{i=k+1}^{n} (m_{ii})^2 + (k-1)(\lambda \sum_{i=1}^{k} m_{ii} \\ & -trace(s|_{\pi_k^{\perp}})) - n(n-1)\lambda\mu. \end{aligned} \tag{48}$$

If the equality case of (48) satisfies at $p \in M$, thus M is minimal and the form of shape operator of M becomes

$$A_\xi^* = \begin{bmatrix} m_{11} & m_{12} & . & . & m_{1k} & \\ m_{21} & m_{22} & . & . & m_{2k} & \\ . & . & . & . & . & 0 \\ . & . & . & . & . & \\ . & . & . & . & . & \\ m_{k1} & m_{k2} & . & . & -\sum_{i=1}^{k-1}(m_{ii}) & \\ & & 0 & & & 0_{n-k} \end{bmatrix}. \tag{49}$$

Proof. One takes

$$\varepsilon = \tau_{S(TM)}(p) - n(n-1)c + (n-1)\alpha + n(n-1)\lambda\mu - \varphi\frac{n^2(n-k)}{(n-k+1)}\mu^2, \tag{50}$$

in (34), then we have

$$\varepsilon = \varphi\frac{n^2}{(n-k+1)}\mu^2 - \varphi\sum_{i,j=1}^{n}(m_{ij})^2.$$

Therefore, we can write

$$\left(\sum_{i=1}^{n} m_{ii}\right)^2 = (n-k+1)\left(\sum_{i=1}^{n}(m_{ii})^2 + \sum_{i\neq j=1}^{n}(m_{ij})^2 + \frac{\varepsilon}{\varphi}\right). \tag{51}$$

From Lemma 4.6 we get

$$2\sum_{1\leq i<j\leq k} m_{ii}m_{jj} \geq \sum_{i\neq j=1}^{n}(m_{ij})^2 + \frac{\varepsilon}{\varphi}. \tag{52}$$

Now, a non-degenerate plane section π_k spanned by $\{e_1, e_2, ..., e_k)$. Then one obtains

$$\begin{aligned}
\tau(\pi_k) &= k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i,e_i) + \lambda\sum_{i=1}^{k} m_{ii}) \\
&+ \varphi\sum_{i,j=1}^{k} m_{ii}m_{jj} - (m_{ij})^2 \\
&= k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i,e_i) + \lambda\sum_{i=1}^{k} m_{ii}) + \varphi\sum_{i=1}^{k}(m_{ii})^2 \\
&+ 2\varphi\sum_{1\leq i<j\leq k} m_{ii}m_{jj} - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 \\
&\geq k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i,e_i) + \lambda\sum_{i=1}^{k} m_{ii}) + \varphi\sum_{i=1}^{k}(m_{ii})^2 \\
&+ \sum_{i\neq j=1}^{n}(m_{ij})^2 + \varepsilon - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 \\
&= k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i,e_i) + \lambda\sum_{i=1}^{k} m_{ii}) + \varepsilon + \varphi\sum_{i,j=1}^{n}(m_{ij})^2 \\
&- \varphi\sum_{i=1}^{n}(m_{ii})^2 - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 + \varphi\sum_{i=1}^{k}(m_{ii})^2 \\
&\geq k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i,e_i) + \lambda\sum_{i=1}^{k} m_{ii}) + \varepsilon \qquad (53) \\
&+ \varphi\sum_{i,j=k+1}^{n}(m_{ij})^2 - \varphi\sum_{i=k}^{n}(m_{ii})^2.
\end{aligned}$$

We remark that

$$\sum_{i=1}^{k} s(e_i, e_i) = \alpha - trace(s|_{\pi_k^\perp}). \tag{54}$$

Using (50), (53) and (54) we get

$$\begin{aligned}\tau(\pi_k) \geq\ & k(k-1)c - (k-1)(\sum_{i,j=1}^{k} s(e_i, e_i) + \lambda \sum_{i=1}^{k} m_{ii}) + \varphi \sum_{i,j=k+1}^{n} (m_{ij})^2 \\ & -\varphi \sum_{i=k+1}^{n} (m_{ii})^2 + \tau_{S(TM)}(p) - n(n-1)c + (n-1)\alpha \\ & + n(n-1)\lambda\mu - \varphi \frac{n^2(n-k)}{(n-k+1)}\mu^2.\end{aligned} \tag{55}$$

From (55) we have (48) and (49) which implies that M is minimal. □

Corollary 4.8. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$, $\Pi_2 = Span\{e_1, e_2\}$ be a 2-dimensional non-degenerate plane section of TpM, $p \in M$. Then we have*

$$\begin{aligned}\delta_M \leq\ & (n-2)\left[\frac{\varphi n^2}{(n-1)}\mu^2 + (n-1)c - \alpha\right] + \varphi \sum_{i=3}^{n} (m_{ii})^2 \\ & + (\sum_{i=1}^{2} m_{ii} - trace(s|_{\pi_2^\perp})) - n(n-1)\lambda\mu.\end{aligned} \tag{56}$$

If the equality case of (56) holds at $p \in M$, then M is minimal and the shape operator of M take the form:

$$A_\xi^* = \begin{bmatrix} m_{11} & m_{12} & . & . & . & 0 \\ m_{21} & -m_{11} & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}. \tag{57}$$

Theorem 4.9. *Let M be a screen homothetic lightlike hypersurface with $\varphi > 0$ of $\bar{M}$. Then we have*

$$\tau_{S(TM)}(p) \leq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \varphi n(n-1)\mu^2. \tag{58}$$

The equality case of (58) holds at $p \in M$ if and only if p is a totally umbilical point.

Proof. From (34) we have

$$\tau_{S(TM)}(p) = n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu + \varphi n^2\mu^2-\varphi\sum_{i=1}^{n}(m_{ii})^2-\varphi\sum_{i\neq j}(m_{ij})^2. \quad (59)$$

Moreover, from (40) we have

$$n\mu^2\leq\sum_{i=1}^{n}(m_{ii})^2. \quad (60)$$

Considering (59) and (60) we obtain (58). Equality case of (58) holds, then

$$m_{11}=m_{22}=...=m_{nn},$$

the shape operator A^*_ξ take the form:

$$A^*_\xi=\begin{bmatrix} m_{11} & 0 & . & . & . & 0 & 0\\ 0 & m_{11} & . & . & . & 0 & 0\\ . & & . & & & & \\ . & & & . & & & \\ . & & & & . & & \\ 0 & 0 & . & . & . & m_{11} & 0\\ 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}, \quad (61)$$

which shows that M is totally umbilical. The proof of the converse part is straightforward. □

Furthermore, the second fundamental form m and the screen second fundamental form D provide

$$\sum_{i,j=1}^{n}m_{ij}D_{ji}=\frac{1}{2}\left\{\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2-\sum_{i,j=1}^{n}(m_{ij})^2+(D_{ji})^2\right\} \quad (62)$$

and

$$\sum_{i,j=1}^{n}m_{ii}D_{jj}=\frac{1}{2}\left\{\left(\sum_{i,j=1}^{n}m_{ii}+D_{jj}\right)^2-\left(\sum_{i=1}^{n}m_{ii}\right)^2-\left(\sum_{j=1}^{n}D_{jj}\right)^2\right\}. \quad (63)$$

Theorem 4.10. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

(*i*)

$$\tau_{S(TM)}(p) \leq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu + n\mu traceA_N+\frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2. \quad (64)$$

The equality case of (64) satisfies for all $p \in M$ if and only if either M is a screen homothetic lightlike hypersurface with $\varphi = -1$ or M is a totally geodesic lightlike hypersurface.

(ii)

$$\tau_{S(TM)}(p) \geq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + n\mu trace A_N - \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} + D_{ji})^2. \tag{65}$$

The equality case of (65) satisfies for all $p \in M$ if and only if either M is a screen homothetic lightlike hypersurface with $\varphi = 1$ or M is a totally geodesic lightlike hypersurface.

(iii) (64) and (65) with equalities if and only if p is a totally geodesic point.

Proof. From (33) and (62), we get

$$\tau_{S(TM)}(p) = n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \sum_{i,j=1}^{n} m_{ii}D_{jj} - \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij} + D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2. \tag{66}$$

Since

$$\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2 = \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} + D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} - D_{ji})^2 \tag{67}$$

one obtains

$$\tau_{S(TM)}(p) = n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \sum_{i,j=1}^{n} m_{ii}D_{jj} - \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} + D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} - D_{ji})^2. \tag{68}$$

From (68) $(i), (ii)$ and (iii) statements are easily obtained. □

Thus we get the following corollary.

Corollary 4.11. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then we have*

(i)

$$\tau_{S(TM)}(p) \leq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \varphi n^2\mu^2 + \frac{(1-\varphi)^2}{4}\sum_{i,j=1}^{n}(m_{ij})^2. \tag{69}$$

(ii)

$$\tau_{S(TM)}(p) \geq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \varphi n^2\mu^2 - \frac{(1+\varphi)^2}{4}\sum_{i,j=1}^{n}(m_{ij})^2. \tag{70}$$

Theorem 4.12. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

$$\tau_{S(TM)}(p) \leq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{1}{2}(trace\bar{A})^2 - \frac{1}{2}(traceA_N)^2 - \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2, \tag{71}$$

where

$$\bar{A} = \begin{bmatrix} m_{11}+D_{11} & m_{12}+D_{21} & . \;.\; . & m_{1n}+D_{n1} \\ m_{21}+D_{12} & m_{22}+D_{22} & . \;. & m_{2n}+D_{n2} \\ . & & . & \\ . & & & \\ . & & & \\ m_{n1}+D_{1n} & m_{n2}+D_{2n} & & m_{nn}+D_{nn} \end{bmatrix}. \tag{72}$$

The equality case of (71) satisfies for all $p \in M$ if and only if M is minimal.

Proof. From (63) and (68) we obtain

$$\tau_{S(TM)}(p) = n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{1}{2}\left(\sum_{i,j=1}^{n} m_{ii}+D_{jj}\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{n} m_{ii}\right)^2 - \frac{1}{2}\left(\sum_{j=1}^{n} D_{jj}\right)^2 - \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2. \tag{73}$$

Assume the equality case of (71) is satisfied, then

$$\sum_{i} m_{ii} = 0.$$

Thus M is minimal. □

Thus we get the following corollary.

Corollary 4.13. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then we have*

$$\tau_{S(TM)}(p) \leq n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{(2\varphi+1)}{4}n^2\mu^2 - \varphi\sum_{i=1}^{n}(m_{ij})^2. \tag{74}$$

The equality case of (74) satisfies for all $p \in M$ if and only if M is minimal.

Theorem 4.14. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then we have*

$$\begin{aligned}\tau_{S(TM)}(p) \leq\ & n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{(2n-1)}{4n}(trace\bar{A})^2 \\ & -\frac{1}{2}(traceA_N)^2 - \frac{1}{2}n^2\mu^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} - D_{ji})^2 \\ & -\frac{1}{4}\sum_{i\neq j}(m_{ij} + D_{ji})^2,\end{aligned} \tag{75}$$

where $\bar{A}$ is equal to (72). The equality case of (75) satisfies for all $p \in M$ if and only if $n\mu = -traceA_N$.

Proof. From (73), we get

$$\begin{aligned}\tau_{S(TM)}(p) =\ & n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{1}{2}(trace\bar{A})^2 \\ & -\frac{1}{2}(traceA_N)^2 - \frac{1}{2}n^2\mu^2 - \frac{1}{4}\sum_{i=1}^{n}(m_{ii} + D_{ii})^2 \\ & -\frac{1}{4}\sum_{i\neq j}(m_{ij} + D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} - D_{ji})^2.\end{aligned} \tag{76}$$

Using Lemma 4.2 in (76), we have

$$\begin{aligned}\tau_{S(TM)}(p) \leq\ & n(n-1)c - (n-1)\alpha - n(n-1)\lambda\mu + \frac{1}{2}(trace\bar{A})^2 \\ & -\frac{1}{2}(traceA_N)^2 - \frac{1}{2}n^2\mu^2 - \frac{1}{4n}\left(\sum_{i=1}^{n} m_{ii} + D_{ii}\right)^2 \\ & -\frac{1}{4}\sum_{i\neq j}(m_{ij} + D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij} - D_{ji})^2\end{aligned} \tag{77}$$

which implies (75). The equality case of (75) holds, then

$$m_{11} + D_{11} = m_{22} + D_{22} = ... = m_{nn} + D_{nn}. \tag{78}$$

From (78) we obtain

$$\begin{aligned}
(1-n)m_{11}+m_{22}+...+m_{nn}+(1-n)D_{11}+D_{22}+...+D_{nn} &= 0,\\
m_{11}+(1-n)m_{22}+...+m_{nn}+D_{11}+(1-n)D_{22}+...+D_{nn} &= 0,\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
m_{11}+m_{22}+...+(1-n)m_{nn}+D_{11}+D_{22}+...+(1-n)D_{nn} &= 0.
\end{aligned}$$

Using last equations, we have

$$(n-1)^2(traceA_N+n\mu)=0. \tag{79}$$

Because of $n\neq 1$, we get $n\mu=-traceA_N$. □

Thus we get the following corollary.

Corollary 4.15. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

$$\begin{aligned}
\tau_{S(TM)}(p) &\leq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{(2n-1)}{4}(\varphi+1)^2n\mu^2\\
&\quad -\frac{(\varphi^2+1)}{2}n^2\mu^2+\frac{(1-\varphi)^2}{2}\sum_{i=1}^{n}(m_{ii})^2-\varphi\sum_{i\neq j}^{n}(m_{ij})^2.
\end{aligned} \tag{80}$$

The equality case of (80) satisfies for all $p\in M$ if and only if either $\varphi=-1$ or M is minimal.

Theorem 4.16. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\overline{\nabla}}$. Then we have*

$$\begin{aligned}
\tau_{S(TM)}(p) &\geq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{1}{2}(trace\bar{A})^2\\
&\quad -\frac{1}{2}(traceA_N)^2-\frac{1}{2}n(n-1)\mu^2-\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2\\
&\quad +\frac{1}{2}\sum_{i,j=1}^{n}\left(D_{ji}\right)^2.
\end{aligned} \tag{81}$$

The equality case of (81) satisfies at $p\in M$ if and only if p is a totally umbilical point.

Proof. Using (63) and (66) we get

$$\begin{aligned}\tau_{S(TM)}(p) &= n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{1}{2}\left(\sum_{i,j} m_{ii}+D_{jj}\right)^2 \\ &-\frac{1}{2}\left(\sum_{i} m_{ii}\right)^2-\frac{1}{2}\left(\sum_{j} D_{jj}\right)^2+\frac{1}{2}\sum_{i}(m_{ii})^2 \\ &+\frac{1}{2}\sum_{i\neq j}(m_{ij})^2+\frac{1}{2}\sum_{i,j=1}^{n}(D_{ji})^2-\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2.\end{aligned} \tag{82}$$

Using Lemma 4.2 in (82), we have

$$\begin{aligned}\tau_{S(TM)}(p) &\geq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{1}{2}(trace\bar{A})^2 \\ &-\frac{1}{2}(traceA_N)^2-\frac{1}{2}n^2\mu^2+\frac{1}{2n}\left(\sum_{i} m_{ii}\right)^2 \\ &+\frac{1}{2}\sum_{i\neq j}(m_{ij})^2+\frac{1}{2}\sum_{i,j=1}^{n}(D_{ji})^2-\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2\end{aligned} \tag{83}$$

which implies (81). The equality case of (81) satisfies if and only if

$$m_{11}=...=m_{nn}$$

and the shape operator A^*_ξ take the form as (61), which shows that M is totally umbilical. The proof of the converse part is straightforward. □

Thus we get the following corollary.

Corollary 4.17. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\bar{\nabla}}$. Then we have*

$$\begin{aligned}\tau_{S(TM)}(p) &\geq n(n-1)c-(n-1)\alpha-n(n-1)\lambda\mu+\frac{(2\varphi+1)}{2}n^2\mu^2 \\ &-\frac{1}{2}n(n-1)\mu^2-\frac{(2\varphi+1)}{2}\sum_{i,j=1}^{n}(m_{ij})^2.\end{aligned} \tag{84}$$

The equality case of (84) satisfies at $p\in M$ if and only if p is a totally umbilical point.

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