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TIMELIKE HELICES IN THE SEMI-EUCLIDEAN SPACE E_2^4

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Abstract. In this paper, we define timelike curves in R_2^4 and characterize such curves in terms of Frenet frame. Also, we examine the timelike helices of R_2^4, taking into account their curvatures. In addition, we study timelike slant helices, timelike B_1-slant helices, timelike B_2-slant helices in four dimensional semi-Euclidean space, R_2^4. And then we obtain an approximate solution for the timelike B_1 slant helix with Taylor matrix collocation method.

1. Introduction

The curves are the common denominator of many different vital necessities such as nature, art, technology and science. It is geometrically important to describe the behavior of the curve in the vicinity a point on the curve. For this, we introduce a frame of mutually orthogonal vectors and curvatures. Thanks to these curvatures and frames that are shaped differently in different spaces, the curves become special. A helix in E^3 is a curve whose tangent vector make a constant angle with a fixed direction (the axis of the helix) [18]. On the other hand, the curves are usually expressed in parametric forms, and arc length of the curve is used for the parameter because of its simplicity of expression, but for practical uses the parameter is changed from arc length s to a more manageable variable parameter t which monotonically increases with arc length [8].

There are many studies about the helices we have discussed in this study [14]. Izumiya and Takeuchi gave a characterization of slant helices [9]. Kula and Yaylı investigated spherical images of a slant helix [12]. In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4-space E^4 and it is called B_2-slant helix [15]. In 2009, Gök et al. generalized the slant helices of E^3 to E^n, $n > 3$, which they called them V_n-slant helix and they gave some characterizations of V_n-slant helix in Euclidean n-space E^n [7]. Altunkaya found position vectors of non-null helices in the n-dimensional Minkowski spacetime

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[2]. Kahraman et al. studied quaternionic slant helices which they called semi-real quaternionic B_2-slant helix in E_2^4 [10]. On the other hand, different approximate solution methods based on matrices for differential equations characterizing special curves were presented by Aydın et al [4], [5]. In addition, the issue of investigating the existence of solutions of different types of equations is still up to date [3], [13].

2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in the semi Euclidean space R_2^4 are briefly presented in this section. A more complete elementary information can be found in [16].

The semi-Euclidean space R_2^4 is an Euclidean space provided with standard flat metric given by

$$g = -da_1^2 - da_2^2 + da_3^2 + da_4^2,$$

where (a_1, a_2, a_3, a_4) is a rectangular coordinate system of the R_2^4. A vector w in R_2^4 is called a spacelike, timelike or null (lightlike) if hold $g(w, w) > 0$, $g(w, w) < 0$ or $g(w, w) = 0$ and $w \neq 0$, respectively. The norm of a vector w is given by $\|w\| = \sqrt{|g(w, w)|}$. Therefore, w is a unit vector if $g(w, w) = \pm 1$. Two vectors u and w are said to be orthogonal if $g(u, w) = 0$ [16]. Also, let u and w be two timelike vectors in R_2^4. Then there is unique real number $0 < \theta < 2\pi$, called angel between u and w, such that

i) if u and w are spacelike, $g(u, w) = \|u\| \|w\| \cos\theta$,

ii) if u and w are timelike, $g(u, w) = -\|u\| \|w\| \cosh\theta$,

iii) if u is spacelike and w is timelike, $|g(u, w)| = \|u\| \|w\| \sinh\theta$.

Similarly, an arbitrary curve $\gamma = \gamma(s)$ in R_2^4 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null (lightlike). The velocity of the curve γ is given by $\|\gamma'\|$. Thus, a timelike curve γ is said to be parametrized by arc length function s if $g(\gamma', \gamma') = -1$ [16].

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ denote the moving Frenet frame along γ in the semi-Euclidean space R_2^4. Then $T(s), N(s), B_1(s)$ and $B_2(s)$ are called the tangent, the principal normal, the first binormal, and the second binormal vector fields of γ, respectively.

A unit speed curve γ is said to be a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. Let γ be a C^∞ special timelike Frenet curve with timelike principal normal, spacelike both first binormal and second binormal vector fields in R_2^4, parametrized by arc length function s. Moreover, non-zero C^∞ scalar functions κ_1, κ_2 and κ_3 are the first, second, and third curvatures of γ, respectively. Then for the C^∞

special timelike Frenet curve γ, the Frenet formula is given by

$$\begin{aligned} T' &= -\kappa_1 N \\ N' &= \kappa_1 T + \kappa_2 B_1 \\ B_1' &= \kappa_2 N + \kappa_3 B_2 \\ B_2' &= -\kappa_3 B_1, \end{aligned} \tag{1}$$

where T, N, B_1 and B_2 mutually orthogonal vector fields satisfying

$$g(T,T) = g(N,N) = -1, g(B_1,B_1) = g(B_2,B_2) = 1 \tag{2}$$

(for the semi-Euclidean space E_v^{n+1}, see [19],[6]).

Definition 2.1. *If the tangent vector T of a curve makes a constant angle with a unit vector U of E^4, then this curve is called a general helix (or inclined curve) in E^4* [17].

Definition 2.2. *A Frenet curve of rank d for which $\kappa_1, \kappa_2, \ldots, \kappa_{d-1,}$ are constant is called (generalized) helix or W-curve* [11].

Definition 2.3. *A unit speed curve $\gamma : I \to E^4$ is called slant helix if its unit principal normal vector N makes a constant angle with a fixed direction U* [1].

Definition 2.4. *Let's consider the differential equation below:*

$$\sum_{k=0}^{m} P_k(s) y^{(k)}(s) = g(s), (a \leq s \leq b).$$

Obviously this is a linear differential equation of order m with variable coefficient. Also, the functions are differentiable functions in the range $a \leq s \leq b$. The Taylor matrix collocation method is developed to find approximate solutions of this equation under certain initial or boundary conditions. Accordingly, the approximate solution can be expressed with Taylor polynomials as follows:

$$y(s) \cong y_N(s) = \sum_{n=0}^{N} a_n s^n, (N \geq m),$$

where the coefficients a_n are defined as Taylor coefficients that must be found. The basis of this method is based on the reduction of the unknown function $y(s)$ to an algebraic system with Taylor coefficient a_n. For this reduction process, the matrix form of the function $y(s)$ and the collocation points

$$s_i = a + \frac{b-a}{N} i, (i = 0, 1, ..., N)$$

are used. Thus, the problem of finding the approximate solutions of a given differential equation or other functional equations becomes the problem of finding the solution of an algebraic matrix equation [4].

3. Timelike Curves in R_2^4

In this section, we give definitions and characterizations of the timelike curves by using Frenet frame in R_2^4.

Theorem 3.1. *Let $\gamma : I \to R_2^4$ be a curve parameterized by arclength. Then, the curve γ is the timelike curve if and only if*

$$T^{(4)} + \lambda_3 T^{(3)} + \lambda_2 T'' + \lambda_1 T' + \lambda_0 T = 0. \tag{3}$$

The coefficient functions $\lambda_i(s), (0 \le i \le 3)$ are as follows:

$$\begin{aligned}
\lambda_0 &= \kappa_1\kappa_2\kappa_3 \left[\frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})'\right]' + \kappa_1^2\kappa_3^2,\\
\lambda_1 &= \kappa_1\kappa_2\kappa_3 \left\{\left[\frac{1}{\kappa_2\kappa_3}(\frac{1}{\kappa_1})''\right]' + (\frac{\kappa_1^2-\kappa_2^2}{\kappa_1\kappa_2\kappa_3})'\right\} + \kappa_1\kappa_3^2(\frac{1}{\kappa_1})' + \kappa_1\kappa_2(\frac{\kappa_1}{\kappa_2})',\\
\lambda_2 &= \kappa_1\kappa_2\kappa_3 \left\{\left[\frac{1}{\kappa_3}(\frac{1}{\kappa_1\kappa_2})'\right]' + \left[\frac{1}{\kappa_2\kappa_3}(\frac{1}{\kappa_1})'\right]'\right\} + \kappa_1(\frac{1}{\kappa_1})'' + \kappa_1^2 - \kappa_2^2 + \kappa_3^2,\\
\lambda_3 &= \kappa_1\kappa_2\kappa_3(\frac{1}{\kappa_1\kappa_2\kappa_3})' + \kappa_1\kappa_2(\frac{1}{\kappa_1\kappa_2})' + \kappa_1(\frac{1}{\kappa_1})'.
\end{aligned} \tag{4}$$

Proof. By using the equations 1 we have

$$\begin{aligned}
N &= -\frac{1}{\kappa_1}T',\\
B_1 &= \frac{1}{\kappa_2}N' - \frac{\kappa_1}{\kappa_2}T,\\
B_2 &= \frac{1}{\kappa_3}B_1' - \frac{\kappa_2}{\kappa_3}N.
\end{aligned} \tag{5}$$

From the first of the equations 5 $N' = -\frac{1}{\kappa_1}T'' - (\frac{1}{\kappa_1})'T'$, and so we get

$$B_1 = -\frac{1}{\kappa_1\kappa_2}T'' - \frac{1}{\kappa_2}(\frac{1}{\kappa_1})'T' - \frac{\kappa_1}{\kappa_2}T. \tag{6}$$

And then we calculate B_1'. With a similar thinking, by using the equations we found, we get B_2 and B_2'. Finally, we use the equality 6 and the expression B_2' in the last equality of Frenet equations 1. Thus the proof is complete. □

Corollary 3.2. *The equation 3 is the differential equation characterizing the timelike curves according to the tangent T field in R_2^4. Similarly, the timelike curves can be characterized according to the principal normal N, the first binormal B_1 and the second binormal B_2 fields.*

3.1. Timelike Helix in R_2^4

Theorem 3.3. *Let $\gamma = \gamma(s) : I \subset R \to R_2^4$ be a regular timelike curve parametrized by arc length s. Then γ is a timelike W- curve or timelike helix if and only if the equality*

$$T^{(4)} + (\kappa_1^2 - \kappa_2^2 + \kappa_3^2)T^{(2)} + (\kappa_1^2\kappa_3^2)T = 0$$

holds.

Proof. A timelike curve $\gamma : I \to R_2^4$ parameterized by arc length provides the differential equation 3 in R_2^4. Since the curve γ is (generalized) helix or W-curve for which $\kappa_1, \kappa_2, \kappa_3$ are constant, with the help of the equations 4 the

equalities

$$\begin{aligned} \lambda_0 &= \kappa_1^2\kappa_3^2, \\ \lambda_2 &= \kappa_1^2 - \kappa_2^2 + \kappa_3^2 \end{aligned}$$

and $\lambda_1 = \lambda_3 = 0$ are obtained. □

3.2. Timelike Slant Helix in R_2^4

Theorem 3.4. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given with arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike slant helix, then their position vector satisfies the equation*

(7) $$\frac{\kappa_1^2 - \kappa_2^2}{\kappa_1\kappa_2\kappa_3}\mu_1'' + [(\frac{\kappa_1}{\kappa_2\kappa_3})' - (\frac{\kappa_2}{\kappa_1\kappa_3})' + \frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})']\mu_1' + [(\frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})')' + \frac{\kappa_1\kappa_3}{\kappa_2}]\mu_1 = 0,$$

where μ_1 is the coefficient function of the tangent of a constant vector taken in the fixed direction studied.

Proof. We call γ as timelike slant helix if its principial normal vector makes a constant angle with a fixed direction. From this definition of the slant helix we can write

(8) $$g(N, U) = -\cosh\theta,$$

where U is a timelike constant vector and we can compose U as

(9) $$U = \mu_1 T + \mu_2 N + \mu_3 B_1 + \mu_4 B_2.$$

The coefficient functions are

$$\mu_1 = -g(T, U), \mu_2 = -g(N, U), \mu_3 = g(B_1, U), \mu_4 = g(B_2, U).$$

Because the vector U is constant, by differentiating the equation 9 and considering Frenet equations we have

(10)
$$(\mu_1' + \kappa_1\mu_2)T + (-\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3)N + (\kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4)B_1 + (\kappa_3\mu_3 + \mu_4')B_2 = 0.$$

Also, the function μ_2 is constant with value $\cosh\theta$ from the equality 8, and so $\mu_2'(s)=0$ for all s. Then we find the following system of ordinary differential equations

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \kappa_2\mu_3 &= 0, \\ \kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4 &= 0, \\ \kappa_3\mu_3 + \mu_4' &= 0. \end{aligned} \tag{11}$$

From the third equation of the equation system 11 $\mu_4 = \frac{\kappa_2}{\kappa_3}\mu_2 + \frac{1}{\kappa_3}\mu_3'$, and so we get

$$[\frac{\kappa_2}{\kappa_3}\mu_2 + \frac{1}{\kappa_3}\mu_3']' = -\kappa_3\mu_3. \tag{12}$$

By using the equalities $\mu_2 = -\frac{1}{\kappa_1}\mu_1'$ and $\mu_3 = \frac{\kappa_1}{\kappa_2}\mu_1$ in the equation 12, we obtain the equation 7 . Thus the proof is complete. □

Corollary 3.5. *The equation 7 is the differential equation characterizing the timelike slant helix according to the coefficient function μ_1 in R_2^4. Obviously, the timelike slant helix can be characterized similarly according to the other coefficient functions μ_3 and μ_4,but, since μ_2 is already fixed, a characterization based on μ_2 cannot be given.*

Theorem 3.6. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given by arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike slant helix, then their position vector satisfies the equations*

$$\begin{aligned} \frac{\kappa_1^2-\kappa_2^2}{\kappa_1^2\kappa_3}\mu_3'' + [(\frac{\kappa_1^2-\kappa_2^2}{\kappa_1^2\kappa_3})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1})']\mu_3' + \{\kappa_3 - [\frac{\kappa_2}{\kappa_3\kappa_1}(\frac{\kappa_2}{\kappa_1})']'\}\mu_3 &= 0, \\ \frac{\kappa_1^2-\kappa_2^2}{\kappa_1\kappa_2\kappa_3}\mu_4'' + [\frac{\kappa_1}{\kappa_2}(\frac{1}{\kappa_3})' - (\frac{\kappa_2}{\kappa_1\kappa_3})']\mu_4' + \frac{\kappa_1\kappa_3}{\kappa_2}\mu_4 &= 0, \end{aligned}$$

where μ_3 and μ_4 are the coefficient functions of the first binormal B_1 and the second binormal B_2, respectively, of a timelike constant vector taken in the fixed direction studied.

Proof. It is obvious from proof of Theorem 3.4. □

3.3. Timelike B_1-Slant Helix in R_2^4

Theorem 3.7. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given by arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike B_1 slant helix, then their position vector satisfies the equation*

$$\frac{\kappa_3^2-\kappa_2^2}{\kappa_1^2\kappa_3^2}\mu_1'' + [\frac{1}{\kappa_1}(\frac{1}{\kappa_1})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1\kappa_3})']\mu_1' + \mu_1 = 0, \tag{13}$$

where μ_1 is the coefficient function of the tangent of a constant vector taken in the fixed direction studied.

Proof. We call γ as B_1 slant helix if its first binormal vector makes a constant angle with a fixed direction. From this definition of the B_1 slant helix, we can write

$$|g(B_1, U)| = \sinh\theta, \tag{14}$$

where U is a timelike constant vector and we can compose U as

$$U = \mu_1 T + \mu_2 N + \mu_3 B_1 + \mu_4 B_2. \tag{15}$$

The coefficient functions are

$$\mu_1 = -g(T, U), \mu_2 = -g(N, U), \mu_3 = g(B_1, U), \mu_4 = g(B_2, U)$$

in R_2^4. Because the vector U is constant, differentiation of the equation 15 and considering Frenet equations, we have

$$(\mu_1' + \kappa_1\mu_2)T + (-\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3)N + (\kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4)B_1 + (\kappa_3\mu_3 + \mu_4')B_2 = 0. \tag{16}$$

Also, the function μ_3 is constant with value $\sinh\theta$ from the equality 14, and so $\mu_3'(s) = 0$ for all s. Then we find the following system of ordinary differential equations

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3 &= 0, \\ \kappa_2\mu_2 - \kappa_3\mu_4 &= 0, \\ \kappa_3\mu_3 + \mu_4' &= 0. \end{aligned} \tag{17}$$

From the second equation of this equation system, the equality

$$\mu_3 = \frac{\kappa_1}{\kappa_2}\mu_1 - \frac{1}{\kappa_2}\mu_2' \tag{18}$$

is obtained. By using the equalities $\mu_2 = -\frac{1}{\kappa_1}\mu_1'$, $\mu_3 = -\frac{1}{\kappa_3}\mu_4'$ and $\mu_4 = \frac{\kappa_2}{\kappa_3}\mu_2$ in the equation 18, we obtain the equation 13. Thus the proof is complete. □

Corollary 3.8. *The equation 13 is the differential equation characterizing the timelike B_1 slant helix according to the coefficient function μ_1 in R_2^4. Obviously, the timelike B_1 slant helix can be characterized similarly according to the other coefficient functions μ_2 and μ_4,but, since μ_3 is already fixed, a characterization based on μ_3 cannot be given.*

Theorem 3.9. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given by arclength parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike B_1 slant helix,* *then their position vector satisfies the equations*

$$\frac{\kappa_3^2-\kappa_2^2}{\kappa_1\kappa_3^2}\mu_2''+[-(\frac{\kappa_2^2}{\kappa_1\kappa_3^2})'-\frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_3})'+(\frac{1}{\kappa_1})']\mu_2'-\{\left[\frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_3})'\right]'+\kappa_1\}\mu_2 = 0,$$
$$\frac{\kappa_3^2-\kappa_2^2}{\kappa_1\kappa_2\kappa_3}\mu_4''+[(\frac{\kappa_3^2-\kappa_2^2}{\kappa_1\kappa_2\kappa_3})'+\frac{1}{\kappa_1}(\frac{\kappa_3}{\kappa_2})']\mu_4'+\{\left[\frac{1}{\kappa_1}(\frac{\kappa_3}{\kappa_2})'\right]'+\frac{\kappa_1\kappa_3}{\kappa_2}\}\mu_4 = 0,$$

where μ_2 and μ_4 are the coefficient functions of the principal normal N and the second binormal B_2, respectively, of a timelike constant vector taken in the fixed direction studied.

Proof. It is obvious from proof of Theorem 3.7. □

3.4. Timelike B_2 Slant Helix in R_2^4

Theorem 3.10. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given by arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike B_2 slant helix,*

i) and if $\kappa_3 = 0, \mu_3 \neq 0$ their position vector satisfies the equation

(19)
$$\mu_1'''+[\kappa_1\kappa_2(\frac{1}{\kappa_1\kappa_2})'+\kappa_1(\frac{1}{\kappa_1})']\mu_1''+\{\kappa_1\kappa_2[\frac{1}{\kappa_2}(\frac{1}{\kappa_1})']'+\kappa_1^2-\kappa_2^2\}\mu_1'+\kappa_1\kappa_2(\frac{\kappa_1}{\kappa_2})'\mu_1 = 0$$

and the curve γ lies in a three-dimensional subspace of the R_2^4, where μ_1 is the coefficient function of the tangent of a constant vector taken in the fixed direction studied.

ii) and if $\kappa_3 \neq 0, \mu_3 = 0$, the coefficient functions of the tangent of the timelike constant vector U satisfy the equalities

$$\begin{aligned} \mu_4 &= c, \\ \mu_1 &= \frac{1}{\kappa_1}(\frac{\kappa_3}{\kappa_2})'c, \\ \mu_2 &= \frac{\kappa_3}{\kappa_2}c, \end{aligned} \tag{20}$$

where c is constant.

iii) and if $\kappa_3 = 0, \mu_3 = 0$, their position vector satisfy the equation

$$\mu_1''+\kappa_1(\frac{1}{\kappa_1})'\mu_1'+\kappa_1^2\mu_1 = 0, \tag{21}$$

for $\mu_2 \neq 0$, and the curve γ is planar. Also, the timelike constant vector U is in the subspace of R_2^4.

Proof. We call γ as B_2 slant helix if its first binormal vector makes a constant angle with a fixed direction. From this definition of the B_2 slant helix, we can write

$$|g(B_2, U)| = \sinh\theta, \tag{22}$$

where U is a timelike constant vector and we can compose U as

$$U = \mu_1 T + \mu_2 N + \mu_3 B_1 + \mu_4 B_2. \tag{23}$$

The coefficient functions are

$$\mu_1 = -g(T, U), \mu_2 = -g(N, U), \mu_3 = g(B_1, U), \mu_4 = g(B_2, U)$$

in R_2^4. Because the vector U is constant, by differentiating of the equation 23 and considering Frenet equations we have

$$(\mu_1' + \kappa_1\mu_2)T + (-\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3)N + (\kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4)B_1 + (\kappa_3\mu_3 + \mu_4')B_2 = 0. \tag{24}$$

Also, the function μ_4 is constant with value $\sinh\theta$ from the equality 22, and so $\mu_4'(s) = 0$ for all s. Then, we find the following system of ordinary differential equations

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3 &= 0, \\ \kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4 &= 0, \\ \kappa_3\mu_3 &= 0, \end{aligned} \tag{25}$$

where i) if $\kappa_3 = 0, \mu_3 \neq 0$, then the equation system 25 is obtained as

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3 &= 0, \\ \kappa_2\mu_2 + \mu_3' &= 0. \end{aligned} \tag{26}$$

We obtain the equation 19, using the equalities $\mu_2 = -\frac{1}{\kappa_1}\mu_1'$ and $\mu_3 = \frac{\kappa_1}{\kappa_2}\mu_1 - \frac{1}{\kappa_2}\mu_2'$, in the last equation of this system. Since $\kappa_3 = 0$, the curve γ lies in a three-dimensional subspace of R_2^4.

ii) if $\kappa_3 \neq 0, \mu_3 = 0$, then the equation system 25 is obtained as

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \mu_2' &= 0, \\ \kappa_2\mu_2 - \kappa_3\mu_4 &= 0. \end{aligned}$$

We obtain the equalities 20, using the equality $\mu_4 = c$ in this system.

iii) if $\kappa_3 = 0, \mu_3 = 0$, then the equation system 25 is obtained as

$$\begin{aligned} \mu_1' + \kappa_1\mu_2 &= 0, \\ -\kappa_1\mu_1 + \mu_2' &= 0, \\ \kappa_2\mu_2 &= 0. \end{aligned} \tag{27}$$

We obtain $\kappa_2 = 0$ f*or* $\mu_2 \neq 0$, and so the curve γ is planar. Also, we get the equality 21 from the equation system 27. □

Corollary 3.11. *The equation 19 is the differential equation characterizing the timelike B_2 slant helix according to the coefficient function μ_1 in R_2^4. Obviously, the timelike B_2 slant helix can be characterized similarly according to the other coefficient functions μ_2 and μ_3,but, since μ_4 is already fixed, a characterization based on μ_4 cannot be given.*

Theorem 3.12. *Let $\gamma : I \to R_2^4$ be a regular timelike curve given by arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a timelike B_2 slant helix and if $\kappa_3 = 0, \mu_3 \neq 0$, then their position vector satisfies the equations*

$$\frac{\frac{1}{\kappa_1}}{(\frac{\kappa_2}{\kappa_1})'}\mu_2''' + [(\frac{\frac{1}{\kappa_1}}{(\frac{\kappa_2}{\kappa_1})'})' + \frac{(\frac{1}{\kappa_1})'}{(\frac{\kappa_2}{\kappa_1})'}]\mu_2'' + [(\frac{(\frac{1}{\kappa_1})'}{(\frac{\kappa_2}{\kappa_1})'})' + \frac{\frac{\kappa_1^2-\kappa_2^2}{\kappa_1}}{(\frac{\kappa_2}{\kappa_1})'}]\mu_2' +$$
$$[(\frac{\frac{\kappa_1^2-\kappa_2^2}{\kappa_1}}{(\frac{\kappa_2}{\kappa_1})'})' - \kappa_2]\mu_2 = 0,$$
$$\mu_3''' + [\kappa_1\kappa_2(\frac{1}{\kappa_1\kappa_2})' + \kappa_2(\frac{1}{\kappa_2})']\mu_3'' + \{\kappa_1\kappa_2[\frac{1}{\kappa_1}(\frac{1}{\kappa_2})']' - \kappa_1^2 - \kappa_2^2\}\mu_3' -$$
$$\kappa_1\kappa_2(\frac{\kappa_2}{\kappa_1})'\mu_3 = 0,$$

and the curve γ lies in a three-dimensional subspace of the R_2^4, where μ_2 and μ_3 are the coefficient functions of the principal normal N and the first binormal B_1, respectively, of a timelike constant vector taken in the fixed direction studied.

Proof. It is obvious from proof of Theorem 3.10. □

4. Approximate solution with Taylor polynomial approach

Firstly, the differential equation 13 is generally expressed as follows: In this section, the approximate solution of the differential equation 13 that characterizes the timelike B_1 slant helix based on the coefficient μ_1, will be obtained by the Taylor matrix collocation method [4]. A similar solution can be applied for the characterizations linked to the coefficients μ_2 and μ_4. Also, since the solution method is valid for all linear differential equations with variable coefficient, it is obviously applicable to all characterizations of the timelike slant helix and the timelike B_2 slant helix.

Firstly, the differential equation 13 is generally expressed as follows:

$$\sum_{k=0}^{2} P_k(s)y^{(k)}(s) = g(s), \tag{28}$$

for the coefficient functions

$$\begin{aligned} P_2(s) &= \frac{\kappa_3^2-\kappa_2^2}{\kappa_1^2\kappa_3^2}, \\ P_1(s) &= \frac{1}{\kappa_1}(\frac{1}{\kappa_1})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1\kappa_3})', \\ P_0(s) &= 1,\ y(s)=\mu_1(s), g(s)=0. \end{aligned}$$

Suppose that this equation has an approximate solution in $(0 \leq s \leq 1)$, under the initial conditions $y^{(k)}(0)=\omega_k$, $(k=0,1)$, in the form of Taylor polynomials as

$$y(s)=\sum_{n=0}^{N} a_n s^n(s). \tag{29}$$

Let $N=3$ for convenience. Here, P_k and g functions are known functions and ω is suitable constant, a_n are unknown coefficients, $y(s)$ can be expanded to Taylor series about $s=0$ in the form, for $N \geq 3$.

4.1. Basic matrix relations

First of all, the approximate solution can be converted into matrix form $y(s)=S(s)A$ with

$$S(s)=\begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix}, A=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$$

for $N=3$. Also, it is clearly seen that the relation between the matrix $S(s)$ and its derivative $S'(s)$ is $S'(s)=S(s)B$ and that repeating the process $S^{(k)}(s)=S(s)B^k$, where

$$B=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and B^0 is unite matrix. $y^{(k)}(s)=S(s)B^kA$ are obtained with the help of these matrices. Also, the matrix relations of the differential part are obtained in the form $\sum\limits_{k=0}^{2} P_kY^{(k)}=G$ by using standard collocation points $s_i=\frac{1}{3}i$ $(i=0,1,2,3)$, in the equation 28, in the range of $0 \leq s \leq 1$ for $N=3$. The matrices

$$\begin{aligned} P_k &= diag[\ P_k(0)\quad P_k(\tfrac{1}{3})\quad P_k(\tfrac{2}{3})\quad P_k(1)\ ], \\ Y^{(k)} &= [\ y^{(k)}(0)\quad y^{(k)}(\tfrac{1}{3})\quad y^{(k)}(\tfrac{2}{3})\quad y^{(k)}(1)\ ]^T \end{aligned}$$

are obvious and the matrix $W=\sum\limits_{k=0}^{2} P_kS(s)B^k$ is calculated, for $WA=G$ and the equation is written as the augmented matrix $[W;G]$.

4.2. Matrix calculations for initial conditions

Under the initial conditions given as $y(0) = 0$, $y'(0) = 1$ the matrix expression of the conditions is calculated as

$$U_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

4.3. The Solution

If the matrix form of conditions is used in the matrix form $[W; G]$ the following matrix is obtained:

$$[W^*; G^*] = \begin{bmatrix} P_0(0) & P_1(0) & 2P_2(0) & 0 & ; & 0 \\ 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 1 & 0 & 0 & ; & 1 \\ P_0(1) & \zeta_{31} & \zeta_{32} & \zeta_{33} & ; & 0 \end{bmatrix},$$

where

$$\begin{aligned} \zeta_{31} &= P_0(1) + P_1(1), \\ \zeta_{32} &= P_0(1) + 2P_1(1) + 2P_2(1), \\ \zeta_{33} &= P_0(1) + 3P_1(1) + 6P_2(1). \end{aligned}$$

Finally, with the help of equality $A = (W^*)^{-1}G$, the unknowns a_n are calculated as follows:

$$\begin{aligned} a_0 &= 0, a_1 = 1, \\ a_2 &= -\frac{P_1(0)}{2P_2(0)}, \\ a_3 &= \frac{P_1(0)[P_0(1) + 2P_1(1) + 2P_2(1)] - 2P_2(0)[P_0(1) + P_1(1)]}{2P_2(0)[P_0(1) + 3P_1(1) + 6P_2(1)]}. \end{aligned}$$

If these values are substituted in the equation 29, the solution is obtained as follows:

$$y(s) = \mu_1(s) = s + a_2 s^2 + a_3 s^3.$$

Corollary 4.1. *The equations found for the special curves we study are generally homogeneous, linear differential equations with variable coefficients. So this solution method we present can be applied to other equations as well.*

Example 4.2. *Let's find the coefficient μ_1 for the timelike semi-B_1 slant helix given with its curvatures $\kappa_1 = \frac{1}{s+1}, \frac{\kappa_2}{\kappa_3} = \sin s$. The vector position of such a curve provides the following differential equation*

$$[(1+s)\cos s]^2\mu_1'' + \{[\cos s - (1+s)\sin s](1+s)\cos s\}\mu_1' + \mu_1 = 0.$$

If the method presented is applied for

$$\begin{aligned}
&P_2(s) = [(1+s)\cos s]^2, \\
&P_1(s) = [\cos s - (1+s)\sin s](1+s)\cos s, \\
&P_0(s) = 1, y(s) = \mu_1(s),\ g = 0,
\end{aligned}$$

the approximate solution is calculated under the initial conditions given as $y(0) = 0,\ y'(0) = 1$, in the range of $0 \leq s \leq 1$ for $N = 3$. Firstly, from the matrix

$$(W^*)^{-1} = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0.5 & -0.5 & -0.5 & 0 \\ -0.100\,65 & -0.131\,79 & 0.155\,20 & 0.232\,44 \end{bmatrix}$$

and $A = (W^*)^{-1}G$, the unknowns a_n are calculated as follows:

$$\begin{aligned}
a_0 &= 0, \\
a_1 &= 1.0, \\
a_2 &= -0.5, \\
a_3 &= 0.155\,20.
\end{aligned}$$

Thus, the solution is obtained as

$$\mu(s) = 0.155\,20s^3 - 0.5s^2 + s.$$

5. Conclusion

In this study, the characterizations are given for the timelike curves according to the Frenet frame in R_2^4. In addition, the timelike slant helix, the timelike B_1 slant helix and the timelike B_2 slant helix concepts are defined in R_2^4 and the differential equations for vector positions are presented. These equations are homogeneous, linear, differential equations with variable coefficients. The Taylor matrix collocation method is given for the approximate solution of such differential equations. This method is applied in the differential equation that characterizes the timelike B_1 slant helix. And an example is presented.

References

[1] T. A. Ahmad and R. Lopez, *Slant helices in Euclidean 4-space* E^4, preprint (2009); arXiv:0901.3324.

[2] B. Altunkaya, *Helices in the n-dimensional Minkowski spacetime*, Results in Phy. **14** (2019), 102445.

[3] R. Ayazoğlu, S. Ş. Şener, and T. A. Aydın, *Existence of solutions for a resonant problem under Landesman-Lazer type conditions involving more general elliptic operators in divergence form*, Trans. Nat. Aca. Sci. Azer. Ser. Phys.-tech. Math. Sci. **40** (2020), 1–14.

[4] T. A. Aydın and M. Sezer, *Taylor-Matrix Collocation Method to Solution of Differential Equations Characterizing Spherical Curves in Euclidean 4-Space*, Celal Bayar Uni. J. Sci. **15** (2019), 1–7.

[5] T. A. Aydın, M. Sezer, and H. Kocayiğit, *An Approximate Solution of Equations Characterizing Spacelike Curves of Constant Breadth in Minkowski 3-Space*, New Trend in Math. Sci. **6** (2018), 182–195.

[6] A. C. Çöken and A. Görgülü, *On Joachimsthal's theorems in semi-Euclidean spaces*, Nonlinear Analysis: Theory, Meth. App. **70** (2009), 3932–3942.

[7] I. Gök, Ç. Camcı, and H. H. Hacısalihoğlu, V_n*-slant helices in Euclidean n-space* E^n, Math. Commun. **14** (2009), 317–329.

[8] M. Hosaka, *Theory of Curves. Modeling of Curves and Surfaces in CAD/CAM.* Computer Graphics-Systems and Applications, Springer, Berlin, Heidelberg, 1992.

[9] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28** (2004), 153–163.

[10] F. Kahraman, I. Gök, and H. H. Hacisalihoğlu, *On the quaternionic* B_2 *slant helices in the semi-Euclidean space* E_2^4, App. Math. Comp. **218** (2012), 6391–6400.

[11] F. Klein and S. Lie, *Uber diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen transformationen in sich übergehen*, Math. Ann. **4** (1871), 50–84.

[12] L. Kula and Y. Yaylı, *On slant helix and its spherical indicatrix*, Appl. Math. Comput. **169** (2005), 600–607.

[13] R. A. Mashiyev, *Three Solutions to a Neumann Problem for Elliptic Equations with Variable Exponent*, Arab. J. Sci. Eng. **36** (2011), 1559–1567.

[14] J. Monterde, *Curves With Constant Curvature Ratios*, Bol. Soc. Mat. Mex. **13** (2007), 177–186.

[15] M. Önder, M. Kazaz, H. Kocayiğit, and O. Kılıç, B_2*-slant helix in Euclidean 4-space* E^4, Int. J. Cont. Math. Sci. **3** (2008), 1433–1440.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Inc., London, 1983.

[17] G. Ozturk, K. Arslan, and H. H. Hacisalihoglu, *A characterization of ccr-curves in* R^m, Proc. Estonian Acad. Sci. **57** (2008), 217–224.

[18] V. Rovenski, *Geometry of curves and surfaces with maple*, Birkhauser, London, 2000.

[19] E. Soley and M. Tosun, *Timelike Bertrand Curves in Semi-Euclidean Space*, Int. J. Math. Stat. **14** (2013), 78–89.

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