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BIFURCATIONS OF STOCHASTIC IZHIKEVICH-FITZHUGH MODEL

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Abstract. Noise is a fundamental factor to increased validity and regularity of spike propagation and neuronal firing in the nervous system. In this paper, we examine the stochastic version of the Izhikevich-FitzHugh neuron dynamical model. This approach is based on techniques presented by Luo and Guo, which provide a general framework for the bifurcation and stability analysis of two dimensional stochastic dynamical system as an Itô averaging diffusion system. By using largest lyapunov exponent, local and global stability of the stochastic system at the equilibrium point are investigated. We focus on the two kinds of stochastic bifurcations: the P-bifurcation and the D-bifurcations. By use of polar coordinate, Taylor expansion and stochastic averaging method, it is shown that there exists choices of diffusion and drift parameters such that these bifurcations occurs. Finally, numerical simulations in various viewpoints, including phase portrait, evolution in time and probability density, are presented to show the effects of the diffusion and drift coefficients that illustrate our theoretical results.

1. Introduction

Usually it is necessary to study a natural phenomena as dynamic systems subjected to stochastic excitations due to the effects of many unknown factors. An example of such systems is neuroscience dynamical models. The field of dynamical systems in neuroscience started with the 1952 paper of Hodgkin and Huxley in which they introduce a nonlinear partial differential equation that describe the genesis of the action potential in the giant axon of the squid [12]. Then a large number of dynamical models have been produced to explain the mechanism of neurons in dynamical viewpoint, for example [5, 10, 13, 19, 20]. Since dynamical behavior of a neural circuit depends on many factors, various researchers tries to describe the action potential of nerves as a stochastic process[6, 15]. In [15] the authors consider experimental measurements of

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synaptic noise and its stochastic processes modeling. Then, review the outcomes of synaptic noise on neuronal integrative properties investigated experimentally using the dynamic-clamp. Yamakou et. al. introduce a stochastic FitzHugh-Nagumo neuron model and prove that Under specific types of noises, there exists a global random attractor for this stochastic system [21]. The stability and bifurcations of such stochastic systems have been of increasing interest to researchers during recent years[4, 7, 9, 11, 16, 17, 18]. Hence, studying the stability and bifurcation in a stochastic dynamical model of neurons action is an interesting topic for many researchers and practitioners in dynamical systems viewpoint.

Specially, Luo and Guo [17] investigated the stability and bifurcation of a two-dimensional stochastic differential equations with multiplicative excitations. They provided some conditions on drift and diffusion coefficient of a two-dimensional nonlinear stochastic system to obtain P-bifurcation and D-bifurcation.

In this paper, we consider a stochastic Izhikevich-FitzHugh dynamical model with multiplicative excitations and proceed to study its stability and bifurcation. Firstly, we present an overview of dynamical behaviour in two-dimensional stochastic systems with multiplicative excitations, that provided by Luo and Guo in [17]. Specially, this section focused on sufficient conditions on drift and diffusion coefficients for stability and P-bifurcation in two dimensional stochastic dynamical systems. Then, we consider deterministic Izhikevich-FitzHugh model that we are going to study its stochastic dynamic. The main part of this paper begins with Section 3 that is devoted to stochastic Izhikevich-FitzHugh model with multiplicative excitations. In Section 4 largest Lyapunov exponent and D-bifurcation for our stochastic model are considered. In Section 5 we consider several conditions on diffusion and drift coefficients that the model undergoes P-bifurcation. Finally using Euler-Maruyama method, we demonstrate some numerical simulation to validate the result.

2. Preliminaries

In this section, we present some preliminaries concepts and definitions that will be used in the sequel. Consider the two-dimensional stochastic differential system with multiplicative excitations

$$\left\{ \begin{array}{l} dx = f_1(x,y)dt + g_1(x,y)dW_1(t), \\ dy = f_2(x,y)dt + g_2(x,y)dW_2(t), \end{array} \right. \tag{1}$$

where $f_i \in C^3(R\times R, R)$, $g_i \in C^1(R\times R, R)$ $(i = 1, 2)$ and $dW_i(t)$ $(i = 1, 2)$ are mutually independent standard real-valued Wiener processes on the complete probability space (Ω, F, P). Most interesting is the problem on conditions for the asymptotic stability with probability of solutions of system 1 , i.e., on conditions for the relation $P\{\lim_{t\to\infty} X^x(t) = 0\} = 1$ to be satisfied for all x, where, $X(t)$ is a solution of 1 satisfying the initial condition $X^x(0) = x$.

Definition 2.1. *$a-$ The equilibrium position of equation 1 is said to be stochastically stable or stable in probability if for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r, t_0) > 0$ such that*

$$P\{|x(t;t_0,x_0)| < r \ for\ all\ t \geq t_0\} \geq 1-\epsilon,$$

whenever $\|x_0\| < \delta_0$. Otherwise, it is said to be stochastically unstable.
$b-$ The equilibrium position is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\epsilon, t_0) > 0$ such that

$$P\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} \geq 1-\epsilon,$$

whenever $\|x_0\| < \delta_0$. The equilibrium position is said to be global stochastically asymptotically stable if it is stochastically stable and, moreover, for all $x_0 \in R^d$

$$P\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} = 1.$$

In the sequel, using Taylor expansion, polar coordinate transformation and stochastic averaging method, a general framework for the stability and bifurcation analysis of the stochastic system 1 is provided [17]. Suppose that $f_i(0,0) = 0$ and $g_i(0,0) = 0$ $(i = 1,2)$.
If in the Taylor expansion of f_i and g_i at the point $O(0,0)$ we ignore the terms higher than third order and rescaling the system as presented in [17], then the following system obtained:

$$(2)\qquad \begin{cases} dx = \epsilon[c_{110}x + c_{101}y + c_{120}x^2 + c_{111}xy + c_{102}y^2 + c_{130}x^3 + c_{121}x^2y \\ +c_{112}xy^2 + c_{103}y^3]dt + \sqrt{\epsilon}[k_{110}x + k_{101}y]dW_1(t), \\ dy = \epsilon[c_{210}x + c_{201}y + c_{220}x^2 + c_{211}xy + c_{202}y^2 + c_{230}x^3 + c_{221}x^2y \\ +c_{212}xy^2 + c_{203}y^3]dt + \sqrt{\epsilon}[k_{210}x + k_{201}y]dW_2(t), \end{cases}$$

In [17] the authors by combining polar coordinate transformation rewrote system 2 to Itô stochastic differential equations

$$(3)\qquad \begin{cases} dr = [(\omega_1 + \frac{1}{16}\omega_2)r + \frac{1}{8}\omega_3 r^3]dt + (\frac{\omega_4}{8}r^2)^{\frac{1}{2}} dW_r(t), \\ d\theta = [\frac{1}{4}\omega_5 + \frac{1}{8}\omega_6 r^2]dt + (\frac{\omega_2}{8})^{\frac{1}{2}} dW_\theta(t), \end{cases}$$

with the following notations:

$$(4)\qquad \begin{cases} \omega_1 = \frac{1}{2}(c_{110} + c_{201}), \\ \omega_2 = k_{110}^2 + b_{201}^2 + b_{101}^2 + 3b_{210}^2, \\ \omega_3 = 3c_{130} + c_{112} + c_{221} + c_{203}, \\ \omega_4 = 3k_{110}^2 + b_{101}^2 + b_{210}^2 + b_{201}^2, \\ \omega_5 = -2c_{101} + 2c_{210} + k_{110}k_{101} - k_{210}k_{201}, \\ \omega_6 = -c_{103} + c_{212} - c_{121} + 3c_{230}. \end{cases}$$

Taking account of the existence of random factors, we assume that ω_2 and ω_4 are positive numbers, in the sequel. Since the modulus equation is uncoupled with the phase equation, we only need the averaging modulus equation:

$$dr = [(\omega_1 + \frac{1}{16}\omega_2)r + \frac{1}{8}\omega_3 r^3]dt + (\frac{\omega_4}{8}r^2)^{\frac{1}{2}} dW_r(t), \tag{5}$$

to investigate the stability and bifurcation of system 7. Now, we wish to describe stability and bifurcation of Equation 5 at the equilibrium point $r = 0$. In the following theorems, by using Equation 5, we investigate the stability conditions of the equilibrium point of System 7.

Theorem 2.2. [17] (i) *When* $\omega_1 + \frac{1}{16}\omega_2 - \frac{1}{16}\omega_4 < 0$, *the trivial solution of the linear Itô stochastic differential Equation 5 is asymptotically stable with probability* 1, *which implies stability of the stochastic system 7 at the equilibrium point* O.
(ii) *When* $\omega_1 + \frac{1}{16}\omega_2 - \frac{1}{16}\omega_4 > 0$, *the trivial solution of the linear Itô stochastic differential Equation 5 is unstable with probability* 1, *consequently the stochastic system 7 is unstable at the equilibrium point* O.

Theorem 2.3. [17] *When* $16\omega_1 + \omega_2 - \omega_4 < 0$ *and* $2\omega_3 < \omega_4$, *the stochastic system 7 is globally stable at the equilibrium point* O.

According to the above theorems, when parameters change, the qualitative behaviour of the system may be changed. Thus, one can expect the bifurcation in the systems. The next two theorems, investigate some conditions that system 5 undergoes stochastic phenomenological bifurcation or P-bifurcation.

Theorem 2.4. [17] *If* $\omega_3 < 0$ *and* $\omega_4 > 0$, *system 5 undergoes stochastic phenomenological bifurcations as the parameter* ω_4 *passes through the values of* $8\omega_1 + \frac{1}{2}\omega_2$ *and* $\frac{16\omega_1+\omega_2}{3}$.

Theorem 2.5. [17] *If* $\omega_3 < 0$ *and* $\omega_4 > 0$, *the stochastic system 7 undergoes phenomenological bifurcations as the parameter* ω_4 *passes through the values of* $\frac{16\omega_1+\omega_2}{3}$ *and* $\frac{16\omega_1+\omega_2}{4}$.

2.1. Deterministic Izhikevich-FitzHugh Model

Here, we provide a brief description of the deterministic Izhikevich-FitzHugh model[13].

$$\begin{cases} \dot{u} = v(\alpha - u)(u-1) - u + I, \\ \dot{v} = \beta u - \gamma v, \end{cases} \tag{6}$$

Here u mimics the membrane voltage and recovery variable v mimics activation of an outward current. Parameter I mimics the injected current, and for the sake of simplicity we set $I = 0$ in our analysis below. Parameter α describes the shape of the cubic parabola $u(\alpha - u)(u - 1)$, and parameters $\beta > 0$ and

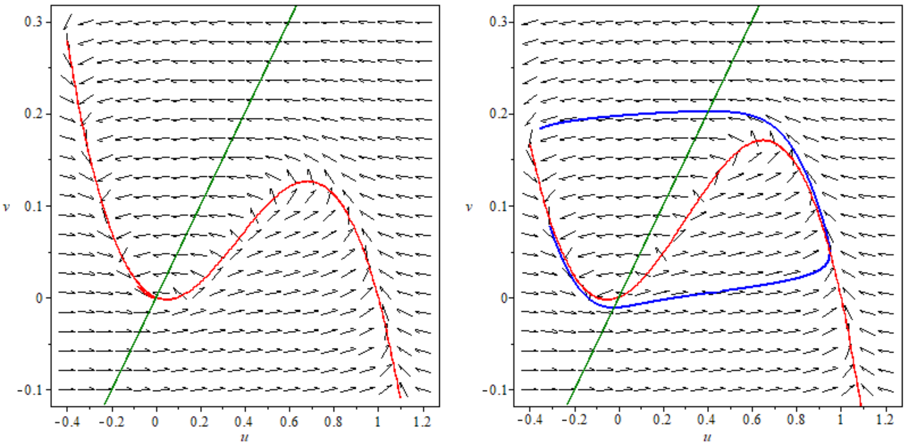

FIGURE 1. Nullclines in System 6 for parameters $I = 0, \beta = 0.01, \gamma = 0.02$, $\alpha = 0.1$ (left) and $\alpha = -0.1$ (right). Red curve is v-nullcline, green line is $w-$nullcline and blue curve is a trajectory.

$\gamma \geq 0$ describe the kinetics of the recovery variable v.
The nullclines of this model have the simple form

$$\begin{cases} v = u(\alpha - u)(u - 1) + I, \\ v = \frac{\beta}{\gamma} u, \end{cases} \tag{7}$$

So, this system has one, two or three equilibria. We suppose that $I = 0$ and consequently the origin $(0, 0)$ is an equilibrium. Indeed, the nullclines of the model, depicted in Fig.1, always intersect at $(0, 0)$ in this case. The stability of the equilibrium $(0, 0)$ depends on the parameters α, β, and γ. For example if $I = 0, \beta = 0.01, \gamma = 0.02$, then the origin for every $\alpha = 0.1$ is stable and corresponding to $\alpha = -0.1$ is unstable. The Jacobian matrix associated with 6 at the equilibrium $(0, 0)$ is given by

$$L = \begin{pmatrix} -\alpha & -1 \\ \beta & -\gamma \end{pmatrix}. \tag{8}$$

The characteristic polynomial of matrix 8 is obtain by

$$F(\lambda) = \lambda^2 + \tau\lambda + \Delta. \tag{9}$$

Where $\tau = tr\ L = -\alpha - \gamma$ and $\Delta = det\ L = \alpha\gamma + \beta$.
Then the equilibrium $(0, 0)$ is stable when $\alpha + \gamma > 0$ and $\alpha\gamma + \beta > 0$. Hence for every $\alpha > 0$ the equilibrium $(0, 0)$ is stable. This model may has two stable equilibria separated by an unstable equilibrium. Depending on the initial condition, the trajectory may approach the left or the right equilibrium.

3. Stochastic Izhikevich-FitzHugh model

In neural systems noise can be generated by a variety of reasons and it may arise from different sources [21]. Thus, mathematical modelling of neural dynamics, as a stochastic process, is interest of a variety of researchers. In this direction, we consider the following stochastic case of Izhikevich-FitzHugh model, presented in Equation 6.

$$\tag{10} \begin{cases} du = (u(\alpha - u)(u-1) - v + I)dt + \sigma_1 u dW_1(t), \\ dv = (\beta u - \gamma v)dt + \sigma_2 v dW_2(t), \end{cases}$$

where, σ_1, σ_2 measure the noise intensity in the system due to the environment and $W_1(t), W_2(t)$ denote the independent standard Wiener processes.
Applying Taylor's expansion, we have the following equivalent system:

$$\tag{11} \begin{cases} du = (-u^3 + (\alpha+1)u^2 - \alpha u - v + I)dt + \sigma_1 u dW_1(t), \\ dv = (\beta u - \gamma v)dt + \sigma_2 v dW_2(t), \end{cases}$$

Let $u = \overline{u}, v = \overline{v}, t = \overline{t}$ and $c_{jis} = \epsilon \overline{c_{jis}},\ k_{jis} = \sqrt{\epsilon}\overline{c_{jis}}$ for all j, i, s. Then

$$\tag{12} \begin{cases} du = \epsilon(-u^3 + (\alpha+1)u^2 - \alpha u - v + I)dt + \sqrt{\epsilon}\sigma_1 u dW_1(t), \\ dv = \epsilon(\beta u - \gamma v)dt + \sqrt{\epsilon}\sigma_2 v dW_2(t), \end{cases}$$

Note that we drop the bars from the scaled variables for simplicity. Now, by Khasminskii limiting theorem, System 12 can be transformed into the following limiting Itô averaging equations via polar coordinate transformation and stochastic differential equations:

$$\tag{13} \begin{cases} dr = [(\omega_1 + \frac{1}{16}\omega_2)r + \frac{1}{8}\omega_3 r^3]dt + (\frac{\omega_4}{8} r^2)^{\frac{1}{2}} dW_r(t), \\ d\theta = [\frac{1}{4}\omega_5]dt + (\frac{\omega_2}{8})^{\frac{1}{2}} dW_\theta(t), \end{cases}$$

where the parameters ω_i arises from Equations 4 and given as follows:

$$\tag{14} \begin{array}{ll} \omega_1 = \frac{1}{2}(-\alpha - \gamma), & \omega_2 = \sigma_1^2 + \sigma_2^2, \\ \omega_3 = -3, & \omega_4 = 3\sigma_1^2 + \sigma_2^2, \\ \omega_5 = 2 + 2\beta. & \end{array}$$

4. Largest Lyapunov exponent and stability

Let λ be the largest Lyapunov exponent of System 6. Oseledec multiplicative ergodic theorem [2] shows that $\lambda < 0$ implies the asymptotically stability of the trivial solution of linearized equation and $\lambda > 0$ implies that our stochastic system is unstable at the equilibrium $(0, 0)$. In Theorem 3.1 of [17], the authors prove that

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \ln \|r(t)\| = \omega_1 + \frac{1}{16}\omega_2 - \frac{1}{16}\omega_4,$$

where $r(t)$ is solution of Equation 13. Then we have the following theorem:

Theorem 4.1. (i) *When $-\gamma - \alpha < \frac{1}{4}\sigma_1^2$, the trivial solution of the linear Itô stochastic differential Equation 10 is asymptotically stable with probability* 1*, thus the stochastic system 10 is stable at the equilibrium point O.*
(ii) *When $-\gamma-\alpha > \frac{1}{4}\sigma_1^2$, the trivial solution of the linear Itô stochastic differential Equation 5 is unstable with probability* 1*, which implies that the stochastic System 10 is unstable at the equilibrium point O.*

Remark 4.2. *Since $\omega_3 = -3$ and $\omega_4 = 3\sigma_1^2 + \sigma_2^2$, Theorem 3.2 of* [17] *implies that if $-(\gamma+\alpha) < \frac{1}{4}\sigma_1^2$, then the stochastic System 10 is globally stable at the equilibrium point O.*

In Fig. 2 we plot largest Lyapunov exponent where σ_1 is variable. It means that if $\gamma = 0.02$, $\beta = 0.01$, $\sigma_2 = 0.2$, $\alpha = -0.2$ then for every $\sigma_1 > \sqrt{0.72}$, the largest Lyapunov exponent is negative and consequently the neuronal activity is stable or periodic. In Fig. 2 we plot largest Lyapunov exponent where α and σ_1 are variable.

Definition 4.3. [9] ***(D-bifurcation:)****Dynamical bifurcation is concerned with a family of random dynamical systems which is differential and has the invariant measure ω_θ. If there exists a constant θ_0 satisfying in any neighbourhood of θ_0, there exists another constant θ and the corresponding invariant measure $\nu_\theta \neq \omega_\theta$ satisfying $\nu_\theta \to \omega_\theta$ as $\theta \to \theta_0$. Then, the constant θ_0 is a point of dynamical bifurcation.*

Then, Theorem 4.1 of [17] and Section 3 of [9] imply that for every parameters γ, α, σ_1 that $-\gamma - \alpha = \frac{1}{4}\sigma_1^2$ the stochastic system 10 undergoes a $D-$bifurcation.

5. P-bifurcation

The stochastic P-bifurcation is a type of stochastic bifurcation that occurs in a stochastic system. This bifurcation describe the mode of the stationary probability density function or the invariant measure of the stochastic process. Stochastic systems undergoes the stochastic P-bifurcation when the mode of the stationary probability density function changes in nature. It indicates the jump of the distribution of the random variable in probability sense. There is no direct relation between D-bifurcation and P-bifurcation [22]. To investigate the P-bifurcation of stochastic system 10 and its polar coordinate transformation 13, we use probability density functions.
According to Section 4 of [17], the stationary probability density function $p(r)$

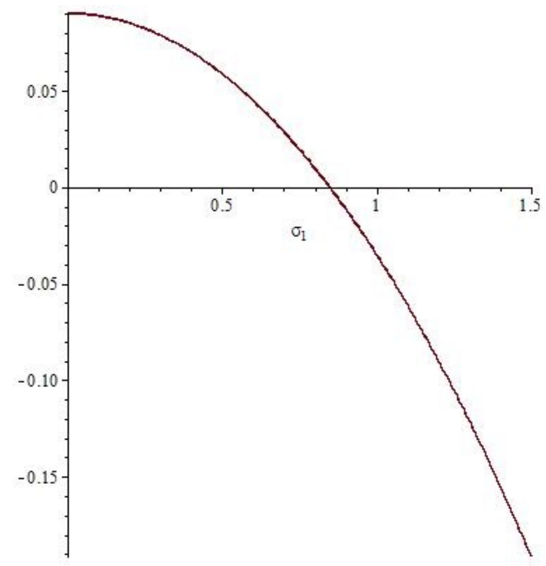

FIGURE 2. Largest Lyapunov exponent of the System 10 where $\gamma = 0.02$, $\beta = 0.01$, $\sigma_2 = 0.2$, $\alpha = -0.2$ and σ_1 is variable.

of random variable r can be given by:

$$(15)\quad P(r) = \begin{cases} \delta(r), & -\alpha-\gamma \le \frac{1}{4}\sigma_1^2, \\ \dfrac{r^{\frac{8(-\gamma-\alpha)-5\sigma_1^2-\sigma_2^2}{3\sigma_1^2+\sigma_2^2}} \exp(\frac{-3}{3\sigma_1^2+\sigma_2^2} r^2)}{\Gamma(\frac{8(-\gamma-\alpha)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2})(\frac{3\sigma_1^2+\sigma_2^2}{-3})^{\frac{8(-\gamma-\alpha)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2}}}, & -\alpha-\gamma > \frac{1}{4}\sigma_1^2, \end{cases}$$

This is clear that the extreme value point of $p(r)$, is $r_0 = 0$ or

$$r_1 = \sqrt{\frac{5\sigma_1^2 + \sigma_2^2 + 8(\gamma+\alpha)}{-6}},$$

when $\frac{5\sigma_1^2+\sigma_2^2}{2} < -4(\alpha+\gamma)$. Consequently, we have the following statements:

(i) If $-\alpha-\gamma \le \frac{5}{8}\sigma_1^2 + \frac{1}{8}\sigma_2^2 \le \frac{\sigma_1^2+\sigma_2^2}{2} - 2(\alpha+\gamma)$, then $\lim_{r\to 0^+} P(r) = \infty$ and the random trajectories of system 13 centralized in a neighborhood of the point $r_0 = 0$.

(ii) If $\frac{-8(\alpha+\gamma)+\sigma_1^2+\sigma_2^2}{3} \le 3\sigma_1^2 + \sigma_2^2 < -4(\alpha+\gamma) + \frac{1}{2}\sigma_2^1 + \frac{1}{2}\sigma_2^2$, then $P(r)$ is minimum at the point r_0 and maximum at the point r_1, but it is

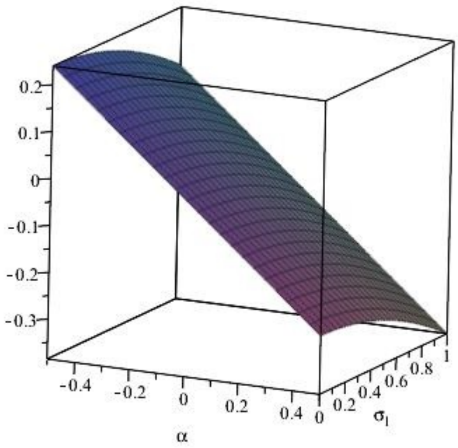

FIGURE 3. Largest Lyapunov exponent of the System 10 where $\gamma = 0.02$, $\beta = 0.01$, $\sigma_2 = 0.2$ and α, σ_1 are variables.

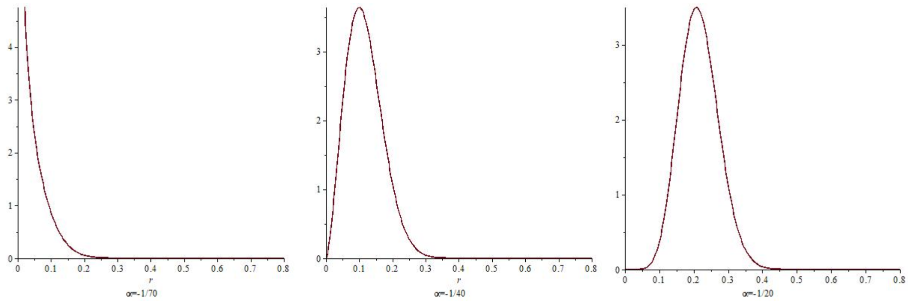

FIGURE 4. Probability density function $P(r)$ of System 13 for parameters $\sigma_1 = \sigma_2 = 0.1, \gamma = 0.01, \beta = 0.01$ and $\alpha = -\frac{1}{70}, -\frac{1}{40}, -\frac{1}{20}$.

not derivable at r_0. Moreover, the random trajectories of system 13 centralized in a neighborhood of the point r_1.

(iii) If $\alpha + \gamma < -\sigma_1^2 - \frac{1}{4}\sigma_2^2$, then $P(r)$ has the minimum value at the point r_0 and the maximum value at the point r_1. In this case, the probability density function $P(r)$ becomes a smooth function at the point r_1.

These results and observations lead us to the following theorem.

Theorem 5.1. *System 7 undergoes stochastic P-bifurcations as the parameter γ passes through the values of $\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \alpha$ and $\frac{5}{8}\sigma_1^2 + \frac{1}{8}\sigma_2^2 + \alpha$.*

Consideration of 15 shows that when the parameter γ passes through the value of $\alpha + \frac{1}{4}\sigma_1^2$, the probability density function $P(r)$ varies from $\delta(r)$ to the other function, which means that $\gamma_0 = \alpha + \frac{1}{4}\sigma_1^2$, is a P-bifurcation value for System 13.

Example 5.2. *As an example, we take $\sigma_1 = \sigma_2 = 0.1, \gamma = 0.01, \beta = 0.01$. By varying parameter α, we can see qualitative changes of density function $P(r)$. Simple calculation implies that:*

(i) *If $-\alpha < \frac{7}{400} < -2\alpha$, then $\lim\limits_{r \to 0^+} P(r) = \infty$, (see Fig. 4, case (a)).*

(ii) *If $\frac{-8\alpha}{3} < \frac{6}{100} < -4\alpha - \frac{1}{100}$, then $P(r)$ has the minimum value at the point $r_0 = 0$ and the maximum value at the point r_1, but the derivative of $P(r)$ at r_0 does not exist, (see Fig. 4, case (b)).*

(iii) *If $\alpha < \frac{-9}{400}$, then $P(r)$ has the minimum value at the point $r_0 = 0$ and the maximum value at the point r_1, (see Fig. 4, case (c)).*

By transforming $P(r)$ to the probability density $\rho(u, v)$ of the stationary distribution in terms of Cartesian coordinates x and y (for more details see [17, 23]), we have:

$$\rho(x,y) = \begin{cases} \delta(r), & -\alpha - \gamma \leq \frac{1}{4}\sigma_1^2, \\ \dfrac{(u^2+v^2)^{\frac{8(\gamma-\alpha)-8\sigma_1^2-2\sigma_2^2}{6\sigma_1^2+2\sigma_2^2}} exp(\frac{-3}{3\sigma_1^2+\sigma_2^2}(u^2+v^2))}{\pi\Gamma(\frac{8(-\gamma-\alpha)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2})(\frac{3\sigma_1^2+\sigma_2^2}{-3})^{\frac{8(-\gamma-\alpha)-2\sigma_1^2}{6\sigma_1^2+2\sigma_2^2}}}, & -\alpha - \gamma > \frac{1}{4}\sigma_1^2, \end{cases}$$

Similar to the above argument for $P(r)$, the extremal value point of $\rho(u, v)$ may be obtained. In this way we need to calculate the gradient of $\rho(u, v)$ in R^2. Hence, we reach the following results:

(i) If $-\gamma \leq \sigma_1^2 + \frac{1}{4}\sigma_2^2 + \alpha$, then $\rho(u, v)$ goes to infinite as $(u, v) \to (0, 0)$.

(ii) If $-24(\alpha + \gamma) \leq 33\sigma_1^2 + 9\sigma_2^2 \leq -32(\alpha + \gamma) + \sigma_1^2 + \sigma_2^2$, then $\rho(u, v)$ has a minimum value point at the origin, but it's partial derivatives are the origin is not continuous. Moreover, It has a maximum value at the point of the stable limit cycle $u^2 + v^2 = \frac{4\sigma_1^2+\sigma_2^2+4(\gamma+\alpha)}{-3}$.

(iii) If $\frac{11}{8}\sigma_1^2 + \frac{3}{8}\sigma_2^2 + \gamma < \alpha$, then $\rho(u, v)$ has a minimum value point at the origin, and a maximum value at the point of the stable limit cycle $u^2 + v^2 = \frac{4\sigma_1^2+\sigma_2^2+4(\gamma+\alpha)}{-3}$. Moreover, $\rho(u, v)$ has continuous partial derivatives.

We can summarize these results to the following theorem.

Theorem 5.3. *The stochastic system 7 undergoes phenomenological bifurcations as the parameter α passes through the values of $-\sigma_1^2 + \frac{1}{4}\sigma_2^2 - \gamma$ and $\frac{-11}{8}\sigma_1^2 + \frac{-3}{8}\sigma_2^2 - \gamma$.*

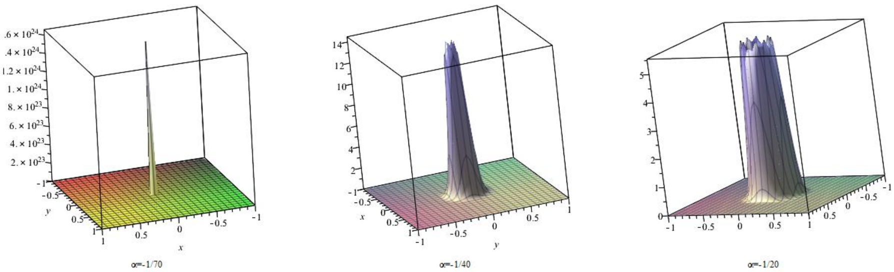

FIGURE 5. Variations of joint probability density $\rho(u,v)$ of system 13 for parameters $\sigma_1 = \sigma_2 = 0.1, \gamma = 0.01, \beta = 0.01$ and $\alpha = -\frac{1}{70}, -\frac{1}{40}, -\frac{1}{20}$.

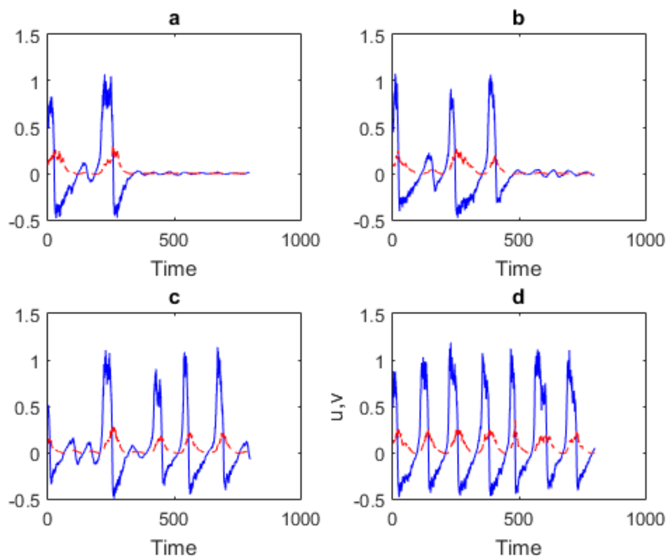

FIGURE 6. The evolution in time of u and v, when $(u_0, v_0) = (0.5, 0.1)$, $\beta = \gamma = 0.01$ $\sigma_1 = \sigma_2 = 0.1$, and (a) $\alpha = -\frac{1}{80}$,(b) $\alpha = -\frac{1}{70}$, (c) $\alpha = -\frac{1}{40}$ (d) $\alpha = -\frac{1}{20}$.

Example 5.4. *As an example, we take* $\sigma_1 = \sigma_2 = 0.1, \gamma = 0.01, \beta = 0.01$. *By varying parameter* γ *for values* $-\frac{1}{70}, -\frac{1}{40}, -\frac{1}{20}$, *we plot qualitative changes of density function* $\rho(u,v)$ *in Fig. 5.*

To verify the result in Figures 4 and 5 more clearly, we give time series evolution of u and v directly for different values of α in Fig 6. When $\alpha = -\frac{1}{70}$

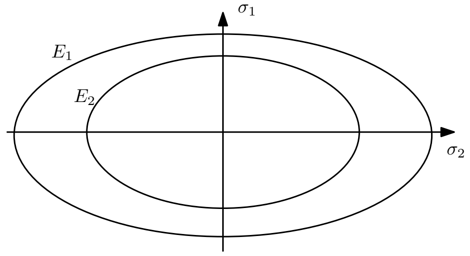

FIGURE 7. If parameters σ_1 and σ_2 pass through two ellipses E_1 and E_2, then system 13 undergoes the P-bifurcations.

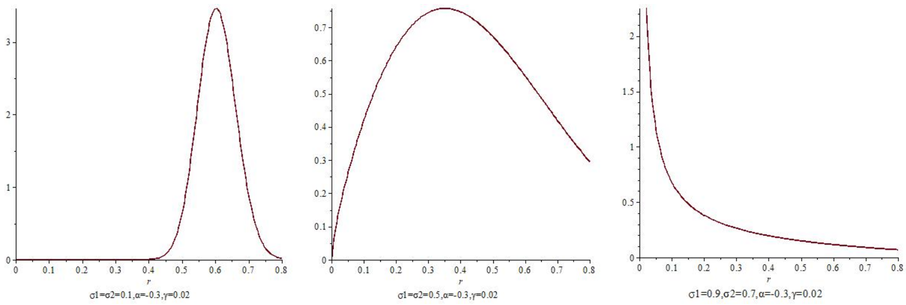

FIGURE 8. Variations of probability density $P(r)$ of system 13 for parameters $\alpha = -0.3, \beta = 0.01$ and $\gamma = 0.02$. (a) $\sigma_1, \sigma_2 \in E_2$, (b) $\sigma_1, \sigma_2 \in E_1 \backslash E_2$, (c) σ_1, σ_2 out of E_1.

or $\alpha = -\frac{1}{80}$ the time series converges to zero. When $\alpha = -\frac{1}{40}$, firstly particle moves near zero and then move periodically nearly 1 and -0.5. For $\alpha = -\frac{1}{20}$ we can see the optimal stochastic resonance.

5.1. P-bifurcation with respect to the noise

Here, we fix α as a constant and investigate the effect of noise intensities on the stationary probability density function. In other words, by changing values of σ_1 and σ_2 the qualitative behaviour of probability density function changes. If parameters σ_1 and σ_2 choose in the ellipse $E_2 := \sigma_1^2 + \frac{1}{4}\sigma_2^2 = -\gamma - \alpha$, then $P(r)$ is a smooth function that has a maximum value at the point $r_1 = \sqrt{\frac{5\sigma_1^2+\sigma_2^2+8(\gamma+\alpha)}{-6}}$ and a minimum value at the point $r = 0$. If σ_1 and σ_2 choose between two ellipses $E_1 := \frac{5}{8}\sigma_1^2 + \frac{1}{8}\sigma_2^2 = -\gamma - \alpha$ and E_2, then $P(r)$ has a maximum value at the point r_1 and minimum at the point $r = 0$. But it has not derivative in $r = 0$. If σ_1 and σ_2 choose out of the ellipse E_1, then $\lim\limits_{r \to 0^+} P(r) = \infty$ (see Fig 7). We can write these results as the following theorem.

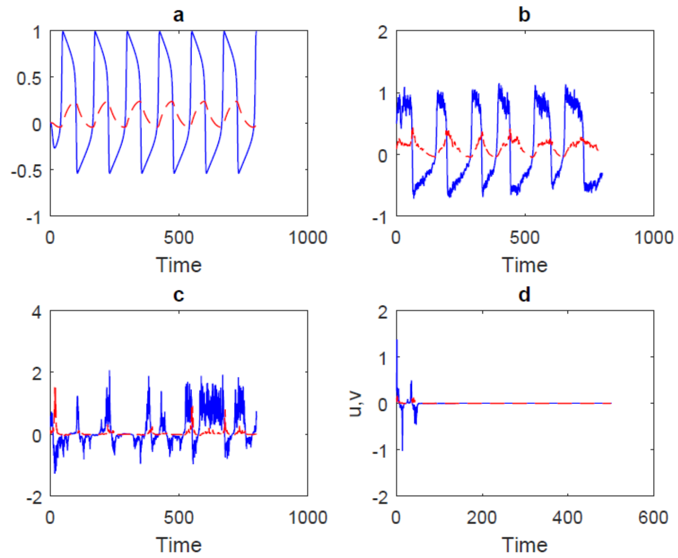

FIGURE 9. The evolution in time of u and v, when $(u_0, v_0) = (0.5, 0.1)$, $\alpha = -0.3, \beta = 0.01,\ \gamma = 0.02$ and (a) $\sigma_1 = \sigma_2 = 0$, (b) $\sigma_1 = \sigma_2 = 0.1$, (c) $\sigma_1 = \sigma_2 = 0.5$ (d) $\sigma_1 = 0.9, \sigma_2 = 0.7$.

Theorem 5.5. *The stochastic system 13 undergoes P-bifurcation as parameters σ_1 and σ_2 passes through two ellipses $E_1 = \frac{5}{8}\sigma_1^2 + \frac{1}{8}\sigma_2^2 = -\gamma - \alpha$ and $E_2 = \sigma_1^2 + \frac{1}{4}\sigma_2^2 = -\gamma - \alpha$.*

Example 5.6. *Let $\alpha = -0.3, \beta = 0.01$ and $\gamma = 0.0.2$. By choosing different values for parameters σ_1 and σ_2 we plot the probability density function $P(r)$ in Fig. 8. If $\sigma_1, \sigma_2 \in E_2$ the probability density is a smooth function with one maximum and one minimum point (Fig. 8 (a)), if σ_1, σ_2 lie between two ellipses E_1 and E_2, then $P(r)$ has one maximum and one minimum point, but it has no derivative in the origin (Fig. 8 (a)). Finally, if σ_1, σ_2 lie out of two ellipses $P(r)$ tend to infinite if $r \to 0^+$. This example confirms P-bifurcation conditions obtained in Theorem 5.5.*

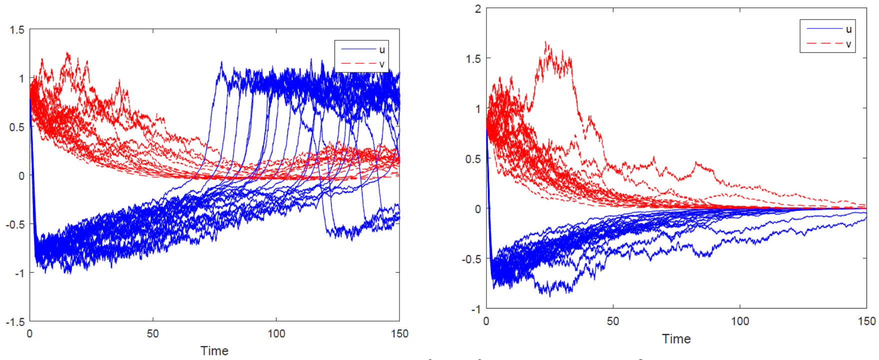

FIGURE 10. 20 trajectories of 10 for $\gamma = 0.02,\ \beta = 0.01,\ \sigma_1 = \sigma_2 = 0.1$ and $\alpha = -0.3$ (left), 0.2 (right).

Fig.9 represent depicts the evolution in time of u and v for constant parameters $\alpha = -0.3, \beta = 0.01,\ \gamma = 0.02$ and variation of diffusion coefficients $\sigma_1,\ \sigma_2$. In (a), $\sigma_1 = \sigma_2 = 0$ and trajectories of system 13 are periodic and concentrate in a neighborhood of the point $r_0 = 0$. In (b), $\sigma_1 = \sigma_2 = 0.1$, therefore the random trajectories of system 13 concentrate in a neighborhood of the point $r_1 \approx 0.6028$. In (c) $\sigma_1 = \sigma_2 = 0.5$, therefore the random trajectories of system 13 concentrate in a neighborhood of the point $r_1 \approx 0.3511$. In (d), $\sigma_1 = 0.9,\ \sigma_2 = 0.7$ which is out of E_1 and E_2. Hence, the random trajectories near origin has maximum probability density. Henceforth, simulation in Fig. 9 agree with the argument in Subsection 4.1.

6. Numerical simulation

Now, we perform a series of numerical simulations including phase portrait, evolution in time and probability density to confirm the analytical results. The time series is generated using Euler–Maruyama method described in [8] to the System 10. Choose the parameter values $\gamma = 0.02,\ \beta = 0.01, \sigma_1 = \sigma_2 = 0.1$. Theorem 4.1 implies that, for every $\alpha > -\frac{9}{400}$, the origin is stable and for every $\alpha < -\frac{9}{400}$ is unstable. In Fig. 10, we plot the time series evaluation of system 10 for initial condition $(u_0, v_0) = (0.8, 0.8)$ which contains 20 trajectories. This figure shows that for $\alpha = 0.2$, all random simulated trajectories convergence to the origin, which coincide the stability the system with these parameters at the origin. For $\alpha = -0.3$ in this case the origin is unstable. This fact, verifies the Theorem 4.1. Figure 11 represents the phase portrait of our stochastic system. It is a helpful evidence to study the qualitative behaviour of trajectories, respect to initial points. This is clear that for all initial points, the phase portrait is distributed between $0.4 < |u| < 1$ that confirm our result in Figures 8 (b), 9 (b)

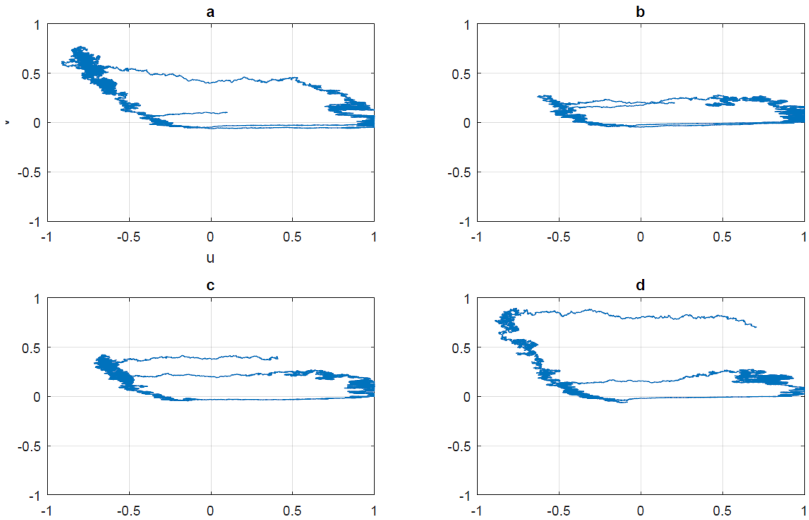

Figure 11. Phase portrait for system 10 for $\gamma = 0.02$, $\beta = 0.01$, $\alpha = -0.3$, $\sigma1 = \sigma2 = 0.1$ and initial points, $(u_0, v_0) = (0.1, 0.1)$ (a), $(u_0, v_0) = (0.2, 0.2)$ (b), $(u_0, v_0) = (0.4, 0.4)$ (c), $(u_0, v_0) = (0.9, 0.7)$ (d).

and 10 (a). In Fig. 12 we plot the phase portrait of system 10 for $\gamma = 0.02$, $\beta = 0.01$, $\alpha = -0.3$, $\sigma1 = 1, \sigma2 = 1.7$ with respect to various initial conditions. For all assumed initial conditions, firstly the trajectory has a complicated behavior and then convergence to origin. Indeed, for these parameters (σ_1, σ_2) is out of E_1 and E_2 and consequently $P(r)$ tend to infinite if $r \to 0^+$.

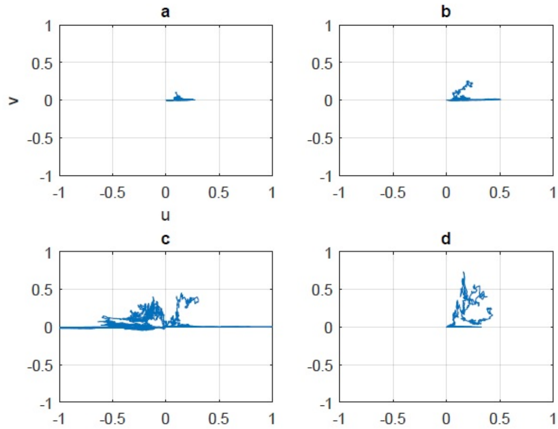

FIGURE 12. Phase portrait for systems 10 for $\gamma = 0.02$, $\beta = 0.01$, $\alpha = -0.3$, $\sigma 1 = 1, \sigma 2 = 1.7$ and initial points $(u_0, v_0) = (0.1, 0.1)$ (a), $(u_0, v_0) = (0.2, 0.2)$ (b), $(u_0, v_0) = (0.2, 0.3)$ (c), $(u_0, v_0) = (0.4, 0.2)$ (d).

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