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MILNE TYPE INEQUALITIES FOR DIFFERENTIABLE s-CONVEX FUNCTIONS

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Abstract. In this paper, a new identity is given. On the basis of this identity, we establish some new estimates of Milne's quadrature rule, for functions whose first derivative is s-convex. We discuss the cases where the derivatives are bounded as well as Lipschitzian. Some illustrative applications are given.

1. Introduction

Definition 1.1. [12] *A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + \left(1-t\right)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

The fundamental inequality for convex functions is undoubtedly the Hermite-Hadamard inequality, which can be stated as follows: For every convex function f on the interval $[a,b]$ with $a < b$, we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f\left(x\right)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction see [5, 7].

The concept of convexity plays an important and very central role in many areas, such as economics, finances, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions. Among those generalization, we note the s-convexity, which is defined as follows

Definition 1.2. [3] *A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

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holds for all $x, y \in I$ and $t \in [0,1]$.

Obviously, convexity has a close relationship in the development of the theory of inequalities, which is an important tool in the study of the properties of solutions of differential equations as well as in the error estimates of quadrature formulas. Indeed, several problems in applied mathematics as well as in sciences engineering come back to evaluation of integrals by adapting some quadrature. The wide family mostly used is the so-called Newton-Cotes quadrature (open or closed), it depends on the endpoints of the given interval if it intervenes in the approximation formula (closed) or not (open). Concerning some papers dealing with some quadrature see [1, 2, 4, 6, 8, 9, 10, 11, 13, 14] and references therein.

The most famous Newton-Cotes quadrature involving three-point is Simpson's inequality which can be described as

$$\left|\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

where f is four-times continuously differentiable function on (a,b), and

$$\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right|.$$

In this paper, we establish a new identity. On the basis of this identity, we derive new estimates of Milne's quadrature rule, for functions whose first derivative is s-convex. We also discuss the cases where the derivatives are bounded as well as Lipschitzian. At the end, some illustrative applications of our results given.

2. Main results

In order to prove our results, we need the following lemma

Lemma 2.1. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L^1[a,b]$, then the following equality holds*

$$\begin{aligned}&\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\\ =&\frac{b-a}{4}\int_0^1\left(t-\frac{4}{3}\right)f'\left((1-t)a+t\frac{a+b}{2}\right)dt\\ &+\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)f'\left((1-t)\frac{a+b}{2}+tb\right)dt.\end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(t - \tfrac{4}{3}\right) f'\left((1-t)\,a + t\tfrac{a+b}{2}\right) dt$$

and

$$I_2 = \int_0^1 \left(t + \tfrac{1}{3}\right) f'\left((1-t)\,\tfrac{a+b}{2} + tb\right) dt.$$

Integrating by parts I_1, we get

$$\begin{aligned} I_1 &= \tfrac{2}{b-a}\left(t - \tfrac{4}{3}\right) f\left((1-t)\,a + t\tfrac{a+b}{2}\right)\Big|_{t=0}^{t=1} - \tfrac{2}{b-a}\int_0^1 f\left((1-t)\,a + t\tfrac{a+b}{2}\right) dt \\ &= -\tfrac{2}{3(b-a)} f\left(\tfrac{a+b}{2}\right) + \tfrac{8}{3(b-a)} f(a) - \tfrac{2}{b-a}\int_0^1 f\left((1-t)\,a + t\tfrac{a+b}{2}\right) dt \\ &= \tfrac{8}{3(b-a)} f(a) - \tfrac{2}{3(b-a)} f\left(\tfrac{a+b}{2}\right) - \left(\tfrac{2}{b-a}\right)^2 \int_a^{\frac{a+b}{2}} f(\xi)\, d\xi. \qquad (2) \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \tfrac{2}{b-a}\left(t + \tfrac{1}{3}\right) f\left((1-t)\,\tfrac{a+b}{2} + tb\right)\Big|_{t=0}^{t=1} - \tfrac{2}{b-a}\int_0^1 f\left((1-t)\,\tfrac{a+b}{2} + tb\right) dt \\ &= \tfrac{2}{b-a}\left(1 + \tfrac{1}{3}\right) f(b) - \tfrac{2}{b-a}\left(\tfrac{1}{3}\right) f\left(\tfrac{a+b}{2}\right) - \tfrac{2}{b-a}\int_0^1 f\left((1-t)\,\tfrac{a+b}{2} + tb\right) dt \\ &= \tfrac{8}{3(b-a)} f(b) - \tfrac{2}{3(b-a)} f\left(\tfrac{a+b}{2}\right) - \left(\tfrac{2}{b-a}\right)^2 \int_{\frac{a+b}{2}}^{b} f(\xi)\, d\xi. \qquad (3) \end{aligned}$$

Summing (2) and (3), and then multiplying the result by $\frac{b-a}{4}$, we get the desired result. □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$ with $0 \le a < b$. If $|f'|$ is s-convex in the second sense for some fixed $s \in (0,1]$, then we have*

$$\begin{aligned} &\left| \tfrac{1}{3}\left(2f(a) - f\left(\tfrac{a+b}{2}\right) + 2f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\, du \right| \\ &\le \tfrac{b-a}{4}\left(\tfrac{4s+5}{3(s+1)(s+2)} |f'(a)| + \tfrac{2s+10}{3(s+1)(s+2)} \left|f'\left(\tfrac{a+b}{2}\right)\right| + \tfrac{4s+5}{3(s+1)(s+2)} |f'(b)| \right). \end{aligned}$$

Proof. From Lemma 2.1, properties of modulus, and s-convexity in the second sense of $|f'|$, we have

$$\begin{aligned}
&\left|\tfrac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\le \tfrac{b-a}{4}\left(\int_0^1\left|t-\tfrac{4}{3}\right|\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|t+\tfrac{1}{3}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|dt\right) \\
&= \tfrac{b-a}{4}\left(\int_0^1\left(\tfrac{4}{3}-t\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left(t+\tfrac{1}{3}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|dt\right) \\
&\le \tfrac{b-a}{4}\left(\int_0^1\left(\tfrac{4}{3}-t\right)\left((1-t)^s|f'(a)|+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)dt\right. \\
&\quad\left.+\int_0^1\left(t+\tfrac{1}{3}\right)\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|+t^s|f'(b)|\right)dt\right) \\
&= \tfrac{b-a}{4}\left(|f'(a)|\int_0^1\left(\tfrac{4}{3}-t\right)(1-t)^s\,dt+\left|f'\left(\tfrac{a+b}{2}\right)\right|\int_0^1\left(\tfrac{4}{3}-t\right)t^s dt\right. \\
&\quad\left.+\left|f'\left(\tfrac{a+b}{2}\right)\right|\int_0^1\left(t+\tfrac{1}{3}\right)(1-t)^s\,dt+|f'(b)|\int_0^1\left(t+\tfrac{1}{3}\right)t^s dt\right)
\end{aligned}$$

$$\begin{aligned}
&= \tfrac{b-a}{4}\left((|f'(a)|+|f'(b)|)\int_0^1\left(t+\tfrac{1}{3}\right)t^s dt+2\left|f'\left(\tfrac{a+b}{2}\right)\right|\int_0^1\left(\tfrac{4}{3}-t\right)t^s dt\right) \\
&= \tfrac{b-a}{4}\left(\tfrac{4s+5}{3(s+1)(s+2)}|f'(a)|+\tfrac{2s+10}{3(s+1)(s+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|+\tfrac{4s+5}{3(s+1)(s+2)}|f'(b)|\right),
\end{aligned}$$

where we have used the fact that

$$\int_0^1\left(\tfrac{4}{3}-t\right)(1-t)^s\,dt=\int_0^1\left(t+\tfrac{1}{3}\right)t^s dt=\tfrac{4s+5}{3(s+1)(s+2)} \tag{4}$$

and

$$\int_0^1 \left(\tfrac{4}{3}-t\right) t^s dt = \int_0^1 \left(t+\tfrac{1}{3}\right)(1-t)^s \, dt = \tfrac{s+5}{3(s+1)(s+2)}. \tag{5}$$

The proof is completed. □

Corollary 2.3. *In Theorem 2.2, if we use the s-convexity of* $|f'|$ *i.e.* $\left|f'\left(\frac{a+b}{2}\right)\right| \leq 2^{1-s}\frac{|f'(a)|+|f'(b)|}{1+s}$*, then we obtain*

$$\left|\tfrac{1}{3}\left(2f(a) - f\left(\tfrac{a+b}{2}\right) + 2f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\, du\right|$$
$$\leq \tfrac{b-a}{4}\left(\tfrac{2^{2-s}(s+5)+(4s+5)(s+1)}{3(s+1)^2(s+2)}\right)(|f'(a)| + |f'(b)|).$$

Corollary 2.4. *In Theorem 2.2, if we take* $s = 1$*, then we get*

$$\left|\tfrac{1}{3}\left(2f(a) - f\left(\tfrac{a+b}{2}\right) + 2f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\, du\right|$$
$$\leq \tfrac{b-a}{24}\left(3|f'(a)| + 4\left|f'\left(\tfrac{a+b}{2}\right)\right| + 3|f'(b)|\right).$$

Moreover, if we use the convexity of $|f'|$*, then we obtain*

$$\left|\tfrac{1}{3}\left(2f(a) - f\left(\tfrac{a+b}{2}\right) + 2f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\, du\right|$$
$$\leq \tfrac{5(b-a)}{24}(|f'(a)| + |f'(b)|).$$

Remark 2.5. *In Corollary 2.7 if we assume that* $|f'|$ *is bounded i.e.*$|f'(x)| \leq \sup_{x\in[a,b]} |f'(x)| = \|f'\|_\infty$*, then we obtain Theorem 3.1 from* [1].

Theorem 2.6. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable function on* (a, b) *such that* $f' \in L^1[a, b]$ *with* $0 \leq a < b$*. If* $|f'|^q$ *is s-convex in the second sense for some fixed* $s \in (0, 1]$ *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then we have*

$$\left|\tfrac{1}{3}\left(2f(a) - f\left(\tfrac{a+b}{2}\right) + 2f(b)\right) - \tfrac{1}{b-a}\int_a^b f(u)\, du\right|$$
$$\leq \tfrac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\tfrac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\tfrac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq\frac{b-a}{4}\left(\int_0^1\left(\frac{4}{3}-t\right)\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|dt\right.$$

$$\left.+\int_0^1\left(t+\frac{1}{3}\right)\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|dt\right)$$

$$\leq\frac{b-a}{4}\left(\left(\int_0^1\left(\frac{4}{3}-t\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(t+\frac{1}{3}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\int_0^1\left((1-t)^s|f'(a)|^q+t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left((1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+t^s|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

□

Corollary 2.7. *In Theorem 2.6, if we use the s-convexity of $|f'|^q$ i.e. $\left|f'\left(\frac{a+b}{2}\right)\right|^q\leq 2^{1-s}\frac{|f'(a)|^q+|f'(b)|^q}{1+s}$, then we obtain*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{\left(2^{1-s}+1+s\right)|f'(a)|+2^{1-s}|f'(b)|}{(s+1)^2}\right)^{\frac{1}{q}}+\left(\frac{2^{1-s}|f'(a)|+\left(2^{1-s}+1+s\right)|f'(b)|}{(s+1)^2}\right)^{\frac{1}{q}}\right).$$

Corollary 2.8. *In Theorem 2.6, if we take $s=1$, then we get*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Moreover, if we use the convexity of $|f'|^q$, then we obtain

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Theorem 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q\geq 1$, then we have*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left(\frac{4s+5}{3(s+1)(s+2)}|f'(a)|^q+\frac{s+5}{3(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{s+5}{3(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{4s+5}{3(s+1)(s+2)}|f'(b)|^q\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{4}\left(\int_0^1\left(\frac{4}{3}-t\right)\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|dt\right.$$
$$\left.+\int_0^1\left(t+\frac{1}{3}\right)\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|dt\right)$$

$$\leq \tfrac{b-a}{4}\left(\left(\int_0^1 \left(\tfrac{4}{3}-t\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\tfrac{4}{3}-t\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left(t+\tfrac{1}{3}\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(t+\tfrac{1}{3}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq \tfrac{b-a}{4}\left(\tfrac{5}{6}\right)^{1-\frac{1}{q}}\left(\left(\int_0^1 \left(\tfrac{4}{3}-t\right)\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left(t+\tfrac{1}{3}\right)\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$=\tfrac{b-a}{4}\left(\tfrac{5}{6}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\left|f'(a)\right|^q\int_0^1\left(\tfrac{4}{3}-t\right)(1-t)^s dt+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1\left(\tfrac{4}{3}-t\right)t^s dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1\left(t+\tfrac{1}{3}\right)(1-t)^s dt+\left|f'(b)\right|^q\int_0^1\left(t+\tfrac{1}{3}\right)t^s dt\right)^{\frac{1}{q}}\right)$$

$$=\tfrac{b-a}{4}\left(\tfrac{5}{6}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{4s+5}{3(s+1)(s+2)}\left|f'(a)\right|^q+\tfrac{s+5}{3(s+1)(s+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\tfrac{s+5}{3(s+1)(s+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+\tfrac{4s+5}{3(s+1)(s+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used (4) and (5). The proof is achieved. □

Corollary 2.10. *In Theorem 2.9, if we use the s-convexity of $|f'|^q$, then we obtain*

$$\left|\tfrac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq \tfrac{b-a}{4}\left(\tfrac{5}{6}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{(4s+5)(s+1)+2^{1-s}(s+5)}{3(s+1)^2(s+2)}\left|f'(a)\right|^q+\tfrac{2^{1-s}(s+5)}{3(s+1)^2(s+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\tfrac{2^{1-s}(s+5)}{3(s+1)^2(s+2)}\left|f'(a)\right|^q+\tfrac{(4s+5)(s+1)+2^{1-s}(s+5)}{3(s+1)^2(s+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 2.11. *In Theorem 2.9, if we take $s=1$, then we get*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\frac{5(b-a)}{24}\left(\left(\frac{3\left|f'(a)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q+3\left|f'(b)\right|^q}{5}\right)^{\frac{1}{q}}\right).$$

Moreover, if we use the convexity of $|f'|^q$, then we obtain

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{6}$$
$$\leq\frac{5(b-a)}{24}\left(\left(\frac{4\left|f'(a)\right|^q+\left|f'(b)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+4\left|f'(b)\right|^q}{5}\right)^{\frac{1}{q}}\right).$$

3. Further results

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If there exist constants $-\infty<m<M<+\infty$ such that $m\leq f'(x)\leq M$ for all $x\in[a,b]$, then we have*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{5(b-a)(M-m)}{24}.$$

Proof. From Lemma 2.1, we have

$$\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du$$
$$=\frac{b-a}{4}\int_0^1\left(t-\frac{4}{3}\right)f'\left((1-t)a+t\frac{a+b}{2}\right)dt$$
$$+\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)f'\left((1-t)\frac{a+b}{2}+tb\right)dt$$
$$=\frac{b-a}{4}\int_0^1\left(t-\frac{4}{3}\right)\left(f'\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dt$$
$$+\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)\left(f'\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)dt$$

$$=\frac{b-a}{4}\int_0^1\left(t-\tfrac{4}{3}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt+\frac{b-a}{4}\frac{m+M}{2}\int_0^1\left(t-\tfrac{4}{3}\right)dt$$
$$+\frac{b-a}{4}\int_0^1\left(t+\tfrac{1}{3}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right)dt+\frac{b-a}{4}\frac{m+M}{2}\int_0^1\left(t+\tfrac{1}{3}\right)$$
$$=\frac{b-a}{4}\int_0^1\left(t-\tfrac{4}{3}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt$$
$$+\frac{b-a}{4}\int_0^1\left(t+\tfrac{1}{3}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right)dt \tag{7}$$

Applying the absolute value to both sides of (7), we get

$$\left|\frac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\frac{b-a}{4}\int_0^1\left(\tfrac{4}{3}-t\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right|dt$$
$$+\frac{b-a}{4}\int_0^1\left(t+\tfrac{1}{3}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right|dt. \tag{8}$$

Obviously, since $m\leq f'(x)\leq M$ for all $x\in[a,b]$, we have

$$\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}, \tag{9}$$

and

$$\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right|\leq\tfrac{M-m}{2}. \tag{10}$$

Using (9) and (10) in (8) we get

$$\left|\frac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\frac{(b-a)(M-m)}{8}\left(\int_0^1\left(\tfrac{4}{3}-t\right)dt+\int_0^1\left(t+\tfrac{1}{3}\right)dt\right)$$
$$=\frac{5(b-a)(M-m)}{24},$$

which is the desired result. □

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If f' is L-Lipschitzian function on $[a,b]$, then we have*

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{7(b-a)^2}{24}L.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\\
=&\frac{b-a}{4}\int_0^1\left(t-\tfrac{4}{3}\right)f'\left((1-t)a+t\tfrac{a+b}{2}\right)dt\\
&+\frac{b-a}{4}\int_0^1\left(t+\tfrac{1}{3}\right)f'\left((1-t)\tfrac{a+b}{2}+tb\right)dt\\
=&\frac{b-a}{4}\left(\int_0^1\left(t-\tfrac{4}{3}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)+f'(a)\right)dt\right.\\
&\left.+\int_0^1\left(t+\tfrac{1}{3}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'(b)+f'(b)\right)dt\right)\\
=&\frac{b-a}{4}\left(\int_0^1\left(t-\tfrac{4}{3}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right)dt\right.\\
&+\int_0^1\left(t+\tfrac{1}{3}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'(b)\right)dt\\
&\left.+f'(b)\int_0^1\left(t+\tfrac{1}{3}\right)dt+f'(a)\int_0^1\left(t-\tfrac{4}{3}\right)dt\right)\\
=&\frac{b-a}{4}\left(\int_0^1\left(t-\tfrac{4}{3}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right)dt\right.\\
&\left.+\int_0^1\left(t+\tfrac{1}{3}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'(b)\right)dt+\frac{5}{6}\left(f'(b)-f'(a)\right)\right).
\end{aligned}$$

(11)

By applying the absolute value of both sides of (11), and using the fact that f' L-Lipschitzian function it yields

$$\begin{aligned}
&\left|\tfrac{1}{3}\left(2f(a)-f\left(\tfrac{a+b}{2}\right)+2f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq&\tfrac{b-a}{4}\left(\int_0^1\left(\tfrac{4}{3}-t\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right|dt\right. \\
&\left.+\int_0^1\left(t+\tfrac{1}{3}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'(b)\right|dt+\tfrac{5}{6}\left|f'(b)-f'(a)\right|\right) \\
\leq&\tfrac{(b-a)^2}{8}L\left(\int_0^1\left(\tfrac{4}{3}-t\right)tdt+\int_0^1\left(t+\tfrac{1}{3}\right)(1-t)\,dt+\tfrac{5}{3}\right) \\
=&\tfrac{7(b-a)^2}{24}L,
\end{aligned}$$

which is the desired result. □

4. Applications

4.1. Milen's quadrature formula

Let Υ be the partition of the points $a = x_0 < x_1 < ... < x_n = b$ of the interval $[a, b]$, and consider the quadrature formula

$$\int_a^b f(u)\,du = \lambda(f,\Upsilon) + R(f,\Upsilon),$$

where

$$\lambda(f,\Upsilon) = \sum_{i=0}^{n-1}\tfrac{x_{i+1}-x_i}{3}\left(2f(x_i)-f\left(\tfrac{x_i+x_{i+1}}{2}\right)+2f(x_{i+1})\right)$$

and $R(f,\Upsilon)$ denotes the associated approximation error

Proposition 4.1. *Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $0 \leq a < b$ and $f' \in L^1[a, b]$. If $|f'|$ is s-convex function in the second sense for some fixed $s \in (0, 1]$, we have*

$$\begin{aligned}
|R(f,\Upsilon)| \leq&\sum_{i=0}^{n-1}\tfrac{(x_{i+1}-x_i)^2}{4}\left(\tfrac{4s+5}{3(s+1)(s+2)}\left|f'(x_i)\right|\right. \\
&\left.+\tfrac{2s+10}{3(s+1)(s+2)}\left|f'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right|+\tfrac{4s+5}{3(s+1)(s+2)}\left|f'(x_{i+1})\right|\right).
\end{aligned}$$

Proof. Applying Theorem 2.2 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Υ, we get

$$\left|\frac{1}{3}\left(2f(x_i) - f\left(\frac{x_i+x_{i+1}}{2}\right) + 2f(x_{i+1})\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(u)\,du\right|$$
$$\leq \frac{b-x_i}{4}\left(\frac{4s+5}{3(s+1)(s+2)}\left|f'(x_i)\right| + \frac{2s+10}{3(s+1)(s+2)}\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \frac{4s+5}{3(s+1)(s+2)}\left|f'(x_{i+1})\right|\right).$$
(12)

Multiplying both sides of (12) by $(x_{i+1} - x_i)$, and then summing the obtained inequalities for all $i = 0, 1, ..., n-1$ and using the triangular inequality, we get the desired result. □

4.2. Application to special means

For arbitrary real numbers a, b we have:
The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.
The Geometric mean: $G(a,b) = \sqrt{ab}$, $a, b > 0$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|4A\left(a^2, b^2\right) - A^2(a,b) - 3L_2^2(a,b)\right| \leq \frac{5(b-a)}{4}\left(\left(\frac{4a^q+b^q}{5}\right)^{\frac{1}{q}} + \left(\frac{a^q+4b^q}{5}\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from inequality (6) of Corollary 2.11, with $q \geq 2$, applied to the function $f(x) = \frac{1}{2}x^2$. □

5. Conclusion

In the study, we have considered the Milne type integral inequalities, which the main results of the paper can be summarized as follows:

1. A new identity regarding Milne type inequalities is proved.
2. Some new Milne type inequalities for functions whose first derivatives are s-convex are established.
3. Some Milne type inequalities for functions whose first derivatives are bounded as well as L-Lipschitzian are discussed.
4. Applications of our findings are provided.

References

[1] M. Alomari, *New error estimations for the Milne's quadrature formula in terms of at most first derivatives.* Konuralp J. Math., **1** (2013), no.1, 17–23.
[2] N. Azzouza and B. Meftah, *Some weighted integral inequalities for differentiable beta-convex functions.* J. Interdiscip. Math. J. Interdiscip. Math. **24** (2021), no. 5, 1–22.

[3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen.* (German) Publ. Inst. Math. (Beograd) (N.S.) **23**(37) (1978), 13–20.

[4] T. Chiheb, N. Boumaza and B. Meftah, *Some new Simpson-like type inequalities via preqausiinvexity.* Transylv. J. Math. Mech. **12** (2020), no.1, 1–10.

[5] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

[6] S. Hamida and B. Meftah, *Some Simpson type inequalities for differentiable h-preinvex functions.* Indian J. Math. **62** (2020), no. 3, 299–319.

[7] C. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883), no. 1, 1–82.

[8] H. Kadakal, M. Kadakal and İ. İşcan, *Some new integral inequalities for n-times differentiable r-convex and r-concave functions.* Miskolc Math. Notes **20** (2019), no. 2, 997–1011.

[9] M. Kadakal, H. Kadakal and İ. İşcan, *Some New Integral Inequalities for n-times Differentiable s-Convex functions in the first sense.* Turkish Journal of Analysis and Number Theory, **5** (2017), no.2, 63–68.

[10] A. Kashuri, B. Meftah and P.O. Mohammed, *Some weighted Simpson type inequalities for differentiable s-convex functions and their applications.* J. Frac. Calc. & Nonlinear Sys. **1** (2020) no. 1, 75–94.

[11] A. Kashuri, P. O. Mohammed, T. Abdeljawad, F. Hamasalh and Y. Chu, *New Simpson type integral inequalities for s-convex functions and their applications.* Math. Probl. Eng. 2020, Art. ID 8871988, 12 pp.

[12] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications.* Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[13] J. Pečarić and S. Varošanec, *A note on Simpson's inequality for functions of bounded variation.* Tamkang J. Math. **31** (2000), no. 3, 239–242.

[14] M. Z. Sarikaya, E. Set and M. E. Ozdemir, *On new inequalities of Simpson's type for s-convex functions.* Comput. Math. Appl. **60** (2010), no. 8, 2191–2199.

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