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A STUDY OF A WEAK SOLUTION OF A DIFFUSION PROBLEM FOR A TEMPORAL FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper, we establish sufficient conditions for the existence and uniqueness of solution for a class of initial boundary value problems with Dirichlet condition in regard to a category of fractional-order partial differential equations. The results are established by a method based on the theorem of Lax Milligram.

1. Introduction

The concepts of fractional differential were examined further over the course of the 18th and 19th centuries [16]. The topic attracted the attention of mathematical giants such as Riemann, Liouville, Abel, Laurent, and Hardy and

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Littlewood. Detailed discussions of the history of fractional calculus were explored via many papers [3, 4, 9, 14]. Here, we wish to focus on a few key points concerning the directions in which the field developed. The "paradoxes"described by Leibniz were resolved by later authors, but this is not to say that the field of fractional calculus is now wholly free of open problems. One recurring issue through the centuries has been the existence of multiple conflicting definitions [2, 8]. In the mid-19th century, several different definitions of fractional calculus had already been proposed, Liouville had created one.

Fractional-order differential equations, which are obtained by generalizing differential equations to an arbitrary order, play a crucial role in engineering, physics and applied mathematics. Complex phenomena can be modeled using these equations. As a result, many applications can be found in the study of viscoelasticity, electrochemistry, signal processing, control theory, porous media, fluid mechanics, rheology, transport by diffusion, electrical networks, electromagnetic and probability, and many other physical processes. The existence and uniqueness results of the weak solution of fractional-order partial differential equations were obtained using the Lax-Milgram theorem, see [1, 5, 13, 15]. This method was adopted by many authors, see for example [6, 7, 10].

In this paper, we apply the Lax-Milgram theorem to the fractional partial differential problems with classical boundary conditions. The first section of this paper is devoted to reminder some basic tools and preliminary results essential to our work. In particular, we present some fundamental results on the properties of fractional derivation, linear operators and functional spaces. The second section is presented our main problem. The third section is devoted to the study of the existence and uniqueness of a weak solution of a fractional diffusion problem using the Lax-Milgram theorem followed by the last section, which summarizes the conclusion of this work.

2. Preliminaries and functional spaces

In this work, we intend to let $\mathbf{\Gamma}(\cdot)$ to denote the gamma function. For any positive integer $0 < \alpha < 1$, Caputo derivative and Riemann Liouville derivative are, respectively, defined as follows:

(1) The left Caputo derivative:

$$ {}_0^C D_t^{\alpha} u\left(x,t\right) := \frac{1}{\mathbf{\Gamma}\left(1-\alpha\right)} \int_0^t \frac{\partial u\left(x,\tau\right)}{\partial \tau} \frac{1}{\left(t-\tau\right)^{\alpha}} d\tau. \tag{2.1} $$

(2) The right Caputo derivative:

$$ {}_{t}^{C}D_{T}^{\alpha}u\left(x,t\right) := \frac{-1}{\boldsymbol{\Gamma}\left(1-\alpha\right)}\int_{t}^{T}\frac{\partial u\left(x,\tau\right)}{\partial\tau}\frac{1}{\left(\tau-t\right)^{\alpha}}d\tau. \tag{2.2} $$

(3) The left Riemann-Liouville derivative:

$$ {}_{0}^{R}D_{t}^{\alpha}u\left(x,t\right) := \frac{1}{\boldsymbol{\Gamma}\left(1-\alpha\right)}\frac{\partial}{\partial t}\int_{0}^{t}\frac{u\left(x,\tau\right)}{\left(t-\tau\right)^{\alpha}}d\tau. \tag{2.3} $$

(4) The right Riemann-Liouville derivative:

$$ {}_{t}^{R}D_{T}^{\alpha}v(t) = \frac{-1}{\Gamma\left(1-\alpha\right)}\frac{\partial}{\partial t}\int\limits_{t}^{T}\frac{u\left(x,\tau\right)}{\left(t-\tau\right)^{\alpha}}d\tau. \tag{2.4} $$

Many authors think that the Caputo's operator version is more natural than the others because it allows the handling of inhomogeneous initial conditions in an easier way. Therefore, the following definitions are linked by the next relationship, which can be verified by a direct calculation:

$$ {}_{0}^{R}D_{t}^{\alpha}u\left(x,t\right) = {}_{0}^{C}D_{t}^{\alpha}u\left(x,t\right) + \frac{u\left(x,0\right)}{\boldsymbol{\Gamma}\left(1-\alpha\right)t^{\alpha}}. \tag{2.5} $$

Definition 2.1. ([11]) For any real $\sigma > 0$, we define the semi-norm

$$ |u|_{{}^{l}H^{\sigma}(I)}^{2} := \left\|{}_{0}^{R}D_{t}^{\sigma}u\right\|_{L_{2}(I)}^{2} \tag{2.6} $$

and the norm

$$ \|u\|_{{}^{l}H^{\sigma}(\Omega)} := \left(\|u\|_{L_{2}(I)}^{2} + |u|_{{}^{l}H_{0}^{\sigma}(I)}^{2}\right)^{\frac{1}{2}}. \tag{2.7} $$

This allows to define ${}^{l}H_{0}^{\sigma}\left(I\right)$ as the closure of $C_{0}^{\infty}\left(I\right)$ with respect to the norm $\|\cdot\|_{{}^{l}H_{0}^{\sigma}(I)}$.

Definition 2.2. For any real $\sigma > 0$, we define the semi-norm

$$ |u|_{{}^{r}H_{0}^{\sigma}(I)}^{2} := \|{}_{t}^{r}D_{T}^{\sigma}u\|_{L_{2}(I)}^{2} \tag{2.8} $$

and the norm

$$ \|u\|_{{}^{r}H_{0}^{\sigma}(I)} := \left(\|u\|_{L_{2}(I)}^{2} + |u|_{{}^{r}H_{0}^{\sigma}(I)}^{2}\right)^{\frac{1}{2}}. \tag{2.9} $$

This allows to define ${}^{r}H_{0}^{\sigma}\left(\Omega\right)$ as the closure of $C_{0}^{\infty}\left(\Omega\right)$ with respect to the norm $\|\cdot\|_{{}^{r}H_{0}^{\sigma}(\Omega)}$.

Definition 2.3. For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, we define the semi-norm

$$|u|_{^cH^\sigma(I)} = \left| \left({}^RD_t^\sigma u,\ {}_t^RD^\sigma u \right)_{L^2(I)} \right|^{1/2} \tag{2.10}$$

and the norm

$$\|u\|_{^cH^\sigma(I)} = (\|u\|^2_{L^2(I)} + |u|^2_{^cH^\sigma(I)})^{1/2}. \tag{2.11}$$

This allows to define ${}^cH^\sigma(I)$ as the closure of $C_0^\infty(I)$ with respect to the norm $\|\cdot\|_{^cH^\sigma(I)}$.

Lemma 2.4. ([11]) *For any real $\sigma \in \mathbb{R}_+$ and $I = (0, T)$, if $u \in {}^lH^\alpha(I)$ and $v \in C_0^\infty(I)$, then*

$$({}^RD_t^\sigma u(t), v(t))_{L^2(I)} = (u(t),\ {}_t^RD^\sigma v(t))_{L^2(I)}. \tag{2.12}$$

Lemma 2.5. ([11, 12]) *For $0 < \sigma < 2$, $\sigma \neq 1$, $u \in H_0^{\frac{\sigma}{2}}(I)$, we have*

$${}^RD_t^\sigma u(t) = {}^RD_t^{\frac{\sigma}{2}}\ {}^RD_t^{\frac{\sigma}{2}} u(t). \tag{2.13}$$

Lemma 2.6. ([11, 12]) *For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, the semi-norms $|.|_{^lH^\sigma(I)}$, $|.|_{^rH^\sigma(I)}$ and $|.|_{^cH^\sigma(I)}$ are equivalent. Then, we pose*

$$|.|_{^lH^\sigma(I)} \cong |.|_{^rH^\sigma(I)} \cong |.|_{^cH^\sigma(I)}\,. \tag{2.14}$$

Lemma 2.7. ([11]) *For any real $\sigma > 0$, the space ${}^rH_0^\sigma(I)$ with respect to the norm (2.9) is complete.*

Definition 2.8. We denote by $L^2(0, T, L_2(0, 1)) := L_2(Q)$ the space of the functions which are square integrable in the Bochner sense, with the scalar product

$$(u, w)_{L_2(0,T,L^2(0,1))} = \int_0^T ((u, \cdot), (w, \cdot))_{L^2(0,1)}\, dt. \tag{2.15}$$

To move forward on this regard, we should mention that the space $L^2(0, T)$ is a Hilbert space, which implies that $L_2(0, T, L_2(0, 1))$ is a Hilbert space as well. Let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3. Study of a weak solution of a diffusion problem

In this section, we aim to present a study on the weak solution of a diffusion problem for a spatio-temporal fractional equation. For this purpose, we divide this part into three main subsections.

3.1. **Position of the problem.** Herein, we consider the area $Q = \Omega \times I$ with $\Omega = (-1,1)^d$ and $I = (0,T)$, where $T < \infty$ and $d \geq 1$. For $0 < \alpha < 1$, $1 < \beta < 2$, we concern with the following problem:

$$\begin{cases} {}^{R}D_t^{\alpha} u(x,t) - p_1\ {}^{R}D_x^{\beta} u(x,t) - p_2\ {}_{x}^{R}D^{\beta} u(x,t) + a(x,t)u(x,t) = f(x,t), \\ u(x,t)\mid_{\partial\Omega} = 0, & \forall t \in I, \\ I_t^{1-\alpha} u(x,0) = 0, & \forall x \in \Omega, \end{cases} \tag{P_1}$$

where $(x,t) \in Q$, and p_1, p_2 are two constants satisfying:

$$p_1 + p_2 = 1, \qquad 0 < p_1, p_2 < 1,$$

and the function a verifies

$$a_0 \leq a(x,t) \leq a_1,\ a_0, a_1 \in \mathbb{R}_*^+,\ \ \forall (x,t) \in Q.$$

3.2. **The variational formulation of the problem.** In this part, we will present the solution of the variational formulation of the problem $(P1)$ by stating and deriving the next proposition.

Proposition 3.1. *If $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ is a solution of (P_1), then it is also a solution of the variational problem that satisfies*

$$\begin{aligned} &\left({}^{R}D_t^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} v \right)_{L^2(Q)} - p_1 \left({}^{R}D_x^{\frac{\beta}{2}} u,\ {}_{x}^{R}D^{\frac{\beta}{2}} v \right)_{L^2(Q)} \\ &- p_2 \left({}_{x}^{R}D^{\frac{\beta}{2}} u,\ {}^{R}D_x^{\frac{\beta}{2}} v \right)_{L^2(Q)} + (au,v)_{L^2(Q)} = \langle f, v \rangle, \end{aligned} \tag{PV_1}$$

for all $v \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ and $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$.

Proof. Suppose that u is a solution of (P_1). Then we have notice that $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ and ${}^{R}D_t^{\alpha} u(x,t) - p_1\ {}^{R}D_x^{\beta} u - p_2\ {}_{x}^{R}D^{\beta} u + au = f$ on Q in $(C_0^{\infty}(Q))'$. For all $v \in C_0^{\infty}(Q)$, we have

$$\begin{aligned} &\left\langle {}^{R}D_t^{\alpha} u - p_1\ {}^{R}D_x^{\beta} u - p_2\ {}_{x}^{R}D^{\beta} u + au, v \right\rangle \\ &= \left\langle {}^{R}D_t^{\alpha} u, v \right\rangle - p_1 \left\langle {}^{R}D_x^{\beta} u, v \right\rangle - p_2 \left\langle {}_{x}^{R}D^{\beta} u, v \right\rangle + \langle au, v \rangle = \langle f, v \rangle . \end{aligned} \tag{3.1}$$

According to the Lemma 2.4, we can find

$$\begin{aligned} &\left({}^{R}D_t^{\frac{\alpha}{2}} u, {}_{t}^{R}D^{\frac{\alpha}{2}} v \right)_{L^2(Q)} - p_1 \left({}^{R}D_x^{\frac{\beta}{2}} u, {}_{x}^{R}D^{\frac{\beta}{2}} v \right)_{L^2(Q)} \\ &- p_2 \left({}_{x}^{R}D^{\frac{\beta}{2}} u, {}^{R}D_x^{\frac{\beta}{2}} v \right)_{L^2(Q)} + (au,v)_{L^2(Q)} = \langle f, v \rangle . \end{aligned} \tag{3.2}$$

$\square$

3.3. **Existence and uniqueness of the weak solution.** This subsection aims to deal with the existence and uniqueness of the weak solution of a diffusion problem for a temporal fractional differential equation. But before this crucial action, we introduce the next definition of the weak solution which would give the way to continue our investigation.

Definition 3.2. Any function $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ satisfies (PV_1) is said to be a weak solution.

Theorem 3.3. *For* $0<\alpha<1$, $1<\beta<2$ *and* $f \in \left(B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)\right)'$, *the problem* (PV_1) *admits a unique solution satisfies:*

$$\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \lesssim \|f\|_{\left(B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)\right)'} \qquad \textit{(a priori estimate of the solution).}$$

Proof. In order to demonstrate the existence and uniqueness of the weak solution of the problem (P_1), we apply the Lax-Milgram theorem. In this regard, we have that $V = B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ is a Hilbert space. Furthermore, we set

$$\begin{aligned} A(u,v) = &\left({}^{R}D_t^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} v\right)_{L^2(Q)} - p_1 \left({}^{R}D_x^{\frac{\beta}{2}} u,\ {}_{x}^{R}D^{\frac{\beta}{2}} v\right)_{L^2(Q)} \\ &- p_2 \left({}_{x}^{R}D^{\frac{\beta}{2}} u,\ {}^{R}D_x^{\frac{\beta}{2}} v\right)_{L^2(Q)} + (au,v)_{L^2(Q)}, \end{aligned} \tag{3.3}$$

where A is a bilinear form on $V \times V$ (according to the bilinearity of the scalar product and the linearity of the fractional-order derivative). Now, let us check the continuity and coercivity of A. To prove the continuity of A, it suffices to verify the following inequality:

$$|A(u,v)| \lesssim \|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \|v\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}.$$

Let $(u,v) \in V \times V$. Then we have

$$\begin{aligned} |A(u,v)| = &\Bigg| \left({}^{R}D_t^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} v\right)_{L^2(Q)} - p_1 \left({}^{R}D_x^{\frac{\beta}{2}} u,\ {}_{x}^{R}D^{\frac{\beta}{2}} v\right)_{L^2(Q)} \\ &- p_2 \left({}_{x}^{R}D^{\frac{\beta}{2}} u,\ {}^{R}D_x^{\frac{\beta}{2}} v\right)_{L^2(Q)} + (au,v)_{L^2(Q)} \Bigg| \\ \le &\left| \left({}^{R}D_t^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} v\right)_{L^2(Q)} \right| + \left| \left({}^{R}D_x^{\frac{\beta}{2}} u,\ {}_{x}^{R}D^{\frac{\beta}{2}} v\right)_{L^2(Q)} \right| \\ &+ \left| \left({}_{x}^{R}D^{\frac{\beta}{2}} u,\ {}^{R}D_x^{\frac{\beta}{2}} v\right)_{L^2(Q)} \right| + a_1 \left| (au,v)_{L^2(Q)} \right|. \end{aligned} \tag{3.4}$$

By using the Cauchy-Schwartzn inequality, we obtain

$$
\begin{aligned}
&|A(u,v)| \\
&\leq \left\| {}^{R}D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)} \left\| {}_t^{R}D^{\frac{\alpha}{2}} v \right\|_{L^2(Q)} + \left\| {}^{R}D_x^{\frac{\beta}{2}} u \right\|_{L^2(Q)} \left\| {}_x^{R}D^{\frac{\beta}{2}} v \right\|_{L^2(Q)} \\
&\quad + \left\| {}_x^{R}D^{\frac{\beta}{2}} u \right\|_{L^2(Q)} \left\| {}^{R}D_x^{\frac{\beta}{2}} v \right\|_{L^2(Q)} + a_1 \|u\|_{L^2(Q)} \|v\|_{L^2(Q)} \\
&\leq \left(\int_{\Omega} |u|^2_{{}^lH^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} \left(\int_{\Omega} |v|^2_{{}^rH^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} + \left(\int_I |u|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \\
&\quad \times \left(\int_I |v|^2_{{}^rH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} + \left(\int_I |u|^2_{{}^rH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \left(\int_I |v|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \\
&\quad + a_1 \left(\int_I \|u\|^2_{L^2(\Omega)} \right)^{1/2} \left(\int_I \|v\|^2_{L^2(\Omega)} \right)^{1/2}.
\end{aligned}
\tag{3.5}
$$

Based on Lemma 2.5 and Lemma 2.6, we obtain

$$
\begin{aligned}
&\left(\int_{\Omega} |u|^2_{{}^lH^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} \left(\int_{\Omega} |v|^2_{{}^rH^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} + \left(\int_I |u|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \\
&\quad \times \left(\int_I |v|^2_{{}^rH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} + \left(\int_I |u|^2_{{}^rH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \left(\int_I |v|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \\
&\quad + a_1 \left(\int_I \|u\|^2_{L^2(\Omega)} \right)^{1/2} \left(\int_I \|v\|^2_{L^2(\Omega)} \right)^{1/2} \\
&\lesssim \left(\int_{\Omega} |u|^2_{H^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} \left(\int_{\Omega} |v|^2_{H^{\frac{\alpha}{2}}(I)} \, dx \right)^{1/2} + \left(\int_I |u|^2_{H_0^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} \\
&\quad \times \left(\int_I |v|^2_{H_0^{\frac{\beta}{2}}(\Omega)} \, dt \right)^{1/2} + a_1 \left(\int_I \|u\|^2_{L^2(\Omega)} \right)^{1/2} \left(\int_I \|v\|^2_{L^2(\Omega)} \right)^{1/2} \\
&\lesssim \|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \|v\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}.
\end{aligned}
\tag{3.6}
$$

On the other hand, to deal with the continuity of A, it suffices to verify that

$$A(u,u) \gtrsim \|u\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}$$

for all $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$. Now, let $u \in V$. Then we have

$$\begin{aligned} A(u,u) = &\left({}^{R}D_t^{\frac{\alpha}{2}}u,\ {}_t^RD^{\frac{\alpha}{2}}u \right)_{L^2(Q)} - p_1 \left({}^{R}D_x^{\frac{\beta}{2}}u,\ {}_x^RD^{\frac{\beta}{2}}u \right)_{L^2(Q)} \\ &- p_2 \left({}_x^RD^{\frac{\beta}{2}}u,\ {}^{R}D_x^{\frac{\beta}{2}}u \right)_{L^2(Q)} + (au,u)_{L^2(Q)}, \end{aligned} \tag{3.7}$$

that is,

$$\begin{aligned} A(u,u) = &\int_\Omega \int_I {}^{R}D_t^{\frac{\alpha}{2}}u\,(x,t)\ {}_t^RD^{\frac{\alpha}{2}}u\,(x,t)\,dtdx \\ &- p_1 \int_\Omega \int_I {}^{R}D_x^{\frac{\beta}{2}}u\,(x,t)\ {}_x^RD^{\frac{\beta}{2}}u\,(x,t)\,dtdx \\ &- p_2 \int_\Omega \int_I {}_x^RD^{\frac{\beta}{2}}u\,(x,t)\ {}^{R}D_x^{\frac{\beta}{2}}u\,(x,t)\,dtdx \\ &+ \int_\Omega \int_I a\,(x,t)\,(u\,(x,t))^2\,dtdx. \end{aligned} \tag{3.8}$$

This implies that

$$\begin{aligned} A(u,u) = &\int_\Omega \int_I {}^{R}D_t^{\frac{\alpha}{2}}u\,(x,t)\ {}_t^RD^{\frac{\alpha}{2}}u\,(x,t)\,dtdx \\ &- p_1 \int_\Omega \int_I {}^{R}D_x^{\frac{\beta}{2}}u\,(x,t)\ {}_x^RD^{\frac{\beta}{2}}u\,(x,t)\,dtdx \\ &+ \int_\Omega \int_I a\,(x,t)\,(u\,(x,t))^2\,dtdx. \end{aligned} \tag{3.9}$$

Using Lemma 2.6 yields

$$\begin{aligned} A(u,u) = &\cos\left(\frac{\alpha}{2}\pi\right) \int_\Omega |u|^2_{{}^lH^{\frac{\alpha}{2}}(I)}\,dx - \cos\left(\frac{\beta}{2}\pi\right) \int_I |u|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)}\,dt \\ &+ \int_\Omega \int_I a\,(x,t)\,(u\,(x,t))^2\,dtdx \end{aligned} \tag{3.10}$$

$$\geq \cos\left(\frac{\alpha}{2}\pi\right) \int\limits_{\Omega} |u|^2_{{}^lH^{\frac{\alpha}{2}}(I)}\, dx - \cos\left(\frac{\beta}{2}\pi\right) \int\limits_{I} |u|^2_{{}^lH^{\frac{\beta}{2}}(\Omega)}\, dt$$
$$+ a_0 \int\limits_{I} \|u\|^2_{L^2(\Omega)}\, dt.$$

Again, by virtue of Lemma 2.6, we can obtain

$$\begin{aligned} A(u,u) \gtrsim \cos\left(\frac{\alpha}{2}\pi\right) \int\limits_{\Omega} |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx - \cos\left(\frac{\beta}{2}\pi\right) \int\limits_{I} |u|^2_{H_0^{\frac{\beta}{2}}(\Omega)}\, dt \\ + a_0 \int\limits_{I} \|u\|^2_{L^2(\Omega)}\, dt, \end{aligned} \tag{3.11}$$

where $0 < \alpha < 1$ and $1 < \beta < 2$ such that

$$0 < \cos\left(\frac{\alpha}{2}\pi\right) < 1 \text{ and } 0 < -\cos\left(\frac{\beta}{2}\pi\right) < 1.$$

This, consequently, yields that there exists $\varepsilon_1, \varepsilon_2 < 1$ such that

$$\cos\left(\frac{\alpha}{2}\pi\right) \geq \varepsilon_1 \text{ and } -\cos\left(\frac{\beta}{2}\pi\right) \geq \varepsilon_2.$$

Therefore, the right hand side of inequality (3.11) becomes

$$\cos\left(\frac{\alpha}{2}\pi\right) \int\limits_{\Omega} |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx - \cos\left(\frac{\beta}{2}\pi\right) \int\limits_{I} |u|^2_{H_0^{\frac{\beta}{2}}(\Omega)}\, dt + a_0 \int\limits_{I} \|u\|^2_{L^2(\Omega)}\, dt$$
$$\geq \varepsilon_1 \int\limits_{\Omega} |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx + \varepsilon_2 \int\limits_{I} |u|^2_{H_0^{\frac{\beta}{2}}(\Omega)}\, dt + a_0 \int\limits_{I} \|u\|^2_{L^2(\Omega)}\, dt.$$

Thus, we have

$$A(u,u) \gtrsim \|u\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}. \tag{3.12}$$

By (3.12) and the Cauchy-Schwarz inequality, we obtain

$$\|u\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \lesssim A(u,u) = \langle f, v\rangle \leq \|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \|f\|_{\left(B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)\right)'}.$$

Hence, we have

$$\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \lesssim \|f\|_{\left(B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)\right)'},$$

which completes the proof. □

4. Conclusion

In this paper, it has been demonstrated that the Lax-Milgram theorem confirms its ability as a way that can be implemented to explore the existence and uniqueness of the weak solution of a diffusion problem for a temporal fractional differential equation. It is valid scheme for coercive the linear operator on Hilbert space.

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