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GENERALIZED α-NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. This paper deals with the new iterative algorithm for approximating the fixed point of generalized α-nonexpansive mappings in a hyperbolic space. We show that the proposed iterative algorithm is faster than all of Picard, Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur and Piri iteration processes for contractive mappings in a Banach space. We also establish some weak and strong convergence theorems for generalized α-nonexpansive mappings in hyperbolic space. The examples and numerical results are provided in this paper for supporting our main results.

1. Introduction

Many researchers attracted in the direction of approximating the fixed points of nonexpansive mapping and its generalized form [4, 5, 10, 12, 14, 15, 18, 20, 21, 29] in a hyperbolic space.

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One of the most used iterative techniques was introduced by Mann [25], which is given as follows: Assume that M is a nonempty, closed and convex subset of a Banach space Y and $S : M \to M$ is self mapping.

For any initial point $t_1 \in M$,

$$t_{m+1} = (1-\alpha_m)t_m + \alpha_m St_m, \ \forall m \in \mathbb{N}, \tag{1.1}$$

where $\{\alpha_m\}$ is real sequence in $(0,1)$. Let S be a nonexpansive mapping and control parameter $\{\alpha_m\}$ satisfies the condition $\sum_{m=0}^{\infty} \alpha_m(1-\alpha_m) = \infty$. Then, the sequence $\{t_m\}$ defined by (1.1) converges weakly to a fixed point of S.

It is well known that the Mann iteration method for the approximation of fixed points of pseudocontractive mappings may not well behave (see [9]). To overcome from this problem, Ishikawa [13] introduced an iterative technique, which is extensively studied for the approximation of fixed points of pseudo-contractive and nonexpansive mappings by many authors in different spaces (see for example Takahashi *et al.* [32], Acedo and Xu [2], Dotson [11]).

The Ishikawa iteration process is defined as

$$\begin{cases} t_{m+1} = (1-\alpha_m)t_m + \alpha_m Su_m \\ u_m = (1-\beta_m)t_m + \beta_m St_m, \ \forall m \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

In 2000, Noor [26] introduced the following three-step iteration process:

$$\begin{cases} t_{m+1} = (1-\alpha_m)t_m + \alpha_m Su_m, \\ u_m = (1-\beta_m)t_m + \beta_m Sv_m, \\ v_m = (1-\gamma_m)t_m + \gamma_m St_m, \ \forall m \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_m\}$, $\{\beta_m\}$ and $\{\gamma_m\}$ are real sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

In 2007, Agarwal *et al.* [3] introduced an iteration method which is called an S-iteration method. Its convergence rate is faster than both Mann and Ishikawa iteration method for contraction mappings. The S-iteration algorithm defined by

$$\begin{cases} t_{m+1} = (1-\alpha_m)St_m + \alpha_m Su_m \\ u_m = (1-\beta_m)t_m + \beta_m St_m, \ \forall m \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences in $(0,1)$ with $\sum_{m=1}^{\infty} \alpha_m\beta_m(1-\beta_m) = \infty$ and $t_1 \in M$ is an arbitrary. The algorithmic design of S-iteration method (1.4) is comparatively different and independent of Mann and Ishikawa iteration methods, that is, neither Mann nor Ishikawa iterative technique can be reduced into S-iteration and vice-versa.

In 2011, Sahu [28] introduced another form of S-iteration, named as normal S-iteration method which is defined by

$$\begin{cases} t_{m+1} = Su_m \\ u_m = (1-\alpha_m)t_m + \alpha_m St_m, \ \forall m \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_m\}$ is sequence in $(0,1)$ and $t_1 \in M$ is an arbitrary. Normal S-iteration (1.5) is also known as Hybrid-Picard Mann iteration method [16]. S-iteration method have attracted many researchers as alternative iteration method for common fixed point problems (see [8, 34]).

In 2014, Abbas and Nazir [1] introduced the following three-step iteration process:

$$\begin{cases} t_{m+1} = (1-\alpha_m)Sv_m + \alpha_m Su_m, \\ u_m = (1-\beta_m)St_m + \beta_m Sv_m, \\ v_m = (1-\gamma_m)t_m + \gamma_m St_m, \qquad \forall m \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_m\}$, $\{\beta_m\}$ and $\{\gamma_m\}$ are real sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

In 2016, Thakur *et al.* [33] introduced the following three-step iteration process:

$$\begin{cases} t_{m+1} = Su_m, \\ u_m = S((1-\alpha_m)t_m + \alpha_m v_m), \\ v_m = (1-\beta_m)t_m + \beta_m St_m, \qquad \forall m \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

In 2018, Piri *et al.* [27] introduced the following three-step iteration process:

$$\begin{cases} t_{m+1} = (1-\alpha_m)Sv_m + \alpha_m Su_m, \\ u_m = Sv_m, \\ v_m = S((1-\beta_m)t_m + \beta_m St_m), \qquad \forall m \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

The following question is quite natural.

Question : Is it possible to develop an iteration process which rate of convergence for contractive maps is faster than the iteration process (1.8) and the other iteration processes?

As a very straight forward answer, we introduce the following three-step iteration process:

$$\begin{cases} t_{m+1} = S((1-\alpha_m)Sv_m + \alpha_m Su_m), \\ u_m = Sv_m, \\ v_m = S((1-\beta_m)t_m + \beta_m St_m), \qquad \forall m \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0,1)$ and $t_1 \in M$ is an arbitrary.

2. Rate of convergence

We now recall the concept which was introduced by Berinde [7] for a comparison of the rates of convergence of different iterative algorithms involving a nonlinear mapping.

Definition 2.1. Let $\{t_m\}$ and $\{u_m\}$ be two iteration processes that both converging to the some fixed point t and u, respectively. Assume that the limit $\lim_{m\to\infty}\frac{||t_m-t||}{||u_m-u||}=l$ exist.

(1) If $l=0$, then we say that $\{t_m\}$ converges to t faster than $\{u_m\}$ to u.
(2) If $0<l<\infty$, then we say that $\{t_m\}$ and $\{u_m\}$ have the same rate of convergence.

Theorem 2.2. *Let M be a nonempty closed convex subset of a Banach space Y and $S: M\to M$ be a contractive mapping with a contraction constant $c\in(0,1)$ and fixed point u. Let $t_1\in M$. Then the rate of convergence of sequence $\{t_m\}$ defined in* (1.9) *is faster than sequence $\{t_m\}$ defined in* (1.8).

Proof. From (1.8) and the contractivity of S,

$$\begin{aligned}\|v_m-u\| &= \|S((1-\beta_m)t_m+\beta_m St_m)-u\|\\ &\le c(1-\beta_m)\|t_m-u\|+c\|St_m-u\|\\ &\le c(1-\beta_m)\|t_m-u\|+c^2\|t_m-u\|\\ &= c(1-(1-c)\beta_m)\|t_m-u\|,\end{aligned}$$

$$\begin{aligned}\|u_m-u\| &= \|Sv_m-u\|\\ &\le c\|v_m-u\|\\ &= c^2(1-(1-c)\beta_m)\|t_m-u\|\end{aligned}$$

and

$$\begin{aligned}\|t_{m+1}-u\| &= \|(1-\alpha_m)Su_m+\alpha_m Sv_m-u\|\\ &\le c(1-\alpha_m)\|u_m-u\|+c\alpha_m\|v_m-u\|\\ &\le c^2(1-\alpha_m)\|v_m-u\|+c^1\alpha_m\|v_m-u\|\\ &\le c(1-(1-c)\alpha_m)\|v_m-u\|\\ &= c^2(1-(1-c)\alpha_m)(1-(1-c)\beta_m)\|t_m-u\|.\end{aligned}$$

Since $\frac{1}{2}<\alpha_m,\beta_m<1$, we have

$$1-(1-c)\alpha_m<c+\frac{1}{2}$$

and

$$1-(1-c)\beta_m<c+\frac{1}{2}.$$

Therefore, we have

$$\|t_{m+1} - u\| \leq c^2 (c + \frac{1}{2})^2 \|t_m - u\|.$$

Let $B_m = c^{2m}(c + \frac{1}{2})^{2m}\|t_m - u\|$. Then

$$\lim_{m\to\infty} \frac{A_m}{B_m} = \lim_{m\to\infty} \frac{c^{3m}(c+\frac{1}{2})^{2m}\|t_1 - u\|}{c^{2m}(c+\frac{1}{2})^{2m}\|t_1 - u\|}$$
$$= 0,$$

so the iterative algorithm (1.9) is faster than (1.8). □

Consider the following example which is given in [27].

Example 2.3. Let $M = [2,5]$ and $S : M \to M$ be a mapping defined by $St = \sqrt{2t+3}$, for any $t \in M$. Then we can show that S is a contractive mapping with contractive constant $\frac{1}{\sqrt{7}}$ and $u^* = 3$ is a fixed point of S. Choose $\alpha_m = 0.7$, $\beta_m = 0.65$ and $\gamma_m = 0.8$. Also the initial value $t_1 = 4$.

We compare the convergence of new iteration process with Picard, Mann, Ishikawa, Noor, Agarwal, Normal S-iteration, Abbas, Thakur and Piri iteration processes for contractive mapping. We select the stopping criteria $||t_m - u^*|| \leq 10^{-9}$.

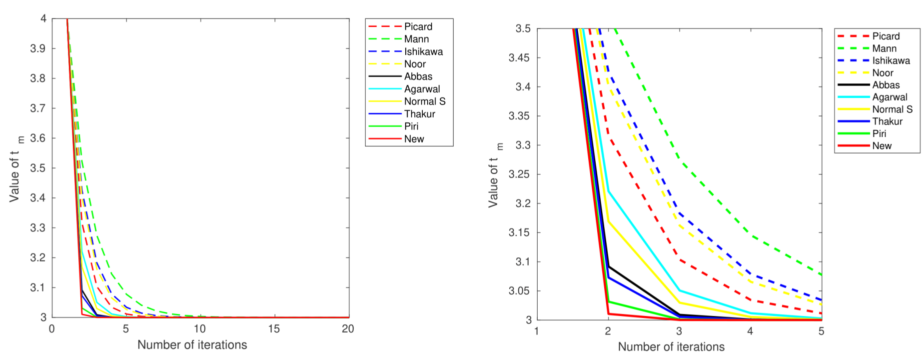

FIGURE 1. Comparison among different iteration processes for Example 2.3

Table 1. Comparison the New iteration process with other processes

m	**Picard**	**Mann**	**Ishikawa**	**Noor**	**Agarwal**
1	4	4	4	4	4
2	3.316624790	3.521637353	3.425913294	3.401577084	3.220900731
3	3.103747667	3.274870030	3.182879466	3.162119876	3.050701106
4	3.034385495	3.145646768	3.078815901	3.065596966	3.011741899
5	3.011440019	3.077407655	3.034023043	3.026566671	3.002725005
6	3.003810919	3.041207070	3.014697475	3.010763574	3.000632714
7	3.001270038	3.021955193	3.006351072	3.004361575	3.000146925
8	3.000423316	3.011703203	3.002744793	3.001767493	3.000034119
9	3.000141102	3.006239935	3.001186308	3.000716280	3.000007923
10	3.000047034	3.003327461	3.000512739	3.000290277	3.000001840
11	3.000015678	3.001774502	3.000221615	3.000117637	3.000000427
12	3.000005226	3.000946360	3.000095787	3.000047673	3.000000099
13	3.000000581	3.000504714	3.000041401	3.000019320	3.000000023
14	3.000000194	3.000269177	3.000017894	3.000007830	3.000000005
15	3.000000065	3.000143560	3.000003343	3.000003173	3.000000001
16	3.000000022	3.000076565	3.000001445	3.000001286	3
17	3.000000007	3.000040835	3.000000625	3.000000521	
18	3.000000002	3.000021779	3.000000270	3.000000211	
19	3.000000001	3.000011615	3.000000117	3.000000086	
20	3	3.000006195	3.000000050	3.000000035	

Table 2. Comparison the New iteration process with other processes

m	**Normal S-it.**	**Abbas**	**Thakur**	**Piri**	**New**
1	4	4	4	4	4
2	3.169112606	3.091925931	3.072949818	3.031619256	3.010521303
3	3.029795197	3.008909405	3.005620822	3.001059616	3.000117687
4	3.005288439	3.000868012	3.000434938	3.000035580	3.000001317
5	3.000939899	3.000084610	3.000033666	3.000001195	3.000000015
6	3.000167085	3.000008248	3.000002606	3.000000040	3
7	3.000029704	3.000000804	3.000000202	3.000000001	
8	3.000005281	3.000000078	3.000000016	3	
9	3.000000939	3.000000008	3.000000001		
10	3.000000167	3.000000001	3		
11	3.000000030	3			
12	3.000000005				
13	3.000000001				
14	3				

From Figure 1, Table 1 and Table 2, it is clear that new iteration process takes less number of iterations compared to other iteration processes to approximate fixed point of the mapping S defined in Example 2.3.

3. Convergence results for generalized α-nonexpansive mappings

3.1. **Basic Definitions and Results.** Throughout this paper, we consider the following definition of a hyperbolic space introduced by Kohlenbach [22].

Definition 3.1. A metric space (Y, d) is said to be a hyperbolic space if there exists a map $W : Y^2 \times [0,1] \to Y$ satisfying

(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha)\, d(u, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\, d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha)\, d(z, w)$

for all $x, y, z, w \in Y$ and $\alpha, \beta \in [0,1]$.

Definition 3.2. ([31]) A metric space is said to be convex, if a triple (Y, d, W) satisfy only (i) in Definition 3.1.

Definition 3.3. ([31]) A subset M of a hyperbolic space Y is said to be convex, if $W(x, y, \alpha) \in M$ for all $x, y \in M$ and $\alpha \in [0,1]$.

If $x, y \in Y$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$. The following holds even for more general setting of convex metric space [31] : for all $x, y \in Y$ and $\lambda \in [0,1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x, y)$$

and

$$d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x, y).$$

Thus

$$1x \oplus 0y = x, \;\; 0x \oplus 1y = y$$

and

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Definition 3.4. ([23]) A hyperbolic space (Y, ∂, W) is said to be uniformly convex, if for any $u, x, y \in Y$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, u\right) \leq (1-\delta)r,$$

whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

Definition 3.5. A map $\eta : (0, \infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

In [23], Luestean proved that every CAT(0) space is a uniformly convex hyperbolic space with modulus of uniform convexity $\eta(r,\varepsilon)=\frac{\varepsilon^2}{8}$ quadratic in ε.

Now we give the concept of Δ-convergence and some of its basic properties.

Let M be a nonempty subset of metric space (Y,d) and $\{y_m\}$ be any bounded sequence in Y while $diam(M)$ denotes the diameter of M. Consider a continuous functional $r_a(.,\{y_m\}):Y\to\mathbb{R}^+$ defined by

$$r_a(y,\{y_m\})=\limsup_{m\to\infty} d(y_m,y),\quad y\in Y.$$

The infimum of $r_a(.,\{y_m\})$ over M is said to be an asymptotic radius of $\{y_m\}$ with respect to M and it is denoted by $r_a(M,\{y_m\})$. A point $z\in M$ is said to be an asymptotic center of the sequence $\{y_m\}$ with respect to M if

$$r_a(z,\{y_m\})=\inf\{r_a(y,\{y_m\}):y\in M\}.$$

The set of all asymptotic center of $\{y_m\}$ with respect to M is denoted by $AC(M,\{y_m\})$. The set $AC(M,\{y_m\})$ may be empty, singleton or have infinitely many points. If the asymptotic radius and asymptotic center are taken with respect to whole space Y, then they are denoted by $r_a(Y,\{y_m\})=r_a(\{y_m\})$ and $AC(Y,\{y_m\})=AC(\{y_m\})$, respectively. We know that for $y\in Y$, $r_a(y,\{y_m\})=0$ if and only if $\lim_{m\to\infty} y_m=y$ and every bounded sequence has a unique asymptotic center with respect to closed convex subset in uniformly convex Banach spaces.

Definition 3.6. The sequence $\{y_m\}$ in Y is said to be Δ-convergent to $y\in Y$, if y is unique asymptotic center of the every subsequence $\{u_m\}$ of $\{y_m\}$. In this case, we write $\Delta-\lim_{m\to\infty} y_m=y$ and call y is the Δ-limit of $\{y_m\}$.

Lemma 3.7. ([24]) *Let (Y,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_m\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset M of X.*

Consider the following lemma of Khan *et al.* [17] which we use in the sequel.

Lemma 3.8. *Let (Y,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x\in Y$ and $\{t_m\}$ be a sequence in $[a,b]$ for some $a,b\in(0,1)$. If $\{x_m\}$ and $\{y_m\}$ are sequences in Y such that*

$$\limsup_{m\to\infty} d(x_m,x)\le c,$$

$$\limsup_{m\to\infty} d(y_m,x)\le c$$

and

$$\limsup_{m\to\infty} d(W(x_m, y_m, t_m), x) = c,$$

for some $c \geq 0$, *then* $\lim_{m\to\infty} d(x_m, y_m) = 0$.

Definition 3.9. Let M be a nonempty convex closed subset of a hyperbolic space Y and $\{x_m\}$ be a sequence in Y. Then $\{x_m\}$ is said to be Fejér monotone with respect to M if for all $x \in M$ and $m \in \mathbb{N}$,

$$d(x_{m+1}, x) \leq d(x_m, x).$$

Now, we list some definitions and results for class of generalized α-nonexpansive mappings.

Assume that M is a nonempty subset of a hyperbolic space (Y, d) and $S : M \to M$ is a mapping and $F(S) = \{t \in M : St = t\}$ is the set of all fixed points of the map S. The mapping $S : M \to M$ is called nonexpansive, if $||St - Su|| \leq ||t - u||$ for all $t, u \in M$ and is called quasi-nonexpansive, if $F(S) \neq \emptyset$ and $||St - q|| \leq ||t - q||$ for all $t \in M$ and $q \in F(S)$.

We can easily prove the following proposition.

Proposition 3.10. *Let* $\{x_m\}$ *be a sequence in* Y *and* M *be a nonempty subset of* Y. *Let* $S : M \to M$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$. *Suppose that* $\{x_m\}$ *is Fejér monotone with respect to* M. *Then we have the followings:*

(1) $\{x_m\}$ *is bounded.*
(2) *The sequence* $\{d(x_m, p)\}$ *is decreasing and converges for all* $p \in F(S)$.
(3) $\lim_{m\to\infty} D(x_m, F(S))$ *exists, where* $D(x, A) = \inf_{y\in A} d(x, y)$.

The mapping satisfies the Condition (C), also known as Suzuki's generalized nonexpansive mapping introduced by Suzuki [30] in 2008.

Definition 3.11. Assume that M is nonempty subset of a Banach space Y. Then a mapping $S : M \to M$ is said to satisfy Condition (C), if

$$\frac{1}{2}||t - St|| \leq ||t - u|| \ \text{ implies } \ ||St - Su|| \leq ||t - u|| \tag{3.1}$$

for all $t, u \in M$.

Aoyama *et al.* [6] introduced the class of α-nonexpansive mappings which is the generalization of the class of nonexpansive mappings.

Definition 3.12. Assume that M is a nonempty subset of a Banach space Y. Then a mapping $S : M \to M$ is said to be α-nonexpansive, if there is $\alpha \in [0, 1)$ such that

$$||St - Su||^2 \leq \alpha||St - u||^2 + \alpha||Su - t||^2 + (1 - 2\alpha)||t - u||^2 \tag{3.2}$$

for all $t, u \in M$.

Definition 3.13. ([29]) Assume that M is a nonempty subset of a Banach space Y. Then a mapping $S : M \to M$ is said to be generalized α-nonexpansive, if $\frac{1}{2}||t - St|| \leq ||t - u||$, then there is $\alpha \in [0, 1)$ such that

$$||St - Su|| \leq \alpha||St - u|| + \alpha||Su - t|| + (1 - 2\alpha)||t - u|| \tag{3.3}$$

for all $t, u \in M$.

Consider the following results of Suanoom *et al.* [29].

Lemma 3.14. ([29]) *Assume that M is a nonempty subset of a hyperbolic space Y and $S : M \to M$ is generalized α-nonexpansive. Then for $t, u \in M$,*

(1) $||St - S^2t|| \leq ||t - St||$,
(2) *either* $\frac{1}{2}||t - St|| \leq ||t - u||$ *or* $\frac{1}{2}||St - S^2t|| \leq ||St - u||$,
(3) *either* $||St - Su|| \leq \alpha||St - u|| + \alpha||Su - t|| + (1 - 2\alpha)||t - u||$ *or* $||S^2t - Su|| \leq \alpha||S^2t - u|| + \alpha||Su - St|| + (1 - 2\alpha)||St - u||$.

Lemma 3.15. ([29]) *Assume that M is a nonempty subset of a hyperbolic space Y and $S : M \to M$ is generalized α-nonexpansive. Then for $t, u \in M$ with $t \leq u$.*

$$||t - St|| \leq \left(\frac{3 + \alpha}{1 - \alpha}\right)||t - St|| + ||t - u||.$$

Proposition 3.16. ([29]) *Assume that M is a nonempty subset of a hyperbolic space Y and $S : M \to M$ is a generalized α-nonexpansive mapping with $F(S) \neq \emptyset$. Then S is quasi-nonexpansive.*

Lemma 3.17. ([29]) *Let Y be complete uniformly convex hyperbolic space with monotone modulus of convexity η, M be a nonempty closed convex subset of Y and $S : M \to M$ be a generalized α-nonexpansive mapping. If $\{t_m\}$ is a bounded sequence in M such that $\lim_{m\to\infty} d(t_m, St_m) = 0$, then S has a fixed point in M.*

Lemma 3.18. ([29]) *Let M be a nonempty, bounded, closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and S be a generalized α-nonexpansive mapping on M. Suppose that $\{t_m\}$ is a sequence in M, with $d(t_m, St_m) \to 0$. If $AC(M, \{t_m\}) = u$, then u is a fixed point of S. Moreover, $F(S)$ is closed and convex.*

3.2. **Convergence Results.** Now, we establish the convergence results for new iteration process for generalized α-nonexpansive mappings in hyperbolic spaces, as follows: Let M be a nonempty, closed and convex subset of a hyperbolic space Y and S be a generalized α-nonexpansive mapping on M. For any $t_1 \in M$ the sequence $\{t_m\}$ is defined by

$$\begin{cases} t_{m+1} = W(Sx_m, 0, 0), \\ x_m = W(Sv_m, Su_m, \alpha_m), \\ u_m = W(Sv_m, 0, 0), \\ v_m = W(Sy_m, 0, 0), \\ y_m = W(t_m, St_m, \beta_m), \qquad \forall m \in \mathbb{N}. \end{cases} \tag{3.4}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0, 1)$.

Lemma 3.19. *Let M be a nonempty, closed and convex subset of a hyperbolic space Y and $S : M \to M$ be a generalized α-nonexpansive mapping. If $\{t_m\}$ is a sequence defined by* (3.4)*, then $\{t_m\}$ is Fejér monotone with respect to $F(S)$.*

Proof. Since S is a generalized α-nonexpansive, for $u \in F(S)$, we have

$$\frac{1}{2}d(u, Su) = 0 \leq d(u, t_m),$$

$$\frac{1}{2}d(u, Su) = 0 \leq d(u, u_m)$$

and

$$\frac{1}{2}d(u, Su) = 0 \leq d(u, v_m),$$

for all $m \in \mathbb{N}$. Now, also we have

$$d(Su, St_m) \leq \alpha d(Su, t_m) + \alpha d(St_m, u) + (1 - 2\alpha)d(u, t_m),$$

$$d(Su, Su_m) \leq \alpha d(Su, u_m) + \alpha d(Su_m, u) + (1 - 2\alpha)d(u, u_m)$$

and

$$d(Su, Sv_m) \leq \alpha d(Su, v_m) + \alpha d(Sv_m, u) + (1 - 2\alpha)d(u, v_m).$$

Now, using (3.4) and Lemma 3.16,

$$d(Su, St_m) \leq d(u, t_m),$$

$$d(Su, Su_m) \leq d(u, u_m)$$

and

$$d(Su, Sv_m) \leq d(u, v_m). \tag{3.5}$$

From (3.4) and Lemma 3.16,

$$\begin{aligned} d(v_m, u) &= d(W(Sy_m, 0, 0), u) \\ &= d(Sy_m, u) \\ &\leq d(y_m, u) \\ &= d(W(t_m, St_m, \beta_m), u) \\ &= (1-\beta_m)d(t_m, u) + \beta_m d(St_m, u) \\ &\leq (1-\beta_m)d(t_m, u) + \beta_m d(t_m, u) \\ &= d(t_m, u). \end{aligned} \tag{3.6}$$

From (3.4), (3.6) and Lemma 3.16,

$$\begin{aligned} d(u_m, u) &= d(W(Sv_m, 0, 0), u) \\ &= d(Sv_m, u) \\ &\leq d(v_m, u) \\ &\leq d(t_m, u). \end{aligned} \tag{3.7}$$

From (3.4), (3.6), (3.7) and Lemma 3.16,

$$\begin{aligned} d(t_{m+1}, u) &= d(W(Sx_m, 0, 0), u) \\ &= d(Sx_m, u) \\ &\leq d(x_m, u) \\ &= d(W(Sv_m, Su_m, \alpha_m), u) \\ &= (1-\alpha_m)d(Sv_m, u) + \alpha_m d(Su_m, u) \\ &\leq (1-\alpha_m)d(v_m, u) + \alpha_m d(u_m, u) \\ &\leq (1-\alpha_m)d(t_m, u) + \alpha_m d(t_m, u) \\ &= d(t_m, u). \end{aligned} \tag{3.8}$$

Hence, $\{t_m\}$ is Fejér monotone with respect to $F(S)$. □

Theorem 3.20. *Let M be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and S be a generalized α-nonexpansive mapping on M. If $\{t_m\}$ is a sequence defined by* (3.4)*, then F(S) is nonempty if and only if the sequence $\{t_m\}$ is bounded and* $\lim_{m\to\infty} d(t_m, St_m) = 0$.

Proof. Assume that $F(S)$ is nonempty and let $u \in F(S)$. From Lemma 3.19 and Proposition 3.10, we have $\{t_m\}$ is Fejér monotone with respect to $F(S)$ and bounded such that $\lim_{m\to\infty} D(t_m, F(S))$ exists, let $\lim_{m\to\infty} d(t_m, u) = l$.
Case I. Let $l = 0$. Then

$$d(t_m, St_m) \leq d(t_m, u) + d(u, St_m),$$

from Lemma 3.16,

$$d(t_m, St_m) \leq 2d(t_m, u).$$

On taking limit as $m \to \infty$ both sides of the inequality,

$$\lim_{m\to\infty} d(t_m, St_m) = 0.$$

Case II. Let $l > 0$. Then, since S is a generalized α-nonexpansive mapping, by Lemma 3.16, for $u \in F(S)$,

$$d(St_m, u) \leq d(t_m, u).$$

On taking lim sup as $m \to \infty$ both sides of the ineqality,

$$\limsup_{m\to\infty} d(St_m, u) \leq l.$$

On taking lim sup as $m \to \infty$ both sides of the (3.7),

$$\limsup_{m\to\infty} d(v_m, u) \leq l. \tag{3.9}$$

From (3.8),

$$\begin{aligned} d(t_{m+1}, u) &= d(W(Sx_m, 0, 0), u) \\ &= d(Sx_m, u) \\ &\leq d(x_m, u) \\ &= d(W(Sv_m, Su_m, \alpha_m), u) \\ &= (1-\alpha_m)d(Sv_m, u) + \alpha_m d(Su_m, u) \\ &\leq (1-\alpha_m)d(v_m, u) + \alpha_m d(u_m, u) \\ &\leq (1-\alpha_m)d(t_m, u) + \alpha_m d(u_m, u) \end{aligned}$$

which provides us

$$\begin{aligned} d(t_{m+1}, u) - d(t_m, u) &\leq \alpha_m(d(u_m, p) - d(t_m, u)), \\ d(t_{m+1}, u) - d(t_m, u) &\leq \frac{d(t_{m+1}, u) - d(t_m, u)}{\alpha_m} \\ &\leq d(u_m, u) - d(t_m, u), \\ d(t_{m+1}, u) &\leq d(u_m, u). \end{aligned}$$

On taking lim inf as $m \to \infty$ both sides of the inequality,

$$l \leq \liminf_{m\to\infty} d(u_m, u). \tag{3.10}$$

From (3.9) and (3.10),

$$\lim_{m\to\infty} d(u_m, u) = l.$$

On taking lim sup as $m \to \infty$ in (3.6),

$$\limsup_{m\to\infty} d(v_m, u) \leq l. \tag{3.11}$$

From (3.8),

$$\begin{aligned} d(t_{m+1}, u) &= d(W(Sx_m, 0, 0), u) \\ &= d(Sx_m, u) \\ &\leq d(x_m, u) \\ &= d(W(Sv_m, Su_m, \alpha_m), u) \\ &= (1-\alpha_m)d(Sv_m, u) + \alpha_m d(Su_m, u) \\ &\leq (1-\alpha_m)d(v_m, u) + \alpha_m d(u_m, u) \\ &\leq (1-\alpha_m)d(v_m, u) + \alpha_m d(v_m, u) \\ &= d(v_m, u). \end{aligned}$$

On taking lim inf as $m \to \infty$ both sides of the inequality,

$$l \leq \liminf_{m\to\infty} d(v_m, u). \tag{3.12}$$

From (3.11) and (3.12),

$$\lim_{m\to\infty} d(v_m, u) = l.$$

Therefore, by (3.6)

$$\begin{aligned} l &= \limsup_{m\to\infty} d(v_m, u) \\ &\leq \limsup_{m\to\infty} d(W(t_m, St_m, \beta_m), u) \\ &= \limsup_{m\to\infty} [(1-\beta_m)d(t_m, u) + \beta_m d(St_m, u)] \\ &\leq \limsup_{m\to\infty} [(1-\beta_m)d(t_m, u) + \beta_m d(t_m, u)] \\ &= \limsup_{m\to\infty} d(t_m, u) = l. \end{aligned}$$

By Lemma 3.8, $\lim_{m\to\infty} d(t_m, St_m) = 0$.

Conversely, assume that $\{t_m\}$ is bounded and $\lim_{m\to\infty} d(t_m, St_m) = 0$. Then, from Lemma 3.17, we have $Su = u$, that is, $F(S)$ is nonempty. □

Theorem 3.21. *Let M be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space Y with monotone modulus of uniform convexity η. Let $S : M \to M$ be a generalized α-nonexpansive mapping with $F(S) \neq \emptyset$. Then the sequence $\{t_m\}$ defined in* (3.4)*, is Δ-convergent to a fixed point of S.*

Proof. From Lemma 3.19, we observe that $\{t_m\}$ is a bounded sequence, therefore $\{t_m\}$ has a Δ-convergent subsequence. Now we will prove that every Δ-convergent subsequence of $\{t_m\}$ has a unique Δ-limit in $F(S)$. For this, let u and v be Δ-limits of the subsequences $\{u_m\}$ and $\{v_m\}$ of $\{t_m\}$ respectively.

Now by Lemma 3.7, $AC(M, \{u_m\}) = \{u_m\}$ and $AC(M, \{v_m\}) = \{v_m\}$. By Lemma 3.20, we have $\lim_{m\to\infty} d(u_m, Su_m) = 0$.

Now we will prove that u and v are fixed points of S and they are same. If not, then by the uniqueness of the asymptotic center

$$\begin{aligned}\limsup_{m\to\infty} d(t_m, u) &= \limsup_{m\to\infty} d(u_m, u)\\ &< \limsup_{m\to\infty} d(u_m, v)\\ &= \limsup_{m\to\infty} d(t_m, v)\\ &= \limsup_{m\to\infty} d(v_m, v)\\ &< \limsup_{m\to\infty} d(v_m, u)\\ &= \limsup_{m\to\infty} d(t_m, u),\end{aligned}$$

which is a contradiction. Hence $u = v$ and sequence $\{t_m\}$ is Δ-convergent to a unique fixed point of S. □

Theorem 3.22. *Let M be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $S : M \to M$ be a generalized α-nonexpansive mapping with $F(S) \neq \emptyset$. Then the sequence $\{t_m\}$ which is defined by* (3.4)*, converges strongly to some fixed point of S if and only if $\liminf_{m\to\infty} D(t_m, F(S)) = 0$, where $D(t_m, F(S)) = \inf_{u\in F(S)} d(t_m, u)$.*

Proof. Assume that $\{t_m\}$ converges strongly to $u \in F(S)$. Therefore we have $\lim_{m\to\infty} d(t_m, u) = 0$. Since $0 \leq D(t_m, F(S)) \leq d(t_m, u)$, we have

$$\liminf_{m\to\infty} D(t_m, F(S)) = 0.$$

Next, we prove sufficient part. From Lemma 3.18, the fixed point set $F(S)$ is closed. Suppose that

$$\liminf_{m\to\infty} D(t_m, F(S)) = 0.$$

Then, from (3.8), we have

$$D(t_{m+1}, F(S)) \leq D(t_m, F(S)).$$

From Lemma 3.19 and Proposition 3.10, we have $\lim_{m\to\infty} d(t_m, F(S))$ exists. Hence

$$\lim_{m\to\infty} D(t_m, F(S)) = 0.$$

Consider the subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $d(t_{m_k}, p_k) < \frac{1}{2^k}$ for all $k \geq 1$, where $\{p_k\}$ is in $F(S)$. From (3.7), we have

$$d(t_{m_{k+1}}, p_k) \leq d(t_{m_k}, p_k) < \frac{1}{2^k}$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, t_{n_{k+1}}) + d(t_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(S)$ is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{t_m\}$ converges to u. Since

$$d(t_{m_k}, u) \leq d(t_{m_k}, p_k) + d(p_k, u),$$

we have

$$\lim_{k\to\infty} d(t_{m_k}, u) = 0.$$

Since $\lim_{m\to\infty} d(t_m, u)$ exists, the sequence $\{t_m\}$ converges to u. □

Recall that a mapping S from a subset of a hyperbolic space Y into itself with $F(S) \neq \emptyset$ is said to satisfy *condition* (A) (see [19]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for $t \in (0, \infty)$ such that

$$d(x, Sx) \quad \geq \quad f(D(x, F(S))),$$

for all $x \in M$.

Theorem 3.23. *Let M be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space Y with monotone modulus of uniform convexity η and $S : M \to M$ be a generalized α-nonexpansive mapping. Moreover, S satisfies the condition (A) with $F(S) \neq \emptyset$. Then the sequence $\{t_m\}$ which is defined by* (3.4)*, converges strongly to some fixed point of S.*

Proof. From Lemma 3.18, we have $F(S)$ is closed. Observe that by Lemma 3.20, we have $\lim_{m\to\infty} d(t_m, St_m) = 0$. It follows from the condition (A) that

$$\lim_{m\to\infty} f(D(t_m, F(S))) \leq \lim_{m\to\infty} d(t_m, St_m).$$

Thus, we get $\lim_{m\to\infty} f(D(t_m, F(S))) = 0$. Since $f : [0, 1) \to [0, 1)$ is a nondecreasing mapping with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{m\to\infty} D(t_m, F(S)) = 0$. Rest of the proof follows in lines of Theorem 3.22. Hence the sequence $\{t_m\}$ is convergent to $p \in F(S)$. This completes the proof. □

4. Numerical results

Example 4.1. Assume that $M = [0,\infty)$ with usual norm $\|.\|$. The map $S : M \to M$ is defined as

$$St = \begin{cases} \frac{t}{2}, & \text{if } t > 4 \\ 0, & \text{if } t \in [0,4]. \end{cases}$$

Then, for $t = \frac{1}{4}$ and $u = \frac{17}{4}$, we have $\frac{1}{2}\|t - St\| = \frac{1}{8}$ and $\|t - u\| = 4$, therefore

$$\frac{1}{2}\|t - St\| < \|t - u\|.$$

Since $\|St - Su\| = \frac{17}{2}$, we have

$$\|St - Su\| > \|t - u\|,$$

which shows that S is not a Suzuki's generalized nonexpansive mapping.

We claim that S is a generalized α-nonexpansive mapping, for this consider the following cases with $\alpha = \frac{1}{3}$:

Case I: If $t > 4$ and $u > 4$, then

$$\begin{aligned} \frac{1}{3}\|St-u\|+\frac{1}{3}\|Su-t\|+\Big(1-\frac{2}{3}\Big)\|t-u\| &= \frac{1}{3}\Big\|\frac{t}{2}-u\Big\| + \frac{1}{3}\Big\|t-\frac{u}{2}\Big\| + \frac{1}{3}\|t-u\| \\ &\geq \frac{1}{3}\Big\|\frac{3t}{2}-\frac{3u}{2}\Big\| + \frac{1}{3}\|t-u\| \\ &= \frac{1}{2}\|t-u\| + \frac{1}{3}\|t-u\| \\ &\geq \frac{1}{2}\Big\|t-u\Big\| \\ &= \|St-Su\|. \end{aligned}$$

Case II: If $0 \leq t \leq 4$ and $0 \leq u \leq 4$, then

$$\begin{aligned} \frac{1}{3}\|St-u\|+\frac{1}{3}\|Su-t\|+\Big(1-\frac{2}{3}\Big)\|t-u\| &= \frac{1}{3}\|u\| + \frac{1}{3}\|t\| + \frac{1}{3}\|t-u\| \\ &\geq \|St-Su\|. \end{aligned}$$

Case III: If $t > 4$ and $0 \leq u \leq 4$, then

$$\begin{aligned}\frac{1}{3}\|St-u\| + \frac{1}{3}\|Su-t\| + \left(1-\frac{2}{3}\right)\|t-u\| &= \frac{1}{3}\left\|\frac{t}{2}-u\right\| + \frac{1}{3}\|t\| + \frac{1}{3}\|t-u\| \\ &\geq \frac{1}{3}\left\|\frac{3t}{2}+u\right\| + \frac{1}{3}\|t-u\| \\ &= \frac{1}{3}\left\|\frac{7t}{2}\right\| \\ &\geq \frac{1}{4}\|t\| \\ &= \|St-Su\|.\end{aligned}$$

Therefore S is a generalized α-nonexpansive mapping with fixed point 0.

We compare the convergence behaviour of new iteration process with Mann, Ishikawa, Noor, Normal S-iteration, Thakur, Piri iteration processes for generalized α-nonexpansive mapping defined in Example 4.1. We select the different set of parameters of α_m, β_m, γ_m and stopping criteria

$$||t_m - u^*|| \leq 10^{-17},$$

where $u^* \in F(S)$. In Figure 2 and Table 3, we examine the influence of initial values of theses iteration processes using $\alpha_m = \left(\frac{1}{m+7}\right)^{\frac{1}{2}}$, $\beta_m = \frac{2m}{5m+2}$ and $\gamma_m = \frac{m}{m+1}$.

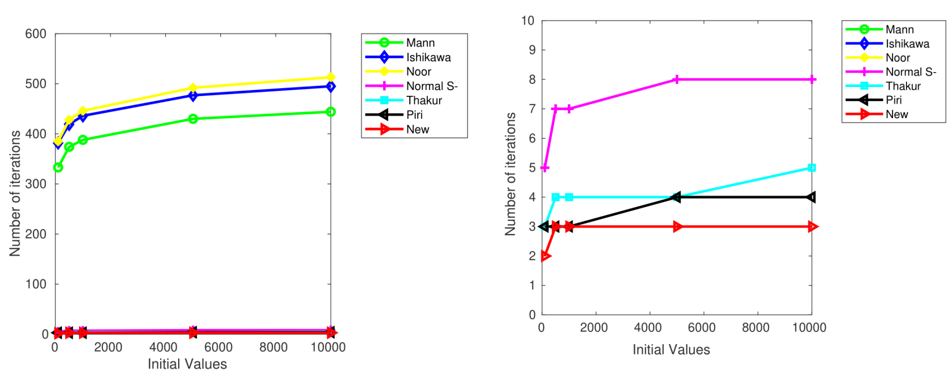

FIGURE 2. Comparison among different iteration processes for Example 4.1

TABLE 3. Comparison of number of iteration of different iteration process for different initial values

Initial Value	**New**	**Piri**	**Thakur**	**Normal S-iter.**	**Noor**	**Ishikawa**	**Mann**
100	2	3	3	5	385	382	333
500	3	3	4	7	428	419	374
1000	3	3	4	7	446	439	388
5000	3	4	4	8	492	477	430
10000	3	4	5	8	513	495	444

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