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GENERALIZED CONTRACTIONS VIA $\mathcal{Z}$-CONTRACTION

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Abstract. In this article, we introduce the concept of contractive mapping, which is generally weak in metric spaces, and show the existence and uniqueness of the fixed point for such mapping in a metric space.

1. Introduction

The metric fixed point theory has been expanded, changed and presented in various forms from Banach's contraction principle (see [1, 2, 3, 11, 12]).

Samet et al. [19] introduced the concept of α-ψ-contractive mapping. It defines the concept of accepting α-admissible and the use of the Bianchini Grandolfi gauge function [4], and the authors examined the existence and uniqueness of fixed points for mapping.

Khojasteh et al. [7] defines the concept of simulation and the new class defining function of nonlinear contraction, namely $\mathcal{Z}$-contractions which outlines Banach contraction principle and combines several known types of contractions. For other results on this interesting approach, see [5, 8, 9, 13, 14, 18].

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2. Preliminaries

Definition 2.1. ([19]) Let $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ be a self-mapping and $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ be a function. $\mathscr{Q}$ is said to be α-admissible if

$$\alpha(\mu, \rho) \geq 1 \;\; \Rightarrow \;\; \alpha(\mathscr{Q}\mu, \mathscr{Q}\rho) \geq 1, \quad \text{for all } \mu, \rho \in \mathscr{X}.$$

Definition 2.2. ([15]) Let $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ be a self-mapping and $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ be a function. $\mathscr{Q}$ is said to be α-orbital admissible if

$$\alpha(\mu, \mathscr{Q}\mu) \geq 1 \;\; \Rightarrow \;\; \alpha(\mathscr{Q}\mu, \mathscr{Q}\mu) \geq 1.$$

Moreover, $\mathscr{Q}$ is called triangular α-orbital admissible if it satisfies the following conditions:

(a) $\mathscr{Q}$ is α-orbital admissible.
(b) $\alpha(\mu, \rho) \geq 1$ and $\alpha(\rho, \mathscr{Q}\rho) \geq 1 \;\; \Rightarrow \;\; \alpha(\mu, \mathscr{Q}\rho) \geq 1$.

Definition 2.3. ([16]) If $\phi^n(\eta) \to 0$ as $n \to \infty$ for every $\eta \in [0, \infty)$, where ϕ^n is the n-th iterate of ϕ then an increasing function $\phi : [0, \infty) \to [0, \infty)$ is a comparison.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) ψ is nondecreasing.
(b) $\sum_{n=1}^{\infty} \psi^n(\eta) < \infty$ for all $\eta > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 2.4. ([16]) *If $\psi \in \Psi$, then the following hold:*

(a) *$\{\psi^n(\eta)\}$ converges to 0 as $n \to \infty$ for all $\eta \in \mathbb{R}^+$;*
(b) *$\psi(\eta) < \eta$, for any $\eta \in \mathbb{R}^+$;*
(c) *ψ is continuous at 0;*
(d) *the series $\sum_{n=1}^{\infty} \psi^n(\eta)$ converges for any $\eta \in \mathbb{R}^+$.*

Karapinar and Samet [6] introduced a generalized α-ψ contractive type mapping which is defined by

$$\alpha(\mu, \rho)\Lambda(\mathscr{Q}\mu, \mathscr{Q}\rho) \leq \psi(\mathscr{M}(\mu, \rho)), \quad \text{for all } \mu, \rho \in \mathscr{X},$$

where

$$\mathscr{M}(\mu, \rho) = \max\left\{\Lambda(\mu, \rho), \frac{\Lambda(\mu, \mathscr{Q}\mu) + \Lambda(\rho, \mathscr{Q}\rho)}{2}, \frac{\Lambda(\mu, \mathscr{Q}\rho) + \Lambda(\rho, \mathscr{Q}\mu)}{2}\right\},$$

$(\mathscr{X}, \Lambda)$ is a metric space, $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ is a given mapping, $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ and $\psi \in \Psi$.

Definition 2.5. ([7]) A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

($\zeta 1$) $\zeta(0,0)=0$;
($\zeta 2$) $\zeta(\eta,\vartheta)<\vartheta-\eta$ for all $\eta,\vartheta>0$;
($\zeta 3$) if $\{\eta_n\},\{\vartheta_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty}\eta_n=\lim_{n\to\infty}\vartheta_n>0$, then

$$\limsup_{n\to\infty}(\eta_n,\vartheta_n)<0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Let $(\mathscr{X},\Lambda)$ be a metric space, $\mathscr{Q}$ be a self-mapping on $\mathscr{X}$ and $\zeta\in\mathcal{Z}$. We say that $\mathscr{Q}$ is a $\mathcal{Z}$-contraction with respect to ζ [7], if

$$\zeta(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho),\Lambda(\mu,\rho))\geq 0,\quad \text{for all } \mu,\rho\in\mathscr{X}.$$

Theorem 2.6. ([7]) *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

Theorem 2.7. ([10]) *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ and a lower semi-continuous function $\varphi:\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho),\Lambda(\mu,\rho)+\varphi(\mu)+\varphi(\rho))\geq 0,$$

for all $\mu,\rho\in\mathscr{X}$. Then $\mathscr{Q}$ has a unique fixed point z such that $\varphi(z)=0$.

3. Main results

Theorem 3.1. *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ and $\varphi:X\to[0,\infty)$, $\psi\in\Psi$ and $\alpha:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)),\psi(\mathscr{M}(\mu,\rho)))\geq 0, \tag{3.1}$$

where

$$\begin{aligned}&\mathscr{M}(\mu,\rho)\\ &=\max\Big\{\Lambda(\mu,\rho)+\varphi(\mu)+\varphi(\rho),\ \Lambda(\mu,\mathscr{Q}\mu)+\varphi(\mu)+\varphi(\mathscr{Q}\mu),\\ &\qquad\Lambda(\rho,\mathscr{Q}\rho)+\varphi(\rho)+\varphi(\mathscr{Q}\rho),\\ &\qquad\frac{1}{2}\{\Lambda(\mu,\mathscr{Q}\rho)+\varphi(\mu)+\varphi(\mathscr{Q}\rho)+\Lambda(\rho,\mathscr{Q}\mu)+\varphi(\rho)+\varphi(\mathscr{Q}\mu)\}\Big\}\end{aligned} \tag{3.2}$$

and satisfies

(1) *$\mathscr{Q}$ is triangular α-orbital admissible*;
(2) *there exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;*
(3) *$\mathscr{Q}$ is continuous.*

Then there exists $z\in\mathscr{X}$ such that $z=\mathscr{Q}z$.

Proof. From the condition (2), there exists $u_0 \in \mathscr{X}$ such that $\alpha(\mu_0, \mathscr{Q}\mu_0) \geq 1$. Starting with this initial point $u_0 \in \mathscr{X}$ an iterative sequence $\{\mu_n\}$ is constructed by $\mu_{n+1} = \mathscr{Q}\mu_n$ for all $n \geq 0$. If $\mu_{m+1} = \mathscr{Q}\mu_m$ for some $m \in \mathbb{N}$, then μ_m is a fixed point of $\mathscr{Q}$. Thus, to continue our proof. Suppose that $\mu_n \neq \mu_{n+1}$ for all $n \in \mathbb{N}$. Using $\mathscr{Q}$ is α-orbital admissible, we obtain

$$\alpha(\mu_0, \mu_1) = \alpha(\mu_0, \mathscr{Q}\mu_0) \geq 1 \quad \Rightarrow \quad \alpha(\mathscr{Q}\mu_0, \mathscr{Q}\mu_1) = \alpha(\mu_1, \mu_2) \geq 1. \tag{3.3}$$

By induction, we get

$$\alpha(\mu_n, \mu_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Using (3.1) and (3.4), it follows that for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0 &\leq \zeta(\alpha(\mu_n, \mu_{n-1})(\Lambda(\mathscr{Q}\mu_n, \mathscr{Q}\mu_{n-1}) + \varphi(\mathscr{Q}\mu_n) + \varphi(\mathscr{Q}\mu_{n-1})), \psi(\mathscr{M}(\mu_n, \mu_{n-1}))) \\ &= \zeta(\alpha(\mu_n, \mu_{n-1})(\Lambda(\mu_{n+1}, \mu_n) + \varphi(\mu_{n+1}) + \varphi(\mu_n)), \psi(\mathscr{M}(\mu_n, \mu_{n-1}))) \\ &< \psi(\mathscr{M}(\mu_n, \mu_{n-1})) - [\alpha(\mu_n, \mu_{n-1})(\Lambda(\mu_{n+1}, \mu_n) + \varphi(\mu_{n+1}) + \varphi(\mu_n))]. \end{aligned} \tag{3.5}$$

The above inequality shows that

$$\begin{aligned} \Lambda(\mu_{n+1}, \mu_n) + \varphi(\mu_{n+1}) + \varphi(\mu_n) &\leq \alpha(\mu_n, \mu_{n-1})(\Lambda(\mu_{n+1}, \mu_n) + \varphi(\mu_{n+1}) + \varphi(\mu_n)) \\ &< \psi(\mathscr{M}(\mu_n, \mu_{n-1})) \\ &< \mathscr{M}(\mu_n, \mu_{n-1}), \end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} &\mathscr{M}(\mu_n, \mu_{n-1}) \\ &= \max\Big\{\Lambda(\mu_n, \mu_{n-1}) + \varphi(\mu_n) + \varphi(\mu_{n-1}), \Lambda(\mu_n, \mathscr{Q}\mu_n) + \varphi(\mu_n) + \varphi(\mathscr{Q}\mu_n), \\ &\qquad \Lambda(\mu_{n-1}, \mathscr{Q}\mu_{n-1}) + \varphi(\mu_{n-1}) + \varphi(\mathscr{Q}\mu_{n-1}), \\ &\qquad \frac{1}{2}\{\Lambda(\mu_n, \mathscr{Q}\mu_{n-1}) + \varphi(\mu_n) + \varphi(\mathscr{Q}\mu_{n-1}) \\ &\qquad + \Lambda(\mu_{n-1}, \mathscr{Q}\mu_n) + \varphi(\mu_{n-1}) + \varphi(\mathscr{Q}\mu_n)\}\Big\} \\ &= \max\Big\{\Lambda(\mu_n, \mu_{n-1}) + \varphi(\mu_n) + \varphi(\mu_{n-1}), \Lambda(\mu_n, \mu_{n+1}) + \varphi(\mu_n) + \varphi(\mu_{n+1}), \\ &\qquad \Lambda(\mu_{n-1}, \mu_n) + \varphi(\mu_{n-1}) + \varphi(\mu_n), \\ &\qquad \frac{1}{2}\{\Lambda(\mu_n, \mu_n) + \varphi(\mu_n) + \varphi(\mu_n) + \Lambda(\mu_{n-1}, \mu_{n+1}) + \varphi(\mu_{n-1}) + \varphi(\mu_{n+1})\}\Big\}. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}&\frac{1}{2}\{\Lambda(\mu_n,\mu_n)+\varphi(\mu_n)+\varphi(\mu_n)+\Lambda(\mu_{n-1},\mu_{n+1})+\varphi(\mu_{n-1})+\varphi(\mu_{n+1})\}\\&\le\frac{1}{2}\{\Lambda(\mu_n,\mu_{n+1})+\varphi(\mu_n)+\varphi(\mu_{n+1})+\Lambda(\mu_{n-1},\mu_n)+\varphi(\mu_{n-1})+\varphi(\mu_n)\}\\&\le\max\{\Lambda(\mu_n,\mu_{n+1})+\varphi(\mu_n)+\varphi(\mu_{n+1}),\Lambda(\mu_{n-1},\mu_n)+\varphi(\mu_{n-1})+\varphi(\mu_n)\},\end{aligned}\tag{3.8}$$

it follows from (3.6) that

$$\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n)<\mathscr{M}(\mu_n,\mu_{n-1}).\tag{3.9}$$

If $\mathscr{M}(\mu_n,\mu_{n-1})=\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n)$, then it follows from inequality (3.9) that

$$\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n)<\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n),$$

which is a contradiction. Therefore, we have

$$\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n)\ge\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n),$$

for all $n\in\mathbb{N}$, and so $\mathscr{M}(\mu_n,\mu_{n-1})=\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1})$. It follows from (3.6) that

$$\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n)<\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1}),$$

which implies that $\{\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1})\}$ is a decreasing sequence and bounded below by zero. Moreover, the inequality (3.6) turns into

$$\begin{aligned}\Lambda(\mu_n,\mu_{n+1})&\le\alpha(\mu_n,\mu_{n-1})(\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n))\\&<\psi(\mathscr{M}(\mu_n,\mu_{n-1}))<\mathscr{M}(\mu_n,\mu_{n-1})\\&<\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1}).\end{aligned}\tag{3.10}$$

Accordingly, there exists $R\ge0$ such that

$$\lim_{n\to\infty}[\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1})]=R\ge0.$$

We will show that have

$$\lim_{n\to\infty}\Lambda(\mu_n,\mu_{n-1})=0\quad\text{and}\quad\lim_{n\to\infty}\varphi(\mu_n)=0.\tag{3.11}$$

Suppose that $R>0$ from the inequality (3.10), we get

$$\lim_{n\to\infty}[\alpha(\mu_n,\mu_{n-1})(\Lambda(\mu_n,\mu_{n-1})+\varphi(\mu_n)+\varphi(\mu_{n-1}))]=R\tag{3.12}$$

and

$$\lim_{n\to\infty}\mathscr{M}(\mu_n,\mu_{n-1})=R.\tag{3.13}$$

It follows from the condition ($\zeta3$), with

$$\vartheta_n=\alpha(\mu_n,\mu_{n-1})(\Lambda(\mu_{n+1},\mu_n)+\varphi(\mu_{n+1})+\varphi(\mu_n))$$

and

$$\eta_n = \mathscr{M}(\mu_n, \mu_{n-1})$$

that

$$0 \le \limsup_{n\to\infty}[\alpha(\mu_n, \mu_{n-1})(\Lambda(\mu_{n+1}, \mu_n) + \varphi(\mu_{n+1}) + \varphi(\mu_n)), \mathscr{M}(\mu_n, \mu_{n-1})] < 0,$$

which is a contradiction. Therefore, we have $R = 0$ and from (3.12), since $\varphi \ge 0$, equation (3.11) holds.

Finally, we will show that $\{\mu_n\}$ is a Cauchy sequence in $\mathscr{X}$. Using the method of Reduction ad absurdum. Suppose to the contrary that $\{\mu_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$, for all $N \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ with $n > m > N$ and $\Lambda(\mu_m, \mu_n) > \varepsilon$. On the other hand, from (3.11), there exists $n_0 \in \mathbb{N}$ such that

$$\Lambda(\mu_n, \mu_{n+1}) < \varepsilon, \quad \text{for all } n > n_0. \tag{3.14}$$

We can find two subsequences $\{\mu_{n_k}\}$ and $\{\mu_{m_k}\}$ of $\{\mu_n\}$ such that

$$n_0 \le n_k \le m_k \quad \text{and} \quad \Lambda(\mu_{m_k}, \mu_{n_k}) > \varepsilon, \quad \text{for all } k, \tag{3.15}$$

where m_k is the smallest index satisfying (3.15). Thus

$$\Lambda(\mu_{m_k-1}, \mu_{n_k}) < \varepsilon, \quad \text{for all } k. \tag{3.16}$$

On account of (3.14), (3.15), and the triangular inequality, we get

$$\begin{aligned} \varepsilon &< \Lambda(\mu_{m_k}, \mu_{n_k}) \\ &\le \Lambda(\mu_{m_k}, \mu_{m_k-1}) + \Lambda(\mu_{m_k-1}, \mu_{n_k}) \\ &\le \Lambda(\mu_{m_k}, \mu_{m_k-1}) + \varepsilon, \text{ for all } k. \end{aligned} \tag{3.17}$$

Taking $k \to \infty$ and using equation (3.11), we obtain

$$\lim_{k\to\infty} \Lambda(\mu_{m_k}, \mu_{n_k}) = \varepsilon. \tag{3.18}$$

Using the triangle inequality, we derive that

$$\Lambda(\mu_{m_k}, \mu_{n_k}) \le \Lambda(\mu_{m_k}, \mu_{m_k+1}) + \Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \Lambda(\mu_{n_k+1}, \mu_{n_k}), \text{ for all } k.$$

So, we we have

$$\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) \le \Lambda(\mu_{m_k+1}, \mu_{m_k}) + \Lambda(\mu_{m_k}, \mu_{n_k}) + \Lambda(\mu_{n_k}, \mu_{n_k+1}), \text{ for all } k.$$

Combining the two inequalities above together with (3.11) and (3.17), we obtain

$$\lim_{k\to\infty} \Lambda(\mu_{m_k+1}, \mu_{n_k+1}) = \varepsilon. \tag{3.19}$$

Using the same reasoning as above, we get

$$\lim_{k\to\infty} \Lambda(\mu_{m_k}, \mu_{n_k+1}) = \lim_{k\to\infty} \Lambda(\mu_{m_k+1}, \mu_{n_k}) = \varepsilon. \tag{3.20}$$

Since $\mathscr{Q}$ is triangular α-orbital admissible, we have

$$\alpha(\mu_{m_k}, \mu_{n_k}) \geq 1. \tag{3.21}$$

Using (3.1), (3.19) and (3.20), we obtain

$$\begin{aligned} 0 &\leq \zeta(\alpha(\mu_{m_k}, \mu_{n_k})(\Lambda(\mathscr{Q}\mu_{m_k}, \mathscr{Q}\mu_{n_k}) + \varphi(\mathscr{Q}\mu_{m_k}) + \varphi(\mathscr{Q}\mu_{n_k})), \psi(\mathscr{M}(\mu_{m_k}, \mu_{n_k}))) \\ &= \zeta(\alpha(\mu_{m_k}, \mu_{n_k})(\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \varphi(\mu_{m_k+1}) + \varphi(\mu_{n_k+1})), \psi(\mathscr{M}(\mu_{m_k}, \mu_{n_k}))) \\ &< \psi(\mathscr{M}(\mu_{m_k}, \mu_{n_k}, \Lambda, \mathscr{Q}, \varphi)) \\ &\quad - [\alpha(\mu_{m_k}, \mu_{n_k})(\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \varphi(\mu_{m_k+1}) + \varphi(\mu_{n_k+1}))]. \end{aligned} \tag{3.22}$$

The above inequality shows that

$$\begin{aligned} &\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \varphi(\mu_{m_k+1}) + \varphi(\mu_{n_k+1}) \\ &\leq \alpha(\mu_{m_k}, \mu_{n_k})(\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \varphi(\mu_{m_k+1}) + \varphi(\mu_{n_k+1})) \\ &< \psi(\mathscr{M}(\mu_{m_k}, \mu_{n_k})) < \mathscr{M}(\mu_{m_k}, \mu_{n_k}), \end{aligned} \tag{3.23}$$

for all $k \geq n_1$, where

$$\begin{aligned} &\mathscr{M}(\mu_{m_k}, \mu_{n_k}) \\ &= \max\Big\{\Lambda(\mu_{m_k}, \mu_{n_k}) + \varphi(\mu_{m_k}) + \varphi(\mu_{n_k}), \Lambda(\mu_{m_k}, \mathscr{Q}\mu_{m_k}) + \varphi(\mu_{m_k}) + \varphi(\mathscr{Q}\mu_{m_k}), \\ &\qquad \Lambda(\mu_{n_k}, \mathscr{Q}\mu_{n_k}) + \varphi(\mu_{n_k}) + \varphi(\mathscr{Q}\mu_{n_k}), \\ &\qquad \frac{1}{2}\{\Lambda(\mu_{m_k}, \mathscr{Q}\mu_{n_k}) + \varphi(\mu_{m_k}) + \varphi(\mathscr{Q}\mu_{n_k}) \\ &\qquad + \Lambda(\mu_{n_k}, \mathscr{Q}\mu_{m_k}) + \varphi(\mu_{n_k}) + \varphi(\mathscr{Q}\mu_{m_k})\}\Big\} \\ &= \max\Big\{\Lambda(\mu_{m_k}, \mu_{n_k}) + \varphi(\mu_{m_k}) + \varphi(\mu_{n_k}), \Lambda(\mu_{m_k}, \mu_{m_k+1}) + \varphi(\mu_{m_k}) + \varphi(\mu_{m_k+1}), \\ &\qquad \Lambda(\mu_{n_k}, \mu_{n_k+1}) + \varphi(\mu_{n_k}) + \varphi(\mu_{n_k+1}), \\ &\qquad \frac{1}{2}\{\Lambda(\mu_{m_k}, \mu_{n_k+1}) + \varphi(\mu_{m_k}) + \varphi(\mu_{n_k+1}) \\ &\qquad + \Lambda(\mu_{n_k}, \mu_{m_k+1}) + \varphi(\mu_{n_k}) + \varphi(\mu_{m_k+1})\}\Big\}. \end{aligned} \tag{3.24}$$

Taking the limit as $k \to \infty$ in (3.24) and using (3.11), (3.18), (3.19) and (3.20), we find that

$$\lim_{k\to\infty} \mathscr{M}(\mu_{m_k}, \mu_{n_k}) = \varepsilon. \tag{3.25}$$

It follows from the condition ($\zeta 3$), with

$$\vartheta_n = \alpha(\mu_{m_k}, \mu_{n_k})(\Lambda(\mu_{m_k+1}, \mu_{n_k+1}) + \varphi(\mu_{m_k+1}) + \varphi(\mu_{n_k+1})) \to \varepsilon$$

and $\eta_n = \mathscr{M}(\mu_{m_k},+\mu_{n_k}) \to \varepsilon$ that

$$0 \leq \limsup_{k\to\infty}[\alpha(\mu_{m_k},\mu_{n_k})(\Lambda(\mu_{m_k+1},\mu_{n_k+1})+\varphi(\mu_{m_k+1})+\varphi(\mu_{n_k+1})), \mathscr{M}(\mu_{m_k}, \mu_{n_k})] < 0,$$

which is a contradiction. Therefore, $\{\mu_n\}$ is a Cauchy sequence. Owing to the fact that $(\mathscr{X}, \Lambda)$ is a complete metric space, there exists $z \in \mathscr{X}$ such that

$$\lim_{n\to\infty} \Lambda(\mu_n, z) = 0. \tag{3.26}$$

Since $\mathscr{Q}$ is continuous, we derive from (3.26) that

$$\lim_{n\to\infty} \Lambda(\mu_{n+1}, \mathscr{Q}z) = \lim_{n\to\infty} \Lambda(\mathscr{Q}\mu_n, \mathscr{Q}z) = 0. \tag{3.27}$$

Taking into account (3.26), (3.27), and the uniqueness of the limit, we conclude that z is a fixed point of $\mathscr{Q}$, that is, $z = \mathscr{Q}z$. □

Theorem 3.2. *Let $(\mathscr{X}, \Lambda)$ be a complete metric space and let $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ, and $\varphi : \mathscr{X} \to [0, \infty)$, $\psi \in \Psi$ and $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ such that*

$$\zeta(\alpha(\mu, \rho)(\Lambda(\mathscr{Q}\mu, \mathscr{Q}\rho) + \varphi(\mathscr{Q}\mu) + \varphi(\mathscr{Q}\rho)), \psi(\mathscr{M}(\mu, \rho))) \geq 0, \tag{3.28}$$

where

$$\begin{aligned}\mathscr{M}(\mu, \rho) &= \max\Big\{\Lambda(\mu, \rho) + \varphi(\mu) + \varphi(\rho), \Lambda(\mu, \mathscr{Q}\mu) + \varphi(\mu) + \varphi(\mathscr{Q}\mu),\\ &\quad \Lambda(\rho, \mathscr{Q}\rho) + \varphi(\rho) + \varphi(\mathscr{Q}\rho),\\ &\quad \frac{1}{2}\{\Lambda(\mu, \mathscr{Q}\rho) + \varphi(\mu) + \varphi(\mathscr{Q}\rho) + \Lambda(\rho, \mathscr{Q}\mu) + \varphi(\rho) + \varphi(\mathscr{Q}\mu)\}\Big\}\end{aligned} \tag{3.29}$$

and satisfies

1. *$\mathscr{Q}$ is triangular α-orbital admissible*;
2. *there exists $\mu_0 \in \mathscr{X}$ such that $\alpha(\mu_0, \mathscr{Q}\mu_0) \geq 1$;*
3. *If $\{\mu_n\}$ is a sequence in $\mathscr{X}$ such that $\alpha(\mu_n, \mu_{n+1}) \geq 1$ for all n and $\mu_n \to \mu \in \mathscr{X}$ as $n \to \infty$, then there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\alpha(\mu_{n_k}, \mu) \geq 1$ for all k.*

Then there exists $z \in \mathscr{X}$ such that $z = \mathscr{Q}z$.

Proof. Similarly, in the proof of Theorem 3.1, we know that the sequence $\{\mu_n\}$ defined by $\mu_{n+1} = \mathscr{Q}\mu_n$ for all $n \in \mathbb{N}$, is a Cauchy sequence in $\mathscr{X}$. Since $(\mathscr{X}, \Lambda)$ is complete, $\{\mu_n\}$ converges for some $z \in \mathscr{X}$. Since φ is lower semicontinuous, we have

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(\mu_n) \leq \lim_{n\to\infty} \varphi(\mu_n) = 0,$$

which implies

$$\varphi(z) = 0. \tag{3.30}$$

By (3.4) and condition (2), there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\alpha(\mu_{n_k}, z) \geq 1$ for all k. Using (3.28), for all k, we get

$$\begin{aligned} 0 &\leq \zeta(\alpha(\mu_{n_k}, z)(\Lambda(\mathscr{Q}\mu_{n_k}, \mathscr{Q}z) + \varphi(\mathscr{Q}\mu_{n_k}) + \varphi(\mathscr{Q}z)), \psi(\mathscr{Q}(\mu_{n_k}, z))) \\ &= \zeta(\alpha(\mu_{n_k}, z)(\Lambda(\mu_{n_k+1}, \mathscr{Q}z) + \varphi(\mu_{n_k+1}) + \varphi(\mathscr{Q}z)), \psi(\mathscr{Q}(\mu_{n_k}, z))) \\ &< \psi(\mathscr{M}(\mu_{n_k}, z)) - [\alpha(\mu_{n_k}, z)(\Lambda(\mu_{n_k+1}, \mathscr{Q}z) + \varphi(\mu_{n_k+1}) + \varphi(\mathscr{Q}z))]. \end{aligned}$$

This inequality shows that

$$\begin{aligned} &\Lambda(\mu_{n_k+1}, \mathscr{Q}z) + \varphi(\mu_{n_k+1}) + \varphi(\mathscr{Q}z) \\ &\leq \alpha(\mu_{n_k}, z)(\Lambda(\mu_{n_k+1}, \mathscr{Q}z) + \varphi(\mu_{n_k+1}) + \varphi(\mathscr{Q}z)) \\ &< \psi(\mathscr{M}(\mu_{n_k}, z)) \\ &< \mathscr{M}(\mu_{n_k}, z), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} &\mathscr{M}(\mu_{n_k}, z) \\ &= \max\Big\{\Lambda(\mu_{n_k}, z) + \varphi(\mu_{n_k}) + \varphi(z), \Lambda(\mu_{n_k}, \mathscr{Q}\mu_{n_k}) + \varphi(\mu_{n_k}) + \varphi(\mathscr{Q}\mu_{n_k}), \\ &\qquad \Lambda(z, \mathscr{Q}z) + \varphi(z) + \varphi(\mathscr{Q}z), \\ &\qquad \frac{1}{2}\{\Lambda(\mu_{n_k}, \mathscr{Q}z) + \varphi(\mu_{n_k})) + \varphi(\mathscr{Q}z)) + \Lambda(z, \mathscr{Q}\mu_{n_k})) + \varphi(z)) + \varphi(\mathscr{Q}\mu_{n_k})\}\Big\}. \end{aligned}$$

Taking $k \to \infty$ in the above equality, we have

$$\lim_{k\to\infty} \mathscr{M}(\mu_{n_k}, z) = \Lambda(z, \mathscr{Q}z) + \varphi(\mathscr{Q}z). \tag{3.32}$$

Suppose that $\Lambda(z, \mathscr{Q}z) > 0$. Taking $k \to \infty$, using (3.31), (3.32) and the continuity of φ, we get

$$\lim_{k\to\infty} \Lambda(\mu_{n_k+1}, \mathscr{Q}z) + \varphi(\mu_{n_k+1}) + \varphi(\mathscr{Q}z) < \lim_{k\to\infty} \mathscr{M}(\mu_{n_k}, z). \tag{3.33}$$

So,

$$\Lambda(z, \mathscr{Q}z) + \varphi(\mathscr{Q}z) < \Lambda(z, \mathscr{Q}z) + \varphi(\mathscr{Q}z), \tag{3.34}$$

which is a contradiction, and hence, $\Lambda(z, \mathscr{Q}z) = 0$, that is, $z = \mathscr{Q}z$ and $\varphi(\mathscr{Q}z) = 0$. Since $z = \mathscr{Q}z$ this implies $\varphi(z) = 0$. □

The following theorem is for the uniqueness of the fixed point of the mapping $\mathscr{Q}$.

Theorem 3.3. *For all $\mu, \rho \in Fix(\mathscr{Q})$, we have $\alpha(\mu, \rho) \geq 1$, where $Fix(\mathscr{Q})$ denotes the set of fixed points of $\mathscr{Q}$. If the hypotheses of Theorem* 3.1 *(resp., Theorem* 3.2*) are hold, then $\mathscr{Q}$ has a unique fixed point in $\mathscr{X}$.*

Proof. Suppose z^* is another fixed point of $\mathscr{Q}$. Then $z^* = \mathscr{Q}z^*$ and $\varphi(z^*) = 0$. From assumption, we have

$$\alpha(z, z^*) \geq 1. \tag{3.35}$$

It follows from equation (3.1) and ($\zeta 2$) that

$$\begin{aligned} 0 &\leq \zeta(\alpha(z, z^*)(\Lambda(\mathscr{Q}z, \mathscr{Q}z^*) + \varphi(\mathscr{Q}z) + \varphi(\mathscr{Q}z^*)), \psi(\mathscr{M}(z, z^*))) \\ &= \zeta(\alpha(z, z^*)(\Lambda(z, z^*) + \varphi(z) + \varphi(z^*)), \psi(\mathscr{M}(z, z^*))) \\ &< \psi(\mathscr{Q}(z, z^*)) - [\alpha(z, z^*)(\Lambda(z, z^*) + \varphi(z) + \varphi(z^*))], \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} &\mathscr{M}(z, z^*) \\ &= \max\Big\{\Lambda(z, z^*) + \varphi(z) + \varphi(z^*), \Lambda(z, \mathscr{Q}z) + \varphi(z) + \varphi(\mathscr{Q}z), \\ &\qquad \Lambda(z^*, \mathscr{Q}z^*) + \varphi(z^*) + \varphi(\mathscr{Q}z^*), \\ &\qquad \frac{1}{2}\{\Lambda(z, \mathscr{Q}z^*) + \varphi(z) + \varphi(\mathscr{Q}z^*) + \Lambda(z^*, \mathscr{Q}z) + \varphi(z^*) + \varphi(\mathscr{Q}z)\}\Big\} \\ &= \max\Big\{\Lambda(z, z^*) + \varphi(z) + \varphi(z^*), \Lambda(z, z) + \varphi(z) + \varphi(z), \\ &\qquad \Lambda(z^*, z^*) + \varphi(z^*) + \varphi(z^*), \\ &\qquad \frac{1}{2}\{\Lambda(z, z^*) + \varphi(z) + \varphi(z^*) + \Lambda(z^*, z) + \varphi(z^*) + \varphi(z)\}\Big\} \\ &= \Lambda(z^*, z). \end{aligned} \tag{3.37}$$

Using (3.36) and (3.37), we obtain

$$0 < \Lambda(z, z^*) - \alpha(z, z^*)\Lambda(z, z^*). \tag{3.38}$$

Therefore, we have

$$\Lambda(z, z^*) \leq \alpha(z, z^*)\Lambda(z, z^*) < \Lambda(z, z^*), \tag{3.39}$$

which is a contradiction. Thus $z = z^*$. This completes the proof for the uniqueness. □

4. Consequences

Corollary 4.1. *Let $(\mathscr{X}, \Lambda)$ be a complete metric space and let $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ be a mapping. Suppose that there exist a function $\varphi : \mathscr{X} \to [0, \infty)$, $\psi \in \Psi$ and $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ such that*

$$\alpha(\mu, \rho)(\Lambda(\mathscr{Q}\mu, \mathscr{Q}\rho) + \varphi(\mathscr{Q}\mu) + \varphi(\mathscr{Q}\rho)) \leq \psi(\mathscr{M}(\mu, \rho)),$$

where

$$\begin{aligned}&\mathscr{M}(\mu,\rho)\\&=\max\Big\{\Lambda(\mu,\rho)+\varphi(\mu)+\varphi(\rho),\Lambda(\mu,\mathscr{Q}\mu)+\varphi(\mu)+\varphi(\mathscr{Q}\mu),\\&\qquad\Lambda(\rho,\mathscr{Q}\rho)+\varphi(\rho)+\varphi(\mathscr{Q}\rho),\\&\qquad\frac{1}{2}\{\Lambda(\mu,\mathscr{Q}\rho)+\varphi(\mu)+\varphi(\mathscr{Q}\rho)+\Lambda(\rho,\mathscr{Q}\mu)+\varphi(\rho)+\varphi(\mathscr{Q}\mu)\}\Big\}\end{aligned}$$

and satisfies

(1) $\mathscr{Q}$ *is triangular* α*-orbital admissible*;
(2) *there exists* $\mu_0\in\mathscr{X}$ *such that* $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;
(3) $\mathscr{Q}$ *is continuous.*

Then there exists $z\in\mathscr{X}$ *such that* $z=\mathscr{Q}z$.

Proof. By taking as simulation function

$$\zeta(\eta,\vartheta)=\psi(\vartheta)-\eta,\quad \text{for all } \eta,\vartheta\geq 0$$

and following the proof of Theorem 3.1, then we can prove the corollary. □

Corollary 4.2. *Let* $(\mathscr{X},\Lambda)$ *be a complete metric space and let* $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ *be a mapping. Suppose that there exist a function* $\varphi:\mathscr{X}\to[0,\infty)$, $\psi\in\Psi$ *such that*

$$\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)\leq\psi(\mathscr{M}(\mu,\rho)),$$

where

$$\begin{aligned}&\mathscr{M}(\mu,\rho)\\&=\max\Big\{\Lambda(\mu,\rho)+\varphi(\mu)+\varphi(\rho),\Lambda(\mu,\mathscr{Q}\mu)+\varphi(\mu)+\varphi(\mathscr{Q}\mu),\\&\qquad\Lambda(\rho,\mathscr{Q}\rho)+\varphi(\rho)+\varphi(\mathscr{Q}\rho),\\&\qquad\frac{1}{2}\{\Lambda(\mu,\mathscr{Q}\rho)+\varphi(\mu)+\varphi(\mathscr{Q}\rho)+\Lambda(\rho,\mathscr{Q}\mu)+\varphi(\rho)+\varphi(\mathscr{Q}\mu)\}\Big\}\end{aligned}$$

and satisfies

(1) $\mathscr{Q}$ *is triangular* α*-orbital admissible*;
(2) *there exists* $\mu_0\in\mathscr{X}$ *such that* $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;
(3) $\mathscr{Q}$ *is continuous.*

Then there exists $z\in\mathscr{X}$ *such that* $z=\mathscr{Q}z$.

Proof. Take $\alpha(\mu,\rho)=1$ for all $\mu,\rho\in\mathscr{X}$ in Corollary 4.1. □

We can easily prove the two corollaries from the Theorem 3.1.

Corollary 4.3. *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ, $\varphi:\mathscr{X}\to[0,\infty)$ and $\alpha:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)),\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)))\geq 0,$$

and satisfies

(1) *$\mathscr{Q}$ is triangular α-orbital admissible*;
(2) *there exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;*
(3) *$\mathscr{Q}$ is continuous.*

Then there exists $z\in\mathscr{X}$ such that $z=\mathscr{Q}z$.

Corollary 4.4. *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ, $\varphi:\mathscr{X}\to[0,\infty)$ and $\alpha:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)),\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)))\geq 0,$$

and satisfies

(1) *$\mathscr{Q}$ is triangular α-orbital admissible*;
(2) *there exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;*
(3) *If $\{\mu_n\}$ is a sequence in $\mathscr{X}$ such that $\alpha(\mu_n,\mu_{n+1})\geq 1$ for all n and $\mu_n\to\mu\in\mathscr{X}$ as $n\to\infty$, then there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\alpha(\mu_{n_k},\mu)\geq 1$ for all k.*

Then there exists $z\in\mathscr{X}$ such that $z=\mathscr{Q}z$.

Corollary 4.5. *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ and $\varphi:\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho),\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)))\geq 0,$$

and satisfies

(1) *$\mathscr{Q}$ is triangular α-orbital admissible*;
(2) *there exists $\mu_0\in\mathscr{X}$ such that $\alpha(\mu_0,\mathscr{Q}\mu_0)\geq 1$;*
(3) *$\mathscr{Q}$ is continuous.*

Then there exists $z\in\mathscr{X}$ such that $z=\mathscr{Q}z$.

Proof. Take $\alpha(\mu,\rho)=1$ for all $\mu,\rho\in\mathscr{X}$ in Corollary 4.3. □

Corollary 4.6. *Let $(\mathscr{X},\Lambda)$ be a complete metric space and let $\mathscr{Q}:\mathscr{X}\to\mathscr{X}$ be a mapping. Suppose that there exist a simulation function ζ and $\varphi:\mathscr{X}\to[0,\infty)$ such that*

$$\zeta(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho),\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)))\geq 0,$$

and satisfies

(1) *$\mathscr{Q}$ is triangular α-orbital admissible*;
(2) *there exists $x_0 \in \mathscr{X}$ such that $\alpha(\mu_0, \mathscr{Q}\mu_0) \geq 1$;*
(3) *If $\{\mu_n\}$ is a sequence in $\mathscr{X}$ such that $\alpha(\mu_n, \mu_{n+1}) \geq 1$ for all n and $\mu_n \to \mu \in \mathscr{X}$ as $n \to \infty$, then there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\alpha(\mu_{n_k}, \mu) \geq 1$ for all k.*

Then there exists $z \in \mathscr{X}$ such that $z = \mathscr{Q}z$.

Proof. Take $\alpha(\mu, \rho) = 1$ for all $\mu, \rho \in \mathscr{X}$ in Corollary 4.4. □

5. Illustrative example

Example 5.1. Let $\mathscr{X} = [0, \infty)$ and the metric be defined by the usual metric. Let $\psi(\eta) = \dfrac{5\eta}{4}$ for $\eta > 0$, and let

$$\varphi(\eta) = \begin{cases} \dfrac{\eta}{6}, & \text{if } 0 \leq \eta \leq 1, \\ \dfrac{\eta}{6} + \dfrac{1}{6}, & \text{if } 1 \leq \eta \leq 6, \\ \eta, & \text{if } \eta \geq 6. \end{cases}$$

Then $\psi \in \Psi$, φ is lower semicontinuous, and $\dfrac{\eta}{6} \leq \varphi(\eta) \leq \eta$, $\eta \geq 0$.

The mapping $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ is defined by $\mathscr{Q}\mu = \dfrac{3\mu^2}{6 + 6\mu}$. Define a function $\alpha : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$\alpha(\mu, \rho) = \begin{cases} 1, & \text{if } 0 \leq \mu, \rho \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\zeta(\mu, \rho) = \lambda\rho - \mu$, $\lambda \in [0, 1)$. We now show that Theorem 3.1 holds. Without loss of generality, assume that $\mu \geq \rho$. Then we obtain

$$\begin{aligned} &\frac{1}{2}\left\{\Lambda(\mu, \mathscr{Q}\rho) + \varphi(\mu) + \varphi(\mathscr{Q}\rho) + \Lambda(\rho, \mathscr{Q}\mu) + \varphi(\rho) + \varphi(\mathscr{Q}\mu)\right\} \\ &\geq \frac{1}{2}\left\{\Lambda(\mu, \mathscr{Q}\rho) + \frac{\mu}{6} + \frac{\mathscr{Q}\rho}{6} + \Lambda(\rho, \mathscr{Q}\mu) + \frac{\rho}{6} + \frac{\mathscr{Q}\mu}{6}\right\} \\ &\geq \frac{1}{2}\left\{\frac{1}{6}\left\{\Lambda(\mu, \mathscr{Q}\rho) + \mu + \mathscr{Q}\rho + \Lambda(\rho, \mathscr{Q}\mu) + \rho + \mathscr{Q}\mu\right\}\right\} \\ &= \begin{cases} \dfrac{1}{6}\left(\mu + \dfrac{\mu^2}{1 + \mu}\right), & \text{if } \rho \leq \dfrac{3\mu^2}{6 + 6\mu} \\ \dfrac{1}{6}\left(\mu + \rho\right), & \text{otherwise} \end{cases} \\ &> \frac{1}{6}\mu. \end{aligned}$$

Also, we obtain

$$\begin{aligned}
&\mathscr{M}(\mu,\rho)\\
&= \max\Big\{\Lambda(\mu,\rho)+\varphi(\mu)+\varphi(\rho),\ \Lambda(\mu,\mathscr{Q}\mu)+\varphi(\mu)+\varphi(\mathscr{Q}\mu),\\
&\qquad \Lambda(\rho,\mathscr{Q}\rho)+\varphi(\rho)+\varphi(\mathscr{Q}\rho),\\
&\qquad \frac{1}{2}\{\Lambda(\mu,\mathscr{Q}\rho)+\varphi(\mu)+\varphi(\mathscr{Q}\rho)+\Lambda(\rho,\mathscr{Q}\mu)+\varphi(\rho)+\varphi(\mathscr{Q}\mu)\}\Big\}\\
&\geq \frac{1}{6}\max\Big\{\Lambda(\mu,\rho)+\mu+\rho,\ \Lambda(\mu,\mathscr{Q}\mu)+\mu+\mathscr{Q}\mu,\\
&\qquad \Lambda(\rho,\mathscr{Q}\rho)+\rho+\mathscr{Q}\rho,\\
&\qquad \frac{1}{2}\{\Lambda(\mu,\mathscr{Q}\rho)+\mu+\mathscr{Q}\rho+\Lambda(\rho,\mathscr{Q}\mu)+\rho+\mathscr{Q}\mu\}\Big\}\\
&= \frac{1}{6}\max\Big\{2\mu,2\mu,2\rho,\frac{1}{6}\mu\Big\}\\
&= \frac{1}{3}\mu
\end{aligned}$$

and

$$\begin{aligned}
\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)) &\leq \Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)\\
&\leq \Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\mathscr{Q}\mu+\mathscr{Q}\rho\\
&\leq \left|\frac{3\mu^2}{6+6\mu}-\frac{3\rho^2}{6+6\rho}\right|+\frac{3\mu^2}{6+6\mu}+\frac{3\rho^2}{6+6\rho}\\
&= \frac{\mu^2}{1+\mu}.
\end{aligned}$$

Hence, for $\lambda\in[0,1)$ we obtain

$$\begin{aligned}
&\zeta(\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho)),\psi(\mathscr{M}(\mu,\rho)))\\
&= \lambda\psi(\mathscr{M}(\mu,\rho)))-\alpha(\mu,\rho)(\Lambda(\mathscr{Q}\mu,\mathscr{Q}\rho)+\varphi(\mathscr{Q}\mu)+\varphi(\mathscr{Q}\rho))\\
&\geq \frac{5}{4}\lambda\left(\frac{1}{3}\mu\right)-\frac{\mu^2}{1+\mu}\\
&= \frac{5\lambda\mu}{12}-\frac{\mu^2}{1+\mu}\geq 0.
\end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied, then $\mathscr{Q}$ has a unique fixed point which is 0.

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