Nonlinear Functional Analysis and Applications
Vol. 27, No. 3 (2022), pp. 499-512
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.03.03
http://nfaa.kyungnam.ac.kr/journal-nfaa

CONVERGENCE THEOREMS FOR TWO NONLINEAR MAPPINGS IN CAT(0) SPACES

Kritsana Sokhuma[1] **and Kasinee Sokhuma**[2]

[1]Department of Mathematics, Faculty of Science and Technology,
Phranakhon Rajabhat University, Bangkok 10220, Thailand
e-mail: k_sokhuma@yahoo.co.th

[2]Department of Educational Research and Measurement, Faculty of Education,
Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand
e-mail: kate_kasinee@hotmail.com

Abstract. In this paper, we construct an iteration scheme involving a hybrid pair of the Suzuki generalized nonexpansive single-valued and multi-valued mappings in a complete $CAT(0)$ space. In process, we remove a restricted condition (called end-point condition) in Akkasriworn and Sokhuma's results [2] in Banach spaces and utilize the same to prove some convergence theorems. The results in this paper, are analogs of the results of Akkasriworn et al. [3] in Banach spaces.

1. Introduction

Fixed point theory in a $CAT(0)$ space was first studied by Kirk [12, 13]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the existence problem of fixed point and the Δ-convergence problem of iterative sequences to a fixed point for nonexpansive mappings, Suzuki generalized nonexpansive mappings in a $CAT(0)$ space have been rapidly developed and have appeared in many papers [4, 10, 11, 18].

[0]Received February 26, 2021. Revised March 19, 2021. Accepted March 31, 2021.
[0]2020 Mathematics Subject Classification: 47H09, 47H10.
[0]Keywords: Ishikawa iteration, $CAT(0)$ spaces, Suzuki generalized nonexpansive mapping, multi-valued mapping.
[0]Corresponding author: Kritsana Sokhuma(k_sokhuma@yahoo.co.th).

Let (X, d) be a geodesic metric space. We denote by 2^K the family of nonempty subsets of K, by $FB(K)$ the collection of all nonempty closed bounded subsets of K, by $KC(K)$ the collection of all nonempty compact convex subsets of K.

A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = \operatorname{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

We denote by $PB(K)$, the collection of all nonempty bounded proximinal subsets of K.

Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\left\{ \sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A) \right\}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(x, B) = \inf\{d(x, y) : y \in B\}$ is the distance from the point x to the subset B.

A mapping $t : K \to K$ is said to be *nonexpansive* if

$$d(tx, ty) \leq d(x, y) \text{ for all } x, y \in K.$$

A mapping $t : K \to K$ is said to be *Suzuki generalized nonexpansive* if

$$\frac{1}{2} d(x, tx) \leq d(x, y) \Rightarrow d(tx, ty) \leq d(x, y) \text{ for all } x, y \in K.$$

A point x is called a fixed point of t if $tx = x$.

A multi-valued mapping $T : K \to FB(K)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K.$$

In 2010, Abkar and Eslamian [1] mentioned the Suzuki generalized multi-valued nonexpansive mapping as follows:

A multi-valued mapping $T : K \to FB(K)$ is said to be a *Suzuki generalized multi-valued nonexpansive mapping* if

$$\frac{1}{2} \operatorname{dist}(x, Tx) \leq d(x, y) \Rightarrow H(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K.$$

Let $T : K \to PB(K)$ be a multi-valued mapping and define the mapping P_T for each x by

$$P_T(x) := \{y \in Tx : d(x, y) = \operatorname{dist}(x, Tx)\}.$$

A point x is called a fixed point for a multivalued mapping T if $x \in Tx$.

We use the notation $Fix(T)$ stands for the set of fixed points of a mapping T and $Fix(t) \cap Fix(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $tx = x \in Tx$.

In 2009, Laokul and Panyanak [14] defined the iterative algorithm and proved the $\Delta-$convergence for nonexpansive mapping in $CAT(0)$ spaces as follows:

Theorem 1.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space and $t : K \to K$ be a nonexpansive mapping with $Fix(t) := \{x \in C : tx = x\} \neq \emptyset$. Suppose the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$,*

$$\begin{cases} y_n = \beta_n t x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n t y_n \oplus (1-\alpha_n)x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ such that one of the following two conditions is satisfied:

(i) *$\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some a,b with $0 < a \le b < 1$,*
(ii) *$\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a,b with $0 < a \le b < 1$.*

Then the sequence $\{x_n\}$ is Δ-convergent to a fixed point of t.

In 2010, Sokhuma and Kaewkhao [21] proved the convergence theorem for a common fixed point in Banach spaces as follows:

Theorem 1.2. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : K \to K$ and $T : K \to KC(K)$ be a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Suppose that the iteration $\{x_n\}$ is generated by $x_1 \in K$,*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \le \alpha_n, \beta_n \le b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2013, Sokhuma [19] proved the convergence theorem for a common fixed point in $CAT(0)$ spaces as follows:

Theorem 1.3. *Let K be a nonempty compact convex subset of a complete $CAT(0)$ space X, and $t : K \to K$ and $T : K \to FC(K)$ a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let the iterative sequence $\{x_n\}$ is generated by $x_1 \in K$,*

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t y_n, \end{cases}$$

for all $n \in \mathbb{N}$, *where* $z_n \in Tx_n$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences of positive numbers satisfying* $0 < a \leq \alpha_n, \beta_n \leq b < 1$. *Then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* t *and* T.

In 2015, Akkasriworn and Sokhuma [2] proved the convergence theorem for a common fixed point in a complete $CAT(0)$ spaces as follow:

Theorem 1.4. *Let* K *be a nonempty bounded closed convex subset of a complete* $CAT(0)$ *space* X, $t : K \to K$ *and* $T : K \to FB(K)$ *an asymptotically nonexpansive mapping and a mult-ivalued nonexpansive mapping, respectively. Assume that* t *and* T *are commuting and* $Fix(t) \cap Fix(T) \neq \emptyset$ *satisfying* $Tw = \{w\}$ *for all* $w \in Fix(t) \cap Fix(T)$ *and* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. *Let* $\{x_n\}$ *be the sequence of the modified Ishikawa iteration defined by*

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t^n y_n, \end{cases}$$

for all $n \in \mathbb{N}$, *where* $z_n \in T(t^n x_n)$ *and* $\{\alpha_n\}$, $\{\beta_n\} \in [0,1]$. *Then* $\{x_n\}$ *is* Δ*-convergent to a common fixed point of* t *and* T.

In 2019, Sokhuma [20] proved the convergence theorem for a common fixed point in $CAT(0)$ spaces as follow:

Theorem 1.5. *Let* K *be a nonempty bounded closed convex subset of a complete CAT(0) space* X, $t : K \to K$ *be a single-valued asymptotically nonexpansive mapping, and* $T : K \to PB(K)$ *be a multi-valued nonexpansive mapping and* $P_T(x) = \{y \in Tx : d(x,y) = dist(x, Tx)\}$. *For fixed* $x_1 \in K$. *The sequence* $\{x_n\}$ *of the Ishikawa iteration is defined by*

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t^n y_n, \end{cases}$$

for all $n \in \mathbb{N}$, *where* $z_n \in P_T(t^n x_n)$ *and* $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$. *Then* $\{x_n\}$ *is* Δ*-convergent to a common fixed point of* t *and* T.

In 2011, Espinola et al. [9] proved the theorem for a common fixed point in $CAT(0)$ spaces as follows:

Theorem 1.6. *Let* X *be a complete uniformly convex space with convex metric and* K *be a bounded closed convex subset of* X. *Suppose* $t : K \to K$ *and* $T : K \to KC(X)$ *be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively. If* t *and* T *are commute, then there exists* $w \in K$ *such that* $tw = w \in Tw$.

The purpose of this paper is to study the iterative process, called the Ishikawa iteration method with respect to the Suzuki generalized nonexpansive single valued and multivalued mapping in a complete $CAT(0)$ space. We also establish the convergence theorem of a sequence from such process in a nonempty bounded closed convex subset of a complete $CAT(0)$ spaces. We remove a restricted condition (called end-point condition) in Akkasriworn et al. [3] and Sokhuma [20] results.

Now, we introduce an iteration method modifying the above ones and call it the Ishikawa iteration method.

Definition 1.7. Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively and

$$P_T(x) = \{y \in Tx : d(x, y) = \operatorname{dist}(x, Tx)\}.$$

For fixed $x_1 \in K$. The sequence $\{x_n\}$ of the Ishikawa iteration is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t y_n, \end{cases} \tag{1.1}$$

for all $n \in \mathbb{N}$, where $z_n \in P_T(tx_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0, 1)$.

2. Preliminaries

With a view to make, our presentation self contained, we collect some relevant basic definitions, results and iterative methods which will be used frequently in the text later.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, s] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(s) = y$, and $d(c(t), c(u)) = |t - u|$ for all $t, u \in [0, s]$. In particular, c is an isometry and $d(x, y) = s$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$. If x, y_1, y_2 are points in a $CAT(0)$ space and

$$y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2,$$

then the $CAT(0)$ inequality implies that

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{2.1}$$

This is the (CN) inequality of Bruhat and Tits [6]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality [5].

The following results and methods deal with the concept of asymptotic centers. Let K be a nonempty closed convex subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

Let

$$r \equiv r(K, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in K\}$$

and

$$A \equiv A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r and the set A are called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to K, respectively.

It is easy to know that if X is complete $CAT(0)$ spaces and K is a closed convex subset of X, then $A(K, \{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. A bounded sequence $\{x_n\}$ is said to be regular with respect to K if for every subsequence $\{x'_n\}$, we get

$$r(K, \{x_n\}) = r(K, \{x'_n\}).$$

We now give the definition of Δ-convergence.

Definition 2.1. ([13, 16]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We now collect some elementary facts about $CAT(0)$ spaces which will be used in the proofs of our main results. The following lemma can be found in [7, 8, 13].

Lemma 2.2. ([7]) *If K is a closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.3. ([8]) *Let (X, d) be a $CAT(0)$ space.*

(i) *For $x, y \in X$ and $u \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = ud(x, y) \quad \textit{and} \quad d(y, z) = (1 - u)d(x, y). \tag{2.2}$$

We use the notation $(1 - u)x \oplus ty$ for the unique point z satisfying (2.2).

(ii) *For $x, y, z \in X$ and $u \in [0, 1]$, we have*

$$d((1 - u)x \oplus uy, z) \leq (1 - u)d(x, z) + ud(y, z).$$

Lemma 2.4. ([13]) *Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence.*

We now collect some basic properties of Suzuki generalized nonexpansive mapping. Although the proofs follow the idea of the proofs in [22]. The following two propositions are very easy to verify.

Proposition 2.5. *Let K be a nonempty subset of a $CAT(0)$ space X and $t : K \to K$ be a nonexpansive mapping. Then t is a Suzuki generalized nonexpansive mapping.*

Proposition 2.6. *Let K be a nonempty subset of a $CAT(0)$ space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping and has a fixed point. Then t is a quasi-nonexpansive mapping.*

Lemma 2.7. *Let K be a nonempty subset of a CAT(0) space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping. Then*

$$d(x, ty) \leq 3d(tx, x) + d(x, y)$$

holds for all $x, y \in K$.

The existence of fixed points for generalized Suzuki nonexpansive mappings in $CAT(0)$ spaces was proved by Nanjaras et al. [17] as the following result.

Theorem 2.8. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mappings. Then t has a fixed point in K.*

Lemma 2.9. *Let K be a closed and convex subset of a complete $CAT(0)$ space X and let $t : K \to X$ be a generalized Suzuki nonexpansive mappings. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\Delta-\lim_{n\to\infty} x_n = w$. Then $tw = w$.*

The following fact is well-known [15].

Lemma 2.10. *Let X be a complete $CAT(0)$ space and let $x \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, x) = r$ for some $r \ge 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 2.11. *Let X be a CAT(0) space, K be a nonempty compact convex subset of X and $\{x_n\}$ be a sequence in K. Then,*

$$dist(y, Ty) \le d(y, x_n) + dist(x_n, Tx_n) + H(Tx_n, Ty),$$

where $y \in K$ and T be a multi-valued mapping from K in to $FB(K)$.

3. Main results

We first prove the following lemmas which play very important roles in this section.

Lemma 3.1. *Let $T : K \to PB(K)$ be a multi-valued mapping and*

$$P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}.$$

Then the followings are equivalent:

(1) *$x \in Fix(T)$, that is $x \in Tx$;*
(2) *$P_T(x) = \{x\}$, that is $x = y$ for each $y \in P_T(x)$;*
(3) *$x \in Fix(P_T)$, that is $x \in P_T(x)$.*

Further, $Fix(T) = Fix(P_T)$.

Proof. (1) $\Longrightarrow$ (2). Since $x \in Tx$, $d(x, Tx) = 0$. Therefore, for any $y \in P_T(x)$, $d(x, y) = \text{dist}(x, Tx) = 0$ and so $x = y$. That is, $P_T(x) = \{x\}$.

(2) $\Longrightarrow$ (3). Since $P_T(x) = \{x\}$, $x \in \text{Fix}(P_T)$ and we get $x \in P_T(x)$.

(3) $\Longrightarrow$ (1). Since $x \in \text{Fix}(P_T)$, $x \in P_T(x)$. Therefore,

$$d(x, x) = \text{dist}(x, Tx) = 0$$

and so $x \in Tx$ by the closedness of Tx. This implies that $\text{Fix}(T) = \text{Fix}(P_T)$. □

Lemma 3.2. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1). *Then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_1 \in K$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Since, $\frac{1}{2}d(tw, w) = 0 \leq d(x_n, w)$, $d(tx_n, tw) \leq d(x_n, w)$. Similarly, we obtain $\frac{1}{2}d(tw, w) = 0 \leq d(y_n, w)$, and then we get $d(ty_n, tw)$ $\leq d(y_n, w)$. Now consider,

$$\begin{aligned} d(y_n, w) &= d((1-\beta_n)x_n \oplus \beta_n z_n, w) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n d(z_n, w) \\ &= (1-\beta_n)d(x_n, w) + \beta_n dist(z_n, P_T(w)) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n H(P_T(tx_n), P_T(w)) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n d(tx_n, w) \\ &= (1-\beta_n)d(x_n, w) + \beta_n d(tx_n, tw) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n d(x_n, w) \\ &= d(x_n, w). \end{aligned}$$

Hence, $d(y_n, w) \leq d(x_n, w)$. It implies that

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\alpha_n)x_n \oplus \alpha_n ty_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(ty_n, tw) \\ &= (1-\alpha_n)d(x_n, w) + \alpha_n d(y_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(x_n, w) \\ &= d(x_n, w). \end{aligned}$$

Since $\{d(x_n, w)\}$ is bounded below and decreasing sequence, we obtain the limit of $\{d(x_n, w)\}$. □

Lemma 3.3. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1). *If $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(ty_n, x_n) = 0$.*

Proof. Let $x_1 \in K$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. From Lemma 3.2, we setting $\lim_{n\to\infty} d(x_n, w) = c$. Recall that, $d(ty_n, w) = d(ty_n, tw) \leq d(y_n, w) \leq d(x_n, w)$. Then we have,

$$\limsup_{n\to\infty} d(ty_n, w) \leq \limsup_{n\to\infty} d(y_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c. \tag{3.1}$$

Since, $c = \lim_{n\to\infty} d(x_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n ty_n, w)$, it follows from the condition of α_n and Lemma 2.10 that $\lim_{n\to\infty} d(ty_n, x_n) = 0$. □

Lemma 3.4. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1). *If $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(x_n, z_n) = 0$.*

Proof. Let $x_0 \in K$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Consider,

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\alpha_n)x_n \oplus \alpha_n ty_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(ty_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(y_n, w) \\ &= d(x_n, w) - \alpha_n d(x_n, w) + \alpha_n d(y_n, w) \\ &= d(x_n, w) + \alpha_n (d(y_n, w) - d(x_n, w)) \end{aligned}$$

and hence

$$\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \leq d(y_n, w) - d(x_n, w).$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \leq d(y_n, w).$$

Thus,

$$\liminf_{n\to\infty} \left\{ \left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \right\} \leq \liminf_{n\to\infty} d(y_n, w).$$

It follows that

$$c \leq \liminf_{n\to\infty} d(y_n, w).$$

Since, from (3.1), $\limsup_{n\to\infty} d(y_n, w) \leq c$, we have

$$c = \lim_{n\to\infty} d(y_n, w) = \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n z_n, w). \tag{3.2}$$

Recall that

$$d(z_n, w) = \mathrm{dist}(z_n, P_T(w)) \leq H(P_T(tx_n), P_T(w)) \leq d(tx_n, w) \leq d(x_n, w).$$

Hence we have

$$\limsup_{n\to\infty} d(z_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c.$$

Since $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and (3.2) we obtain

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

□

Lemma 3.5. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1)*. If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(tx_n, x_n) = 0$.*

Proof. By Lemma 2.7, we obtain

$$\begin{aligned}
d(tx_n, x_n) &\leq d(tx_n, y_n) + d(y_n, x_n) \\
&\leq 3d(ty_n, y_n) + d(x_n, y_n) + d(x_n, y_n) \\
&= 3d(ty_n, y_n) + 2d(x_n, y_n) \\
&\leq 3d(ty_n, x_n) + 3d(x_n, y_n) + 2d(x_n, y_n) \\
&= 3d(ty_n, x_n) + 5d(x_n, y_n) \\
&= 3d(ty_n, x_n) + 5d(x_n, (1-\beta_n)x_n \oplus \beta_n z_n) \\
&\leq 3d(ty_n, x_n) + 5(1-\beta_n)d(x_n, x_n) + 5\beta_n d(x_n, z_n) \\
&= 3d(ty_n, x_n) + 5\beta_n d(x_n, z_n).
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} d(tx_n, x_n) \leq \lim_{n\to\infty} 3d(ty_n, x_n) + \lim_{n\to\infty} 5\beta_n d(x_n, z_n).$$

Hence, by Lemma 3.3 and Lemma 3.4, $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Theorem 3.6. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1)*. If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is Δ-convergent to y in $Fix(t) \cap Fix(T)$.*

Proof. Since $\{x_n\}$ is $\Delta-$convergent to y, from Lemma 3.5, $\lim_{n\to\infty} d(tx_n, x_n) = 0$. By Lemma 2.9, $y \in K$ and $ty = y$, it follows that $y \in \text{Fix}(t)$. By Lemma 2.11, which implies that

$$\begin{aligned}\operatorname{dist}(y, P_T(y)) &\leq d(y, x_n) + \operatorname{dist}(x_n, P_T(tx_n)) + H(P_T(tx_n), P_T(y))\\ &\leq d(y, x_n) + d(x_n, z_n) + d(tx_n, y)\\ &\leq d(x_n, y) + d(x_n, z_n) + d(tx_n, x_n) + d(x_n, y) \to 0 \text{ as } n \to \infty.\end{aligned}$$

It follows that, $y \in \operatorname{Fix}(P_T)$ then $y \in \operatorname{Fix}(T)$. Therefore $y \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ as desired. □

Theorem 3.7. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by* (1.1)*. If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is Δ-convergent to a common fixed point of t and T.*

Proof. Since Lemma 3.5 guarantees that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(tx_n, x_n) = 0$. We now let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Since $\lim_{n\to\infty} d(tv_n, v_n) = 0$, then $v \in \operatorname{Fix}(t)$. Since,

$$\begin{aligned}\operatorname{dist}(v, P_T(v)) &\leq \operatorname{dist}(v, P_T(tv_n)) + H(P_T(tv_n), P_T(v))\\ &\leq d(v, z_n) + d(tv_n, v)\\ &\leq d(v, v_n) + d(v_n, z_n) + d(tv_n, v) \to 0 \ \text{ as } n \to \infty.\end{aligned}$$

It follows that $v \in \operatorname{Fix}(P_T)$, we get $v \in \operatorname{Fix}(T)$ by Lemma 3.1. Therefore $v \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ as desired. We claim that $u = v$. If not, since t is a single-valued Suzuki generalized mapping and $v \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ such that $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.2, then by the uniqueness of asymptotic center,

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u)\\ &\leq \limsup_{n\to\infty} d(u_n, u)\\ &< \limsup_{n\to\infty} d(u_n, v)\\ &= \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

which is a contradiction, and hence $u = v \in \operatorname{Fix}(t) \cap \operatorname{Fix}(T)$.

To show that $\{x_n\}$ is $\Delta-$convergent to a common fixed point, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.

We can complete the proof by showing that $x = v$. If not, since $\lim_{n\to\infty} d(x_n, v)$ exists, by the uniqueness of asymptotic center,

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x)\\ &\le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v)\end{aligned}$$

which is a contradiction, and hence the conclusion follows. □

Acknowledgments: The author thanks for the support of the Institute for Research and Development, Phranakhon Rajabhat University.

References

[1] A. Abkar and M. Eslamian, *Fixed Point Theorems for Suzuki Generalized Nonexpansive Multivalued Mappings in Banach Space,* Fixed Point Theory Appl., (2010); doi:10.1155/2010/457935.

[2] N. Akkasriworn and K. Sokhuma, *Convergence theorem for a pair of asymptotically and multivalued nonexpansive mapping,* Commun. Korean Math. Soc., **30** (2015), 177–189.

[3] N. Akkasriworn, K. Sokhuma and K. Chuikamwong, *Ishikawa iterative process for a pair of Suzuki generalized nonexpansive single valued and multivalued mappings in Banach spaces,* Int. J. Math. Anal., **19** (2012), 923–932.

[4] M. Asadi, Sh. Ghasemzadehdibagi, S. Haghayeghi and N. Ahmad, *Fixed point theorems for (α, p)-nonexpansive mappings in $CAT(0)$ spaces,* Nonlinear Funct. Anal. Appl., **26**(5) (2021), 1045-1057.

[5] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[6] F. Bruhat and J. Tits, *Groupes reductifs sur un corps local I. Donnees radicielles valuees*, Inst. Hautes Etudes Sci. Publ. Math., **41** (1972), 5–251.

[7] S. Dhompongsa, W.A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces,* J. Nonlinear Convex Anal., **8** (2007), 35–45.

[8] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly Lipschitzian mappings,* Nonlinear Anal., **65** (2006), 762–772.

[9] R. Espinola, P. Lorenzo and A. Nicolae, *Fixed points, selections and common fixed points for nonexpansive-type Mappings*, J. Math. Anal. Appl., **382** (2011), 503–515.

[10] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R.L. Gupta, *Demiclosedness principle and convergence theorems for Lipschitzian type nonself-mappings in $CAT(0)$ spaces,* Nonlinear Funct. Anal. Appl., **23**(1) (2018), 73-95.

[11] K.S. Kim, *Existence theorem of a fixed point for asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces,* Nonlinear Funct. Anal. Appl., **25**(2) (2020), 355-362.

[12] W.A. Kirk, *Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003).* Colecc. Abierta, vol.64, Univ. Sevilla Secr. Publ., Seville, **6** (2003), 195–225.

[13] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal., **68** (2008), 3689–3696.

[14] T. Laokul and B. Panyanak, *Approximating Fixed Points of Nonexpansive Mappings in $CAT(0)$ Spaces,* Int. J. Math. Anal., **3** (2009), 1305–1315.

[15] W. Laowang and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in $CAT(0)$ spaces,* Fixed Point Theory Appl., (2010); doi:10.1155/2010/367274.

[16] T.C. Lim, *Remarks on some fixed point theorems,* Proc. Amer. Math. Soc., **60** (1976), 179–182.

[17] B. Nanjaras, B. Panyanaka and W. Phuengrattana, *Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in $CAT(0)$ spaces,* Nonlinear Analysis: Hybrid Syst., **4** (2010), 25–31.

[18] G.A. Okeke, M. Abbas and M. de la Sen, *Fixed point theorems for convex minimization problems in complex valued $CAT(0)$ spaces,* Nonlinear Funct. Anal. Appl., **25**(4) (2020), 671-696.

[19] K. Sokhuma, Δ-*Convergence Theorems for a Pair of Single valued and Multivalued Nonexpansive Mappings in $CAT(0)$ spaces,* J. Math. Anal., **4** (2013), 23–31.

[20] K. Sokhuma, *An Ishikawa Iteration Scheme for two Nonlinear Mappings in $CAT(0)$ Spaces,* Kyungpook Math. J., **59** (2019), 665–678.

[21] K. Sokhuma and A. Kaewkhao, *Ishikawa Iterative Process for a Pair of Single-valued and Multivalued Nonexpansive Mappings in Banach Spaces,* Fixed Point Theory Appl., (2010); doi:10.1155/2010/618767.

[22] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mapping*, J. Math. Anal. Appl., **340** (2008), 1088–1095.