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SOME FIXED POINT THEOREMS FOR GENERALIZED KANNAN TYPE MAPPINGS IN RECTANGULAR b-METRIC SPACES

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Abstract. This present paper extends some fixed point theorems in rectangular b-metric spaces using subadditive altering distance and establishing the existence and uniqueness of fixed point for Kannan type mappings. Non-trivial examples are further provided to support the hypotheses of our results.

1. Introduction

In 1968, Kannan proved that a contractive mapping with a fixed point need not be necessarily continuous and presented the following fixed point result.

Theorem 1.1. ([13]) *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a mapping such that there exists* $0 < k < \frac{1}{2}$ *satisfying*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)], \ \ \forall x,y \in X.$$

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Then, T has a unique fixed point $u \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to u and

$$d(T^{n+1}x, u) \leq k(\frac{k}{1-k})^n d(x, Tx), \quad n = 0, 1, 2, \cdots .$$

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions. In particular, b-metric spaces were introduced by Bakhtin [1] and Czerwik [2], in such a way that triangle inequality is replaced by the b-triangle inequality:

$$d(x,y) \leq b\,(d(x,z) + d(z,y))$$

for all pairwise distinct points x, y, z and $b \geq 1$. Various fixed point results were established on such spaces, see [3, 4, 5, 10, 11, 14, 15, 17, 18, 19, 20].

In this paper, we provide some fixed point results for generalized Kannan type mapping in rectangular b-metric spaces. Moreover, an illustrative examples is presented to support the obtained results.

2. Preliminaries

Combining conditions used to define b-metric and rectangular metric spaces, George et al. [9] announced the notions of b-rectangular metric space as follow:

Definition 2.1. ([9]) Let X be a nonempty set, $b \geq 1$ be a given real number, and let $d : X \times X \to [0, +\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(1) $d(x,y) = 0$, if only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq b\,[d(x,u) + d(u,v) + d(v,y)]$ (b-rectangular inequality).

Then (X, d) is called a b-rectangular metric space.

Example 2.2. Let $X = \mathbb{R}$. Define $d(x,y) = |x - y|$ where $x, y \in \mathbb{R}$. It is easy to verify that d is a rectangular b-metric and $(X, \mathbb{R}, d)$ is a complete rectangular b-metric space.

We try to extend the result of Kannan using the following class of subadditive altering distance functions.

Definition 2.3. ([12]) A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a subadditive altering distance function if

(1) φ is an altering distance function (that is, φ is continuous, strictly increasing and $\varphi(t) = 0$ if and only if $t = 0$),
(2) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$, $\forall x, y \in [0, \infty)$.

Example 2.4. The functions $\varphi_1(x) = \sqrt{x}$, $\varphi_2(x) = 3x$ and $\varphi_3(x) = \log(1+x)$ are subadditive altering distance functions.

We note that, if φ is subadditive, then for any non negative real number $k < 1$, $\varphi(d(x,y)) \leq k\varphi(d(a,b))$ implies $d(x,y) \leq k'd(a,b)$ for some $k' < 1$.

3. Main result

Consider φ as a subadditive altering distance function and the b-metric d is assumed to be continuous in the topology generated by it, we give some new fixed point results.

Theorem 3.1. *Let (X,d) be a complete rectangular b-metric space with coefficient $b \geq 1$ and $T : X \to X$ be a mapping such that there exists $p < \frac{1}{2b+1}$ satisfying:*

$$\varphi(d(Tx,Ty)) \leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))],\ \forall x,y \in X. \quad (3.1)$$

Then, T has a unique fixed point $u \in X$, the sequence $\{T^n x\}$ converges to u and for $q = \frac{2p}{1-p} < 1$ we have

$$d(T^{n+1}x, T^n x) \leq q^n d(x,Tx),\ n = 0,1,2,3,\cdots .$$

Proof. Let $z = Tx$ for an arbitrary element $x \in X$. Then

$$\begin{aligned}\varphi(d(z,Tz)) &= \varphi(d(Tx,Tz)) \\ &\leq p[\varphi(d(x,z)) + \varphi(d(x,Tx)) + \varphi(d(z,Tz)).\end{aligned}$$

Hence we have

$$\varphi(d(z,Tz)) \leq q\varphi(d(x,Tx)),$$

where $q = \frac{2p}{1-p} < 1$, it implies that

$$d(z,Tz) \leq q'd(x,Tx) \quad (3.2)$$

for $q' < 1$.

Without loss of generality, we assume $q = q'$. Let $x_0 \in X$, consider the sequence $\{x_n\} \subset X$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_n = Tx_n$. Then x_n is a fixed point of T and the proof is finished. Hence, we assume that $x_n \neq Tx_n$ for all $n \in \mathbb{N}$. Then for $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned}
&d(x_{m+r}, x_m)\\
&\le b[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + d(x_{m+r-2}, x_m)]\\
&\le bd(x_{m+r}, x_{m+r-1}) + bd(x_{m+r-1}, x_{m+r-2})\\
&\quad + b[b[d(x_{m+r-2}, x_{m+r-3}) + d(x_{m+r-3}, x_{m+r-4}) + d(x_{m+r-4}, x_m)]]\\
&= bd(x_{m+r}, x_{m+r-1}) + bd(x_{m+r-1}, x_{m+r-2}) + b^2 d(x_{m+r-2}, x_{m+r-3})\\
&\quad + b^2 d(x_{m+r-3}, x_{m+r-4}) + b^2 d(x_{m+r-4}, x_m)\\
&\le bd(x_{m+r}, x_{m+r-1}) + bd(x_{m+r-1}, x_{m+r-2}) + b^2 d(x_{m+r-2}, x_{m+r-3})\\
&\quad + b^2 d(x_{m+r-3}, x_{m+r-4}) + \cdots + b^{\frac{r-1}{2}} d(x_{m+3}, x_{m+2})\\
&\quad + b^{\frac{r-1}{2}} d(x_{m+2}, x_{m+1}) + b^{\frac{r-1}{2}} d(x_{m+1}, x_m)\\
&\le d(x_1, x_0)(bq^{m+r-1} + b^2 q^{m+r-3} + \cdots + b^{\frac{r-1}{2}} q^{m+2} + bq^{m+r-2}\\
&\quad + b^2 q^{m+r-4} + \cdots + b^{\frac{r-1}{2}} q^{m+1} + b^{\frac{r-1}{2}} q^m)\\
&= \sum_{k=1}^{\frac{r-1}{2}} b^k q^{m+r-(2k-1)} d(x_1, x_0) + \sum_{k=1}^{\frac{r-1}{2}} b^k q^{m+r-2k} d(x_1, x_0) + b^{\frac{r-1}{2}} q^m d(x_1, x_0)\\
&\to 0 \text{ as } m \to \infty.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists an $x \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

Since

$$d(Tx, x) \le b[d(Tx, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x)],$$

$$\begin{aligned}
\varphi(d(Tx, x)) \le bp[&\varphi(d(x, x_n)) + \varphi(d(x, Tx)) + \varphi(x_n, x_{n+1}) + \varphi(d(x_n, x_{n+1})\\
&+ \varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))] + b\varphi(d(Tx_{n+1}, x)).
\end{aligned}$$

Then

$$\begin{aligned}
(1 - bp)\varphi(d(Tx, x)) &\le bp[\varphi(d(x, x_n)) + \varphi(x_n, x_{n+1}) + \varphi(d(x_n, x_{n+1})\\
&\quad + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))] + b\varphi(d(Tx_{n+1}, x))\\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This implies that $Tx = x$, it means that that x is a fixed point of T.

Now if $y (\neq x)$ is an another fixed point of T, then

$$\varphi(d(x, y)) \le p[\varphi(d(x, y)) + \varphi(d(x, Tx)) + \varphi(d(y, Ty))],$$

it implies that

$$\varphi(d(x,y)) \leq p\varphi(d(x,y)).$$

Since φ is strictly increasing and $p < \frac{1}{2b+1}$, $d(x,y) = 0$, therefore the fixed point of T is unique. From (3.2) we have

$$d(T^{n+1}x, T^n x) \leq qd(T^{n-1}x, T^n x),$$

where $q = \frac{2p}{1-p} < 1$, that is,

$$d(T^{n+1}x, T^n x) \leq q^n d(x, Tx)$$

for all $n = 0, 1, 2, \cdots$. This completes the proof. □

Example 3.2. Let $X = \mathbb{R}$ and (X,d) the complete rectangular b-metric space as given in Example 2.2.

Define $T : X \to X$, by $Tx = \dfrac{x}{3}$ for all $x \in X$ and $\varphi(t) = 2t$, we have

$$\varphi(d(Tx,Ty)) < \frac{1}{6}(\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \ \forall x,y \in X.$$

Then T is a continuous map satisfying (3.1) and 0 is a unique fixed point of T and the sequence $\{T^n x\} = \{\dfrac{x}{3^n}\}$ for any point $x \in X$ converges to 0.

Corollary 3.3. *Let (X,d) be a complete rectangular b-metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq p[d(x,y) + d(x,Tx) + d(y,Ty))], \ \ \forall x,y \in X,$$

where $p < \frac{1}{2b+1}$. Then, T has a fixed point in X.

Proof. From Theorem 3.1 if we take $\varphi(x) = x$, we obtain the result. □

Theorem 3.4. *Let (X,d) be a complete rectangular b-metric space with coefficient $b \geq 1$ and $T : X \to X$ be a mapping such that there exists p_1, p_2, p_3 with $p_1 + p_2 + p_3 < 1$ and $bp_2 < 1$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p_1\varphi(d(x,y)) + p_2\varphi(d(x,Tx)) + p_3\varphi(d(y,Ty)), \ \forall x,y \in X. \quad (3.3)$$

Then T has a unique fixed point $u \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to u and for $q = \dfrac{p_1 + p_2}{1 - p_3}$,

$$d(T^{n+1}x, T^n x) \leq q^n d(x, Tx), \ n = 0, 1, 2, \cdots .$$

Proof. Similary to the proof of Theorem 3.1 if we consider a metric space (X,d) and $\varphi(x) = x$. □

Example 3.5. Let $X = [0,1]$ and $d : X \times X \to [0,\infty[$ defined as $d(x,y) = |x-y|^2$ is a rectangular b-metric and $T : X \to X$ defined by $Tx = \frac{x}{2}$; if $x \in [0,1[$ and $T1 = \frac{1}{3}$. If we put $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{9}$ and $\varphi(x) = x$, we obtain that T satisfies (3.3) then T has a unique fixed point.

We can easily prove the following two theorems.

Theorem 3.6. *Let (X,d) be a rectangular b-metric space with coefficient $b \geq 1$, if every mapping $T : X \to X$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))], \ \ \forall x,y \in X,$$

for some $0 \leq p < \frac{1}{2b+1}$, then X is complete.

Theorem 3.7. *Let (X,d) be a complete rectangular b-metric space with coefficient $b \geq 1$, and $T : X \to X$ be a mapping such that there exists $0 \leq p < \frac{1}{2b+1}$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \ \forall x,y \in X.$$

Then T has a unique fixed point $u \in X$ and the sequence $\{T^n x\}$ converges to u.

By the proof of Theorem 3.1, we get the following result which is the Kannan theorem as a consequence.

Theorem 3.8. *Let (X,d) be a complete rectangular b-metric space with coefficient $b \geq 1$, and $T : X \to X$ be a mapping such that there exists $p < \frac{1}{2b}$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \ \forall x,y \in X. \tag{3.4}$$

Then T has a unique fixed point $u \in X$, and for all $x \in X$ the sequence $\{T^n x\}$ converges to u and for $q = \frac{p}{1-p} < 1$,

$$d(T^{n+1}x,u) \leq q^n d(x,Tx), \ n = 0,1,2,\cdots.$$

Proof. Let x_0 be an arbitrary point of X. Consider the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\varphi(d(x_n,x_{n+1})) &= \varphi(d(Tx_{n-1},Tx_n)) \\ &\leq p[\varphi(d(x_{n-1},Tx_{n-1})) + \varphi(d(x_n,Tx_n))] \\ &\leq p[\varphi(d(x_{n-1},x_n)) + \varphi(d(x_n,x_{n+1}))].\end{aligned}$$

Hence, we get

$$(1-p)\varphi(d(x_n,x_{n+1}))\le p\varphi(d(x_{n-1},x_n)),$$

that is,

$$\varphi(d(x_n,x_{n+1}))\le \frac{p}{1-p}\varphi(d(x_{n-1},x_n)).$$

From (3.2), we get

$$\begin{aligned}d(x_n,x_{n+1}) &\le \frac{p}{1-p}d(x_{n-1},x_n)=qd(x_{n-1},x_n)\\ &\le q^n d(x_0,x_1)\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

For $m\ge 1$ and $r\ge 1$, it follows that

$$\begin{aligned}&d(x_{m+r},x_m)\\ &\le b[d(x_{m+r},x_{m+r-1})+d(x_{m+r-1},x_{m+r-2})+d(x_{m+r-2},x_m)]\\ &\le bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})\\ &\quad +b[b[d(x_{m+r-2},x_{m+r-3})+d(x_{m+r-3},x_{m+r-4})+d(x_{m+r-4},x_m)]]\\ &= bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &\quad +b^2d(x_{m+r-3},x_{m+r-4})+b^2d(x_{m+r-4},x_m)\\ &\le bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &\quad +b^2d(x_{m+r-3},x_{m+r-4})+....+b^{\frac{r-1}{2}}d(x_{m+3},x_{m+2})\\ &\quad +b^{\frac{r-1}{2}}d(x_{m+2},x_{m+1})+b^{\frac{r-1}{2}}d(x_{m+1},x_m)\\ &\le d(x_1,x_0)(bq^{m+r-1}+b^2q^{m+r-3}+...+b^{\frac{r-1}{2}}q^{m+2}\\ &\quad +bq^{m+r-2}+b^2q^{m+r-4}+...+b^{\frac{r-1}{2}}q^{m+1}+b^{\frac{r-1}{2}}q^m\\ &=\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-(2k-1)}d(x_1,x_0)+\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-2k}d(x_1,x_0)+b^{\frac{r-1}{2}}q^md(x_1,x_0)\\ &\to 0 \quad \text{as } m\to\infty.\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists an $x\in X$ such that

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}Tx_{n-1}=x.$$

From

$$d(Tx,x)\le b[d(Tx,Tx_n)+d(Tx_n,x_n)+d(x_n,x)],$$

we have

$$\begin{aligned}\varphi(d(Tx,x)) &\leq bp[\varphi(d(Tx,Tx_n)) + \varphi(Tx_n,x_n) + \varphi(d(x_n,x)).\\ &\leq bp[\varphi(d(x,Tx)) + \varphi(d(x_n,Tx_n))]\\ &\quad + b\varphi(d(Tx_n,x_n)) + b\varphi(d(x_n,x)).\end{aligned}$$

Hence, we have

$$\begin{aligned}(1-bp)\varphi(d(Tx,x)) &\leq b(p+1)\varphi(d(Tx_n,x_n)) + b\varphi(d(x_n,x))\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This implies that $Tx = x$, it means that x is a fixed point of T.

Now, if $y(\neq x)$ is an another fixed point of T, then

$$\varphi(d(x,y)) \leq p[\varphi(d(x,Tx)) + \varphi(d(y,Ty))].$$

Hence,

$$\varphi(d(x,y)) \leq p(\varphi(d(x,x)) + \varphi(d(y,y))) = 0,$$

then $d(x,y) = 0$. Therefore, the fixed point of T is unique. From (3.2), we have

$$d(T^{n+1}x, T^n x) \leq q d(T^{n-1}x, T^n x),$$

where $q = \frac{p}{1-p} < 1$, that is,

$$d(T^{n+1}x, T^n x) \leq q^n d(x,Tx)$$

for all $n = 0,1,2,\cdots$. □

Example 3.9. Consider the complete rectangular b-metric space (X,d), where $X = \mathbb{R}$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Define the mapping $T : X \to X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \leq 1,\\ -\dfrac{1}{3}, & \text{if } x > 1.\end{cases}$$

Then T is not continous at 1. For $\varphi(x) = 3x$, we have

$$3d(Tx,Ty) \leq 3p(d(x,Tx) + d(y,Ty)).$$

For $x \leq 1$ and $y \leq 1$,

$$\begin{aligned}d(Tx,Ty) = 0 &\leq p[d(x,Tx) + d(y,Ty)]\\ &= p[|x| + |y|]\end{aligned}$$

and

$$\varphi(d(Tx,Ty)) \leq p[\varphi(|x|) + \varphi(|y|)].$$

For $x > 1$ and $y > 1$,

$$\begin{aligned}d(Tx,Ty) = 0 &\leq p[d(x,Tx) + d(y,Ty)]\\ &= p\left[\left|x+\frac{1}{3}\right| + \left|y+\frac{1}{3}\right|\right],\end{aligned}$$

$$0 \leq p\left(x+y+\frac{2}{3}\right)$$

and

$$\varphi(d(Tx,Ty)) \leq 3p\left(x+y+\frac{2}{3}\right).$$

Thus, T satisfies (3.4). Therefore, T has a unique fixed point $x = 0$.

Theorem 3.10. *Let (X,d) be a rectangular b-metric space with coefficient $b \geq 1$, if every mapping $T : X \to X$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \ \forall x,y \in X$$

for some $p < \frac{1}{2b}$, has a unique fixed point, then X is complete.

In 1975, Subrahmanyam [21] proved that a metric space (X,d) is complete if and only if every Kannan mapping has a unique fixed point in X. Later on, Fisher [7] and Khan [16] proved two important fixed point results related to contractive type mappings on compact metric spaces. They proved that a continuous mapping on a compact metric space (X,d) has a unique fixed point if T satisfies

$$d(Tx,Ty) < \frac{1}{2}(d(x,Ty) + d(y,Tx))$$

or

$$d(Tx,Ty) < (d(x,Tx)d(y,Ty))^{\frac{1}{2}}$$

for all $x, y \in X$ with $x \neq y$ respectively.

Since sequentially compact rectangular b-metric spaces are complete, the completeness condition in Theorem 3.8 may be replaced by sequential compactness.

A bounded compact metric space [6] is a metric space X in which every bounded sequence in X has a convergent subsequence. The same notion may be defined in the case of rectangular b-metric spaces. The class of bounded compact rectangular b-metric spaces is larger than that of sequentially compact spaces as the rectangular b-metric space $\mathbb{R}$ of real numbers with the usual metric is not sequentially compact but bounded compact. In the next result, p is independent of the coefficient b of the rectangular b-metric space.

Theorem 3.11. *Let (X,d) be a bounded compact rectangular b-metric space and $T : X \to X$ be a continuous mapping satisfying (3.4) for some $0 \leq p < \frac{1}{2}$. Then T has a unique fixed point $u \in X$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to u.*

Proof. Let $x_0 \in X$ be an arbitrary point. Consider a sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Then by (3.4) we have

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &= \varphi(d(T^n x_0, T^{n+1} x_0)) \\ &= \varphi(d(T(T^{n-1} x_0), T(T^n x_0))) \\ &\leq p(\varphi(d(T^{n-1} x_0, T^n x_0)) + \varphi(d(T^n x_0, T^{n+1} x_0))) \\ &= p(\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1}))).\end{aligned}$$

It implies that

$$(1-p)\varphi(d(x_{n-1}, x_n)) < p\varphi(d(x_n, x_{n+1})), \ \ \forall n \in \mathbb{N}.$$

Since $1 - p \geq p$,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \ \ \forall n \in \mathbb{N}.$$

This means that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing and hence convergent, so there exists $t \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = t$.

For $m, n \in \mathbb{N}$ with $n < m$, we have

$$\varphi(d(x_m, x_n)) \leq \varphi(d(x_{m-1}, x_m) + \varphi(d(x_{n-1}, x_n))),$$

and hence $\varphi(d(x_m, x_n)) \leq \varphi(t)$ as $m, n \to \infty$. This implies that $d(x_m, x_n) \leq t$ as $m, n \to \infty$, therefore, $\{x_n\}$ is a bounded sequence. Hence, $\{x_n\}$ has a subsequence which converges to u, that is, $\lim_{k\to\infty} x_{n_k} = u$. By the continuity of T we have $Tu = T(\lim_{k\to\infty} T^{n_k} x_0) = \lim T x_{n_k+1} x_0 = u$, thus, u is a fixed point of T.

Next, we show the uniqueness of the fixed point ot T. Let $z(\neq u)$ be an another fixed point of T. Then

$$\varphi(d(Tz, Tu)) \leq p(\varphi(d(z, Tz)) + \varphi(d(u, Tu))),$$

it implies that

$$\varphi(d(z, u)) \leq p(\varphi(d(z, z)) + \varphi(d(u, u))),$$

which is a contradiction. Hence, $u = z$. This completes the proof. □

Example 3.12. Let (X, d) a bounded compact rectangular b-metric space, where $X = [0, \infty[$ and

$$d(x, y) = \begin{cases} (x+y)^2, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Define $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{1}{3}, & \text{if } 0 \leq x \leq 2, \\[2ex] \dfrac{1}{x}, & \text{if } x > 2. \end{cases}$$

Then, for $\varphi(t) = 3t$, we have

$$d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty)).$$

For $x \neq y$ and $x, y > 2$, we have

$$d(Tx, Ty) = \left(\frac{1}{x} + \frac{1}{y}\right)^2 < 1$$

and

$$\frac{1}{2}(d(x, Tx) + d(y, Ty)) = \frac{1}{2}\left(\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2\right) > 1.$$

Similary, for $0 \leq x \leq 2$ and $y > 2$, we have

$$d(Tx, Ty) = \left(\frac{1}{3} + \frac{1}{y}\right)^2$$

and

$$\frac{1}{2}(d(x, Tx) + d(y, Ty)) = \frac{1}{2}\left((x + \frac{1}{3})^2 + (y + \frac{1}{y})^2\right) > \left(\frac{1}{3} + \frac{1}{y}\right)^2.$$

Thus, T has a unique fixed point $x = 3$.

Garai et al. [8] defined T-orbitally compact metric spaces and derived a fixed point result for the same. The definition of T-orbitally compactness can be extended to rectangular b-metric spaces as follows.

Definition 3.13. Let (X, d) be a rectangular b-metric space and T be a self-mapping on X. The orbit of T at $x \in X$ is defined as

$$O_x(T) = \{x, Tx, T^2x, T^3x, ...\}.$$

If every sequence in $O_x(T)$ has a convergent subsequence for all $x \in X$, X is said to be T-orbitally compact.

It is easy to see that every compact rectangular b-metric space is T-orbitally compact. Also the bounded compactness and T-orbitally compactness are totally independent. Moreover, T-orbitally compactness of X does not give to be complete.

Theorem 3.14. *Let (X, d) be a T-orbitally compact rectangular b-metric space and T satisfying (3.4) with $p < \frac{1}{2}$ and $bp < 1$. Then T has a unique fixed point u and*

$$\lim_{n\to\infty} T^n x = u, \quad \forall x \in X.$$

Proof. Let $x_0 \in X$ be arbitrary but fixed, and consider the iterative sequence $\{x_n\}$, where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. We denote $d_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Then, by (3.4) we have

$$\varphi(d_n) \leq p(\varphi(d_{n-1}) + \varphi(d_n)),$$

it implies that

$$(1-p)\varphi(d_n) \leq p\varphi(d_{n-1}).$$

Since $1-p \geq p$, $p < \frac{1}{2}$ and φ is strictly increasing, we get $d_n < d_{n-1}$, this show that $\{d_n\}$ is a strictly decreasing sequence of non negative real numbers and hence convergent. Since X is T-orbitally compact, so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $\lim_k x_{n_k} = u$

$$\lim_k d_{n_k} = \lim_k d(x_{n_k}, x_{n_{k+1}}) = d(\lim_k x_{n_k}, \lim_k x_{n_{k+1}}) = 0.$$

Therefore, $\lim_{n\to\infty} d_n = 0$.

We have for $n, m \in \mathbb{N}$,

$$\begin{aligned}\varphi(d(x_n, x_m)) &\leq p(\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_{m-1}, x_m))) \\ &= p(\varphi(d_{n-1}) + \varphi(d_{m-1})) \\ &\to 0 \quad \text{as } n, m \to \infty,\end{aligned}$$

this implies $d(x_n, x_m) \to 0$ as $n \to \infty$. This means that $\{x_n\}$ is a Cauchy sequence and $x_n \to u$ as $n \to \infty$. Also we have

$$\begin{aligned}\varphi(d(u, Tu)) &\leq \varphi(b(d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu))) \\ &\leq b\varphi(d(u, x_n)) + bp[\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1})) \\ &\quad + \varphi(d(x_n, x_{n+1}) + \varphi(d(u, Tu)))].\end{aligned}$$

This implies that

$$\begin{aligned}(1-bp)\varphi(d(u, Tu)) &\leq b\varphi(d(u, x_n)) + bp[\varphi(d(x_{n-1}, x_n)) \\ &\quad + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_n, x_{n+1})] \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, $Tu = u$.

Next, let u^* be an another fixed point of T. Then, we have

$$d(u, u^*) = d(Tu, Tu^*) < \frac{1}{2}(d(u, Tu) + d(Tu^*, Tu^*)) < 0,$$

which is a contradiction. Hence, T has a unique fixed point. □

Let us point out that Theorem 3.14 does not hold for $p \geq \dfrac{1}{2}$.

To find a solution we assume that T is an asymptotically regular mapping, that is, $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$.

Theorem 3.15. *Let (X,d) be a complete rectangular b-metric space and $T : X \to X$ be an asymptotically regular mapping satisfying* (3.4) *for some p with $bp < 1$. Then T has a unique fixed point.*

Proof. Let $x \in X$ and define the sequence $x_n = T^n x,\ n \in \mathbb{N}$. Since T is an asymptotically regular mapping, we get for $m > n$,

$$\begin{aligned}\varphi(d(T^{n+1}x, T^{m+1}x)) &\le p(\varphi(d(T^n x, T^{n+1}x)) + \varphi(d(T^m x, T^{m+1}x))) \\ &\to 0 \quad \text{as } n \to \infty,\end{aligned}$$

it implies that

$$d(T^{n+1}x, T^{m+1}x) \to 0 \ \text{ as } n \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence and convergent in X with $\lim_{n\to\infty} x_n = u$. Hence, we have

$$\begin{aligned}\varphi(d(u,Tu)) &\le \varphi(b[d(u,T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tu)]) \\ &\le b\varphi(d(u,T^n x)) + b\varphi(d(T^n x, T^{n+1}x)) + b\varphi(d(T^{n+1}x, Tu))) \\ &\le b\varphi(d(u,T^n x)) + bp[\varphi(d(T^{n-1}x, T^n x)) + \varphi(d(T^n x, T^{n+1}x)) \\ &\quad + \varphi(d(T^n x, T^{n+1}x)) + \varphi(d(u,Tu)),\end{aligned}$$

this implies that

$$\begin{aligned}(1-bp)\varphi(d(u,Tu)) &\le b\varphi(d(u,T^n x)) \\ &\quad + bp[\varphi(d(T^{n-1}x, T^n x)) + 2\varphi(d(T^n x, T^{n+1}x)))].\end{aligned}$$

When $n \to \infty$, we obtain $d(u,Tu) = 0$. Therefore, u is a fixed point of T.

Let u^* be an another fixed point of T. Then

$$d(u,u^*) = d(Tu,Tu^*) < P(d(u,Tu) + d(Tu^*,Tu^*)) = 0,$$

which is a contradiction. Hence T has a unique fixed point. □

Example 3.16. Let (X,d) be a complete rectangular b-metric space and $T : X \to X$ be an asymptotically regular mapping satisfying $Tx = \dfrac{x}{3}$ for all $x \in X$ and $d(x,y) = |x-y|^2$, $b = 2$ and $p < \dfrac{1}{2}$. Then for $\varphi(t) = \sqrt{t}$, we have $|x-y| < 2(|x| + |y|)$. Therefore, T has a unique fixed point $x = 0$.

Theorem 3.17. *Let (X,d) be a complete rectangular b-metric space and $T : X \to X$ be an asymptotically regular mapping satisfying:*

$$\varphi(d(Tx,Ty)) \le p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))],\ \forall x,y \in X$$

for some p with $bp < 1$. Then T has a unique fixed point.

Proof. Let $x \in X$ and define the sequence $x_n = T^n x, \quad n \in \mathbb{N}$. Since T is an asymptotically regular mapping, we get for $m > n$,

$$\begin{aligned}\varphi(d(T^{n+1}x, T^{m+1}x)) &\leq p(\varphi(d(T^n x, T^m x)) + \varphi(d(T^n x, T^{n+1}x)) \\ &\quad + \varphi(d(T^m x, T^{m+1}x))) \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence and convergent in X with $\lim_{n\to\infty} x_n = u$. Also, we have

$$\begin{aligned}\varphi(d(u, Tu)) &\leq \varphi(b[d(u, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tu)]) \\ &\leq b\varphi(d(u, T^n x)) + b\varphi(d(T^n x, T^{n+1}x)) + b\varphi(d(T^{n+1}x, Tu))) \\ &\leq b\varphi(d(u, T^n x)) + bp[\varphi(d(T^{n-1}x, T^n x)) \\ &\quad + \varphi(d(T^{n-1}x, T^n x)) + \varphi(d(T^n x, T^{n+1}x)) \\ &\quad + \varphi(d(T^n x, u)) + \varphi(d(T^n x, T^{n+1}x)) + \varphi(d(u, Tu))],\end{aligned}$$

this implies that

$$\begin{aligned}(1 - bp)\varphi(d(u, Tu)) &\leq b(1 + p)\varphi(d(u, T^n x)) + 2bp[\varphi(d(T^{n-1}x, T^n x)) \\ &\quad + 2\varphi(d(T^n x, T^{n+1}x)))].\end{aligned}$$

When $n \to \infty$, we obtain $d(u, Tu) = 0$. Therefore, u is a fixed point of T.

Let $u^*(\neq u)$ be an another fixed point of T. Then

$$d(u, u^*) = d(Tu, Tu^*) < P(d(u, Tu) + d(Tu^*, Tu^*)) = 0,$$

which is a contradiction. Hence, T has a unique fixed point. □

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